國立中央大學數學系

博士班資格考試《數理統計》試題，2001 年 9 月

Qualifying Exam (Mathematical Statistics)

Sep. 3, 2001

1. a) Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be independently distributed as $E(a, b)$ (exponential with location and scale parameters a and b, respectively) and $E(a', b')$, respectively.

a) If a, b, a', b' are all unknown, show that $X_{(1)}$ (the smallest among $X_1, \ldots, X_n$), $Y_{(1)}$, $\sum_{i=1}^{n}(X_i - X_{(1)})$, and $\sum_{i=1}^{m}(Y_i - Y_{(1)})$ are jointly sufficient and complete. [10%]

b) Find the UMVUE's of $a' - a$ and b'/b. [10%]

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on $(0, \theta)$. Show that

a) For testing $H_0 : \theta \leq \theta_0$ against $\theta > \theta_0$ any test ϕ is UMP at level α for which $E_{\theta_0}\phi(\underline{X}) = \alpha$, $E_\theta \phi(\underline{X}) \leq \alpha$, for $\theta \leq \theta_0$, and $\phi(\underline{X}) = 1$ when $\max(X_1, \ldots, X_n) > \theta_0$. [10%]

b) For testing $H_0 : \theta = \theta_0$ against $\theta \neq \theta_0$ a unique UMP level α test exists, and is given by $\phi(\underline{X}) = 1$ when $\max(X_1, \ldots, X_n) > \theta_0$, or $\max(X_1, \ldots, X_n) \leq \theta_0 \sqrt[n]{\alpha}$, and $\phi(\underline{X}) = 0$ otherwise. [10%]

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $N(\theta, 1)$ random variables where $\theta \in R$. Let $\bar{X} = \sum_{i=1}^{n} X_i/n$ and consider the squared error loss.

a) Show that $\bar{X}$ is admissible. [10%]

b) State the Stein's Paradox and give an example. [10%]

4. Let the random variable X have probability mass function

$$P_\theta(x) = \binom{r + x - 1}{x} \theta^x (\theta + 1)^{-(r+x)}, \quad x = 0, 1, 2, \ldots,$$

where $r > 0$ is a known constant, and $\theta > 0$ is the unknown parameter. Consider the loss function $L(\theta, a) = (\theta - a)^2/[\theta(\theta + 1)]$.

a) Find the Bayes estimator of θ with respect to the prior $\pi(\theta) = \lambda/(\theta + 1)^{(\lambda+1)}$, where $\lambda > 0$. [10%]

b) Show that $\delta(X) = X/(r + 1)$ is a minimax estimator of θ. [10%]

5. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be two independent random samples from $N(0, \sigma_1^2)$ and $N(0, \sigma_2^2)$, respectively, where $\sigma_1^2, \sigma_2^2 > 0$ are both unknown. Find the $1 - \alpha$ (uniformly) most accurate unbiased confidence interval for σ_1^2/σ_2^2. [10%]

6. Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli (θ) distributed. Find $P(\bar{X}(1 - \bar{X}) \leq t)$ asymptotically as $n \to \infty$, where $\bar{X}$ is the sample mean. [10%]