# Qualifying Exam for Numerical Analysts at Math.NCU

## September 1, 2009

Throughout this problem sheet, $\Omega$ denotes some bounded polygonal convex domain in the two dimensional space. Let $L^2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ be the function spaces with their standard definitions. Let the bilinear form $(\cdot,\cdot)$ be the standard $L^2$-inner product and $\|\cdot\|$ be the standard $L^2$-norm.

**1. Basic Models**

Let $V$ be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_V$ and corresponding norm $\|\cdot\|_V$. Suppose that $a(\cdot,\cdot)$ is a bilinear form on $V\times V$ and $\ell(\cdot)$ a linear form on $V$ with the following properties.

(i) $a(\cdot,\cdot)$ is symmetric.

(ii) $a(\cdot,\cdot)$ is continuous; that is, there is a positive constant $\gamma$ such that

$$|a(v,w)| \le \gamma\|v\|_V\,\|w\|_W, \quad \forall v,w\in V.$$

(iii) $a(\cdot,\cdot)$ is $V$-elliptic; that is, there is a positive constant $\alpha$ such that

$$a(v,v)\ge \alpha\|v\|_V^2, \quad \forall v\in V.$$

(iv) $\ell$ is continuous; that is, there is a positive constant $\lambda$ such that

$$|\ell(v)|\le\lambda\|v\|_V.$$

Show that the *minimization problem*

$$\text{Find } u\in V \text{ such that } F(u)=\min_{v\in V}F(v) \text{ where } F(v)=\frac{1}{2}a(v,v)-\ell(v) \tag{1.1}$$

and the abstract *variational problem*

$$\text{Find } u\in V \text{ such that } a(u,v)=\ell(v), \quad \forall v\in V \tag{1.2}$$

are equivalent, and there is a unique solution $u\in V$ such that

$$\|u\|_V\le\frac{\lambda}{\alpha}.$$

**2. Finite Element Spaces**

Let $T_h$ be a triangulation of $\Omega$, and $K\in T_h$ be a triangle with vertices $a^1$, $a^2$, $a^3$ where $a^i=(a_1^i,a_2^i)$. Let $V_h$ be the standard finite element space of piecewise linear functions on triangles $K$, that is,

$$V_h=\{v\in H^1(\Omega)\mid v|_K\in\Pi_1(K),\ \forall K\in T_h\}$$

where $\Pi_1$ is the space of polynomials of degree 1 in variables $x=(x_1,x_2)$. What is the dimension of $\Pi_1(K)$ and what are the basis functions $\lambda_i(x)$ for $\Pi_1(K)$?