GRAPH THEORY

In the following, graph means finite simple graph.

1. (20 points) Let G be a graph of order $n \geq 3$. Suppose that $deg\ x + deg\ y \geq n$ for every pair of nonadjacent vertices x, y in G. Show that G contains a Hamiltonian cycle.

2. (10 points) Let G be a graph of order $n \geq 3$. Suppose that $deg\ x + deg\ y \geq n-1$ for every pair of nonadjacent vertices x, y in G. Show that G contains a Hamiltonian path.

3. (10 points) Let diam G denote the maximum of distance between two vertices in a graph G. Show that if diam G ≥ 3, then diam $G^c \leq 3$.

4. (15 points) Let G be a bipartite graph with bipartition (X,Y) such that $|N(S)| \geq |S|$ for every $S \subset X$. Show that G contains a matching M with $|M| = |X|$.

5. (10 points) Let G be a k-regular bipartite graph (k $\geq$ 1). Show that G contains a perfect matching.

6. (10 points) Show that every graph is a vertex-induced subgraph of a regular graph.

7. (15 points) For $k, l \in N$, let $r(k, l)$ denote the minimum n such that any graph on n vertices contains either a clique of order k or an independent set of order l. Show that

$$r(k+1, l+1) \leq r(k+1, l) + r(k, l+1).$$

8. (10 points) Let $d_1, d_2, ..., d_n (n \geq 2)$ be the sequence of positive integers such that $d_1 + d_2 + ... + d_n = 2(n-1)$. Show that $d_1, d_2, ..., d_n$ is a degree sequence of a tree.