Department of Mathematics, National Central University
Ph.D. Qualifying Examination
Graph Theory, 2011

Problem 1.(10%) Let G be a connected graph. Show that the eigenvalue of G with largest absolute value is $\Delta(G)$ if and only if G is $\Delta(G)$-regular.

Problem 2.(10%) Prove that if T is an m-vertex tree, then $R(T, K_n) = (m-1)(n-1)+1$.

Problem 3.(10%) Let G be a simple graph with n vertices. Let u, v be distinct non-adjacent vertices of G with $d(u) + d(v) \geq n$. Show that G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Problem 4.(10%) Let G be the simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ whose edges are $\{v_i v_j : |i-j| \leq 3\}$. Prove that G is a maximal planar graph.

Problem 5.(10%) Prove that the chromatic polynomial of an n-vertex graph has no real root larger than $n-1$.

Problem 6.(10%) Let G be a bipartite graph. Show that the maximum size of a matching in G equals the minimum size of a vertex cover of G.

Problem 7.(10%) Let G be a graph with $\chi(G) > k$, and let X, Y be a partition of $V(G)$. Show that if $G[X]$ and $G[Y]$ are k-colorable, then the edge cut $[X, Y]$ has at least k edges.

Problem 8.(10%) Show that a graph G having at least three vertices is 2-connected if and only if for each pair $u, v \in V(G)$ there exist internally disjoint u, v-paths in G.

Problem 9.(10%) State and prove the Matrix Tree Theorem.

Problem 10.(10%) For $n \geq 4$, let G be a simple n-vertex graph with $e(G) \geq 2n-3$. Prove that G has two cycle of equal length.