Qualify Exam for Differential Equations

2011. Spring semester

1. (a) (10%) Suppose that $x : [0,\infty) \to [0,\infty)$ and $y : [0,\infty) \to [0,\infty)$ are both continuous and satisfies

$$x'(t) + y(t) \le \sqrt{y(t)}x(t) \qquad \forall t \in [0,\infty).$$

Show that there exists a $T > 0$, depending only on $x(0)$, such that

$$x(t) + \int_0^t y(s)ds \le M \qquad \forall t \in [0,T]$$

for some M depending on $x(0)$.

(b) (5%) Suppose that $x : [0,\infty) \to [0,\infty)$ is continuous and satisfies

$$x(t) \le M + Ctx(t)^2 \qquad \forall t \in [0,\infty)$$

for some constant M and C. Show that there exists a $T > 0$, independent of x, such that

$$x(t) \le 2M \qquad \forall t \in [0,T].$$

Note: (a) and (b) are **NOT** related.

2. Consider the 2-d system

$$x' = x + e^y, \quad y' = -y.$$

(a) (10%) Show that the system has a saddle point at $(-1,0)$ and its unstable manifold is the x-axis.

(b) (10%) Let (x,y) be a point on the stable manifold and close to $(-1,0)$. Introduce a new variable $u = x+1$ and write the stable manifold as

$$y = a_1 u + a_2 u^2 + \mathcal{O}(u^3).$$

Determine the coefficients a_1, a_2.

3. (15%) Show that $x' = y$, $y' = -x + x^3$ has a fixed point at the origin that is a center. Show that all trajectories in a neighborhood of the origin is closed (i.e., periodic orbits).

4. Consider the operator

$$(Lx)(t) = \Big(-\frac{d^2}{dt^2} + q(t)\Big)x(t) \quad \text{or} \quad Lx = -x'' + qx,$$

in which $q(t)$ is a smooth periodic function with period 1. Let $y_1(t,\lambda)$, $y_2(t,\lambda)$ be solutions of the equation $Ly = \lambda y$ with the condition

$$\big(y_1(0), y_1'(0)\big) = (1,0), \quad \big(y_2(0), y_2'(0)\big) = (0,1). \tag{1}$$

(a) (10%) Show that $Ly = \lambda y$ has at least one periodic solution of period 1 if and only if

$$y_1(1,\lambda) + y_2'(1,\lambda) = 2\,.$$

(b) (10%) Show that $Ly = \lambda y$ has at least one periodic solution of period 2, but none of period 1, if and only if

$$y_1(1,\lambda) + y_2'(1,\lambda) = -2\,.$$

5. (15%) Show that $(0,0)$ is a globally asymptotically stable equilibrium to the system

$$\begin{aligned} x' &= -x^3 + 2y^3\,, \\ y' &= -xy^2\,. \end{aligned}$$

6. Let Ω be a bounded smooth domain of $\mathbb{R}^2$, and $u : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth vector field depending only on the spacial variables. Consider the dynamical system $x' = u(x)$.

   (a) (10%) Suppose that $u \cdot \mathrm{N} = 0$ on $\partial\Omega$, where N is the outward-pointing normal of $\partial\Omega$, and there is no equilibrium point on $\partial\Omega$. Show that $\partial\Omega$ is an orbit.

   (b) (5%) Assume conditions in (a) and suppose further that u is a divergence-free vector field, that is, if $u = (u_1, u_2)$, then $\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} = 0$. Show that there is a neighborhood U of $\partial\Omega$ such that the trajectory starting from x is a close orbit for every $x \in U$.

Hint: For (b), you might need to recall the proof of Poincaré-Bendixon Theorem and the divergence theorem.