For a given triangle K, let h be the longest side of K, ρ be the diameter of the circle inscribed in K. For a given continuously differentiable function v on K, let the interpolant $\hat{v} \in \Pi_1(K)$ be defined by

$$\hat{v}(a^i) = v(a^i), \quad i = 1, 2, 3.$$

Show that

$$\|v - \hat{v}\|_\infty \le 2h^2 \max_{|\alpha|=2} \|D^\alpha v\|_\infty \tag{2.1}$$

and

$$\max_{|\alpha|=1} \|D^\alpha (v - \hat{v})\|_\infty \le 6\frac{h^2}{\rho} \max_{|\alpha|=2} \|D^\alpha v\|_\infty \tag{2.2}$$

where α is a multi-index (α_1, α_2) with whole number elements and $|\alpha| = \alpha_1 + \alpha_2$ and

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}.$$

And $\|\cdot\|_\infty$ is an abbreviation of

$$\|v\|_{L^\infty(K)} = \max_{x \in K} |v(x)|.$$

3. Systems of Linear Equations

Let A be an $n \times n$ square matrix. When A is symmetric and positive definite, the LU-factorization of A becomes the *Cholesky* decomposition $A = LL^T$, where L is a lower triangular matrix with positive diagonal elements. Describe an algorithm for the Cholesky decomposition and discuss its computational complexities.

4. Parabolic Problems

Consider the model parabolic problem

$$\begin{aligned} \dot{u} - \Delta u &= f \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{on } \Gamma \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{in } \Omega \end{aligned} \tag{4.1}$$

where $u = u(x,t)$ for $x \in \Omega$ and $t \in (0,T)$, $\dot{u} = \partial u/\partial t$ and Δu is the Laplacian of u in space.

The *semi-discretization* (in space) for (4.1) is based on the variational formulation:

$$\text{Find } u(t) \in H_0^1(\Omega) \text{ such that } \begin{cases} (\dot{u}(t), v) + a(u(t), v) = (f(t), v) \\ u(0) = u_0 \end{cases} \quad \forall v \in H_0^1(\Omega) \tag{4.2}$$

What is the proper meaning of $a(\cdot,\cdot)$ in (4.2)?

Let V_h be a finite element space in $H_0^1(\Omega)$, describe the semi-discretization of the problem (4.1). And, let $u_h(t) \in V_h$ be the finite element solution of your semi-discretization problem, show the stability property that

$$\|u_h(t)\| \le \|u_0\|, \quad \text{for } t \in (0,T).$$