Qualify Exam for ODE

01/19/2010

1. (15 pts) Let $x : [a,b] \to \mathbb{R}^+$ be a continuous function that satisfies

$$x(t) \le M + \int_a^t \Psi(s) g(x(s)) ds \text{ for } t \in [a,b]$$

where $M \ge 0$, $\Psi : [a,b] \to \mathbb{R}^+$ is continuous, $g : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and monotonic increasing. Show that

$$x(t) \le \Phi^{-1}\Big(\Phi(M) + \int_a^t \Psi(s) ds\Big) \text{ for } t \in [a,b]$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is given by $\Phi(u) = \int_{u_0}^u \frac{1}{g(s)} ds$ for some constant $u_0, u \in \mathbb{R}$.

2. (15 pts) Find the general solution of $x' = Ax + g(t)$, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 4 & 3 & -4 \\ 1 & 2 & -1 \end{pmatrix}, \quad g(t) = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}.$$

3. Consider the equation

$$x' = A(t)x$$

where $A(t)$ is a continuous T-periodic $n \times n$-matrix. Prove the following statements.

   (a) (10 pts) Each fundamental matrix $\Phi(t)$ can written as $\Phi(t) = P(t)e^{Bt}$ where $P(t)$ is T-periodic and B is a constant $n \times n$-matrix.

   (b) (5 pts) Let m_i, $i = 1, \cdots, n$ be the Floquet multipliers of $A(t)$. Then

$$\prod_{i=1}^n m_i = \exp \int_0^T \mathbf{tr}\, A(s) ds$$

(c) (5 pts) If $A(t) = \begin{pmatrix} \frac{1}{2} - \cos t & b \\ a & \frac{3}{2} + \sin t \end{pmatrix}$, where a and b are constants, then there exists at least a one-parameter family of solutions which becomes unbounded as $t \to \infty$.

4. (15 pts) Solve the system

$$\begin{cases} x' = -x \\ y' = -y + x^2 \\ z' = z + x^2 \end{cases}$$

and find the stable and unstable manifolds for the equilibrium 0.

5. Consider the following system

$$\begin{cases} x' = -x + 2y \\ y' = -2x - y + 2x^2y^2 + 2x^4 \end{cases} \tag{1}$$

(a) (7 pts) Use a suitable Lyapunov function to show that (1) has an asymptotically stable equilibrium 0.

(b) (8 pts) Estimate the domain of attraction for 0 by La Salle's invariant principle.

6. Consider the system

$$\begin{cases} x' = -y + x + x(x^2 + y^2)^2(x^4 + y^4 + 2x^2y^2 - 3) \\ y' = x + y + y(x^2 + y^2)^2(x^4 + y^4 + 2x^2y^2 - 3) \end{cases} \tag{2}$$

(a) (10 pts) Show that there is a stable limit cycle in the region

$$A_1 = \{x \in \mathbb{R}^2 | 1 < |x| < 2\}.$$

(b) (10 pts) Show that the origin is an unstable focus for (2) and there is an unstable limit cycle in the region

$$A_2 = \{x \in \mathbb{R}^2 | 0 < |x| < 1\}.$$