5. Hyperbolic Problems

Consider the *reduced model* problem

$$\begin{cases} u_\beta + u = f & \text{in } \Omega \\ u = g & \text{on } \Gamma_- \end{cases} \tag{5.1}$$

It comes from setting $\epsilon = 0$ in the stationary part

$$\operatorname{div}(\beta u) + \sigma u - \epsilon \Delta u = f \quad \text{in } \Omega$$

of the *convection-diffusion* equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(\beta u) + \sigma u - \epsilon \Delta u = 0 \quad \text{in } \Omega \times (0, T).$$

Here Γ is the boundary of Γ, while Γ_- is the inflow part of the boundary. Let β be a unit constant vector and $v_\beta = \beta \cdot \nabla v$ denote the derivative of v in the direction of β. Consider β as the direction of flow, then

$$\Gamma_- = \{x \in \Gamma \mid n(x) \cdot \beta < 0\}$$

where $n(x)$ is the outward unit normal vector at $x \in \Gamma$.

Let $\Gamma_+ = \Gamma / \Gamma_-$, define

$$\langle v, w \rangle = \int_\Gamma vw(\beta \cdot n)\, ds, \quad \langle v, w \rangle_- = \int_{\Gamma_-} vw(\beta \cdot n)\, ds \quad \text{and} \quad \langle v, w \rangle_+ = \int_{\Gamma_+} vw(\beta \cdot n)\, ds,$$

and let $|v|^2 = \langle v, v \rangle$. With the notation

$$\begin{aligned} b(w, v) &= (w_\beta + w, v) - \langle w, v \rangle_- \\ \ell(v) &= (f, v) - \langle g, v \rangle_- \end{aligned}$$

we can formulate the standard Galerkin method with weakly imposed boundary conditions:

$$\text{Find } u^h \in V_h \text{ such that } b(u^h, v) = \ell(v), \quad \forall v \in V_h. \tag{5.2}$$

Show the stability property of the problem: For any $v \in H^1(\Omega)$, we have

$$b(v, v) = \|v\|^2 + \frac{1}{2}|v|^2.$$

And argue that the problem (5.2) has a unique solution.