Qualifying Exam for Numerical Analysis

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Problem 1. Let V be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_V$ and corresponding norm $\|\cdot\|_V$, $S_h \subseteq V$ is a finite dimensional subspace, $a : V \times V \to \mathbb{R}$ be a continuous, V-elliptic bilinear form, and $\ell : \mathcal{V} \to \mathbb{R}$ be a bounded linear form.

1. (10%) State the Lax-Milgram theorem and show that there exist unique $u \in V$ and $u_h \in S_h$ satisfying
$$a(u,v) = \ell(v) \quad \forall\, v \in V \qquad \text{and} \qquad a(u_h,v) = \ell(v) \quad \forall\, v \in S_h\,.$$
2. (10%) Show that there exists $C > 0$ such that $\|u - u_h\|_V \leqslant C\|u - v_h\|_V$ for all $v_h \in S_h$.

Problem 2. Let $\{x_i\}_{i=0}^N$ be a partition of $I = [a,b]$; that is, $a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$, and $h = \max\{x_i - x_{i-1} \,\big|\, 1 \leqslant i \leqslant N\}$. Let $\{\phi_i\}_{i=1}^N$ be a collection of continuous functions satisfying

(a) for $i,j \in \{0,1,\cdots,N\}$, $\phi_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta;

(b) for $i,j \in \{0,1,\cdots,N\}$, ϕ_i is a polynomial of degree 1 in $[x_{j-1},x_j]$.

Define $S_h = \operatorname{span}(\phi_1,\phi_2,\cdots,\phi_N)$ and $\Pi : C^0(I) \to S_h$ by
$$(\Pi v)(x) = \sum_{j=1}^N v(x_j)\phi_j(x)\,.$$

1. (10%) Show that there exists a constant $C > 0$ such that
$$\|v - \Pi v\|_{L^2(I)} \leqslant Ch^2\|v''\|_{L^2(I)} \ \text{ and } \ \|v' - (\Pi v)'\|_{L^2(I)} \leqslant Ch\|v''\|_{L^2(I)}$$
for all $v \in H^2(I)$ satisfying $v(a) = 0$.
2. (10%) Let $f \in L^2(I)$, and $u \in H^1(I)$ be the unique weak solution to the differential equation
$$-u'' = f \ \text{ in } \ (a,b)\,, \quad u(a) = u'(b) = 0\,,$$
and $u_h \in S_h$ satisfies
$$\int_I u_h'(x)v_h'(x)\,dx = \int_I f(x)v_h(x)\,dx \qquad \forall\, v_h \in S_h\,.$$
Show that there exists a constant C independent of f (and u) such that
$$\|u - u_h\|_{H^1(I)} \leqslant Ch\|f\|_{L^2(I)}\,.$$

Problem 3. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, convex, polygonal domain. Consider the parabolic equation

$$\partial_t u - \Delta u = f \qquad \text{in} \quad \Omega \times (0,T)\,, \tag{0.1a}$$
$$\frac{\partial u}{\partial \mathrm{N}} + u = g \qquad \text{on} \quad \partial\Omega \times (0,T)\,, \tag{0.1b}$$
$$u(x,0) = u_0(x) \qquad \text{for all } x \in \Omega\,, \tag{0.1c}$$

where N denotes the outward-pointing unit normal of Ω. The semi-discretization (in space) for (0.1) is based on the variational formulation: Find $u : [0,T] \to H^1(\Omega)$ such that $u(0) = u_0$ and

$$(\dot{u}(t), v)_{L^2(\Omega)} + a(u(t), v) = (f(t), v)_{L^2(\Omega)} + (g(t), v)_{L^2(\partial\Omega)} \qquad \forall\, v \in H^1(\Omega)\,, \tag{0.2}$$

where $(\cdot,\cdot)_{L^2(\Omega)}$ and $(\cdot,\cdot)_{L^2(\partial\Omega)}$ denote the inner product in $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively.

1. (5%) Provide the explicit form of $a(\cdot,\cdot)$ in (0.2).

2. (10%) Let V_h be a finite element space in $H^1(\Omega)$ consisting of piecewise linear functions on a quasi-uniform triangulation of Ω with mesh size h, and $u_h : [0,T] \to V_h$ satisfies

$$(\dot{u_h}(t), v)_{L^2(\Omega)} + a(u_h(t), v) = (f(t), v)_{L^2(\Omega)} + (g(t), v)_{L^2(\partial\Omega)} \qquad \forall\, v \in V_h\,. \tag{0.3}$$

   Show the stability property that

$$\|u_h(t)\|_{L^2(\Omega)} \leqslant \|u_0\|_{L^2(\Omega)} \qquad \forall\, t \in (0,T)$$

   if $f = g = 0$.

3. (10%) Suppose that one use the discontinuous Galerkin method for solving the semi-discrete problem (0.3) by assuming that

$$u_h(t) = (u_0 + tv_1)\mathbf{1}_{[0,\frac{T}{2})}(t) + (v_2 + tv_3)\mathbf{1}_{(\frac{T}{2},T]}(t)$$

   for some $v_1, v_2, v_3 \in V_h$. Write down explicitly the relation among v_1, v_2, v_3.

Problem 4. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded strictly convex domain with outward-pointing unit normal n (which is defined except on the vertices), $\beta \in \mathbb{R}^2$ being a constant vector, and

$$\Gamma_+ = \big\{x \in \partial\Omega \,\big|\, n(x)\cdot\beta > 0\big\}\,, \quad \Gamma_- = \big\{x \in \partial\Omega \,\big|\, n(x)\cdot\beta < 0\big\}\,.$$

Define $V = \big\{v \in H^1(\Omega) \,\big|\, v = 0 \text{ on } \Gamma_-\big\}$ and a bilinear form $b : V \times V \to \mathbb{R}$ by

$$b(u,v) = \int_\Omega \big(u + \beta\cdot\nabla u\big) v\, dx - \int_{\Gamma_-} uv(n\cdot\beta)\, dS \quad \text{and} \quad \ell(v) = \int_\Omega fv\, dx\,.$$

1. (10%) Suppose that V_h is a finite dimensional subspace of V such that for all $v \in V$ there exists $\widetilde{v}_h \in V_h$ satisfying

$$\|v - \widetilde{v}_h\|_{H^1(\Omega)} + \|v - \widetilde{v}_h\|_{L^2(\partial\Omega)} \leqslant Ch\|v\|_{H^2(\Omega)}\,,$$

   where C is a generic constant independent of v. Show that if $u \in V$, $u_h \in V_h$ are solutions to

$$b(u + \beta\cdot\nabla u, v) = \ell(v) \quad \forall\, v \in V \qquad \text{and} \qquad b(u_h + \beta\cdot\nabla u_h, v_h) = \ell(v_h) \quad \forall\, v_h \in V_h\,,$$

   then

$$\|u - u_h\|_{L^2(\Omega)} + \|u - u_h\|_{L^2(\partial\Omega)} \leqslant Ch\|u\|_{H^2(\Omega)}\,.$$

2. (10%) Let T_h be a triangulation of Ω, and for $K \in T_h$ we use $P_r(K)$ to denote the collection of polynomials of degree r defined on K. The discontinuous Galerkin method of solving for

$$\begin{aligned} u + \operatorname{div}(\beta u) &= f \qquad \text{in} \quad \Omega\,, \\ u &= 0 \qquad \text{on} \quad \partial\Omega\,, \end{aligned}$$

can be written as

$$B(u_h, v_h) = (f, v_h)_{L^2(\Omega)} \qquad \forall\, v \in W_h$$

for some finite dimensional subspace W_h of V. Write down the bilinear form B and the subspace W_h.

Problem 5. (15%) For a triangle $K \subseteq \mathbb{R}^2$, let h_K denote the maximum length of sides of K, and ρ_K be the radius of the maximum disc which is contained in K. Suppose that $K \subseteq \mathbb{R}^2$ is a triangle such that $\rho_K \geqslant \beta h_K$ for a certain $\beta > 0$. Show that there exists a constant C, depending only on β, such that

$$\|\nabla v\|_{L^2(K)} \leqslant C h_K^{-1} \|v\|_{L^2(K)} \qquad \forall\, v \in P_1(K)\,,$$

where $P_1(K)$ is the collection of polynomials of degree 1 defined on K.