Qualify Exam for ODE

08/27/2010

1. (15 pts) Consider the following initial value problem

$$\begin{cases} \dot{x} = f(x,t) + \displaystyle\int_0^t g(x(s),s)ds \\ x(0) = x_0 \end{cases}$$

where f, g are Lipschitz continuous with respect to x in some neighborhood containing $(x_0, 0)$. Show that there exists $b > 0$ such that the solution exists uniquely in $(-b, b)$.

2. (15 pts) Let $x : [a,b) \to \mathbb{R}^+$ be a continuous function satisfying

$$x(t) \leq \lambda(t) + \eta(t) \int_\alpha^t K(s, x(s))ds, \ t \in [a,b)$$

where λ, η are nonnegative continuous functions on $[a,b)$, $K : [a,b) \times \mathbb{R}^+ \to \mathbb{R}^+$ be continuous such that

$$0 \leq K(t,u) - K(t,v) \leq M(t,v)(u,v), \ t \in [\alpha, \beta), \ u \geq v \geq 0,$$

for some continuous nonnegative function M on $[a,b) \times \mathbb{R}^+$. Show that

$$x(t) \leq \lambda(t) + \eta(t) \int_\alpha^t K(u, \lambda(u)) \exp\Big(\int_u^t M(s, \lambda(s))\eta(s)ds \Big) du$$

for all $t \in [\alpha, \beta)$.

3. (15 pts) Consider the initial value problem

$$\dot{x} = Ax + B(t)x + h(x,t), \quad x(t_0) = x_0, \quad x \in \mathbb{R}^n$$

where A is a constant matrix which all real part of eigenvalues are negative, and $B(t)$ is a continuous $n \times n$ matrix with $\|B(t)\| \to 0$ as $t \to \infty$. And $h(x,t)$ is a smooth function such that

$$|h(x,t)| \le k|x|^2, \ \forall t \ge 0, \ |x| < a$$

for some positive constants a, k. Show that there are constants $C > 1, \ \delta, \lambda > 0$ such that

$$|x(t)| \le C|x_0|e^{-\lambda(t-t_0)}, \quad t \ge t_0$$

whenever $|x_0| \le \frac{\delta}{C}$. In particular, the zero solution is asymptotically stable.

4. Let $A(t)$ be a continuous periodic $n \times n$ matrix with period ω.

   (a) (7 pts) Show that the linear system $\dot{x} = A(t)x$ has periodic solution of period ω (or 2ω) if and only if 1 (or -1) is a Floquet multiplier of $A(t)$.

   (b) (7 pts) Show that the inhomogeneous system $\dot{x} = A(t)x + g(t)$, where $g(t)$ is continuous periodic with period ω, has periodic solution of period ω if 1 is not a Floquet multiplier of $A(t)$.

   (c) (6 pts) Consider the equation

   $$\ddot{x} + (a + b\cos 2t)x = 0$$

   where a, b are real parameters. Give some conditions on a, b such that the equation has periodic solutions.

5. Consider the predator-prey system

$$\begin{cases} \dot{x} = \gamma x\Big(1 - \dfrac{x}{K}\Big) - \dfrac{mx}{a+x}y \\ \dot{y} = \Big(\dfrac{mx}{a+x} - d\Big)y, \qquad\qquad m > d \\ x(0) > 0, \ y(0) > 0 \end{cases}$$

(a) (6 pts) Show that the solutions $x(t)$, $y(t)$ are positive whenever they are defined, and there exists an interior equilibrium (x^*, y^*) for the system.

(b) (7 pts) If $\dfrac{K-a}{2} > x^*$. Use Poincaré-Bendixson Theorem to show that there exists a limit cycle for the system in the first quadrant of phase plane.

(c) (7 pts) If $\dfrac{K-a}{2} < x^*$. Use Dulac criterion to show that (x^*, y^*) is globally asymptotically stable in the first quadrant of phase plane.

6. (15 pts)Consider the nonlinear system

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y + z^2 \\ \dot{z} = -z - x^2 \end{cases}$$

Use the method of successive approximations to find the stable manifolds for the equilibrium 0.