Qualifying Exam for Numerical Analysts at Math.NCU

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Throughout this problem sheet, Ω denotes some bounded polygonal convex domain in the two dimensional space with boundary $\Gamma = \partial\Omega$. Let $L^2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ be the function spaces with their standard definitions. Let the bilinear form $(\cdot,\cdot)$ be the standard L^2-inner product and $\|\cdot\|$ be the standard L^2-norm.

1. A Model Problem

In this problem and for this problem only, let Ω be the domain bounded by x-axis, y-axis, and the line $x+y=1$. Describe a triangulation of Ω by 25 equivalent right triangles. Set up a linear system of equations for the boundary value problem

$$\begin{cases} -u_{xx} - u_{yy} = 1, & (x,y) \in \Omega \\ u = 0, & (x,y) \in \Gamma \end{cases}$$

by a *finite element method* with piecewise linear elements.

2. The Fundamental Theory

Let V be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_V$ and corresponding norm $\|\cdot\|_V$. Suppose that $a(\cdot,\cdot)$ is a bilinear form on $V \times V$ and $\ell(\cdot)$ a linear form on V with the following properties.

(i) $a(\cdot,\cdot)$ is symmetric.

(ii) $a(\cdot,\cdot)$ is continuous; that is, there is a positive constant γ such that

$$|a(v,w)| \le \gamma \|v\|_V \, \|w\|_W, \quad \forall v, w \in V.$$

(iii) $a(\cdot,\cdot)$ is V-elliptic; that is, there is a positive constant α such that

$$a(v,v) \ge \alpha \|v\|_V^2, \quad \forall v \in V.$$

(iv) ℓ is continuous; that is, there is a positive constant λ such that

$$|\ell(v)| \le \lambda \|v\|_V.$$

Show that the *minimization problem*

$$\text{Find } u \in V \text{ such that } F(u) = \min_{v \in V} F(v) \text{ where } F(v) = \frac{1}{2} a(v,v) - \ell(v) \tag{2.1}$$

and the abstract *variational problem*

$$\text{Find } u \in V \text{ such that } a(u,v) = \ell(v), \quad \forall v \in V \tag{2.2}$$

are equivalent, and there is a unique solution $u \in V$ such that

$$\|u\|_V \le \frac{\lambda}{\alpha}.$$

3. Finite Element Spaces

Let T_h be a triangulation of Ω and $K \in T_h$ be a triangle with vertices a^1, a^2, a^3 where $a^i = (a^i_1, a^i_2)$. Let V_h be the standard finite element space of piecewise linear functions on triangles K, that is,

$$V_h = \{v \in H^1(\Omega) \mid v|_K \in \Pi_1(K),\ \forall K \in T_h\}$$

where Π_1 is the space of polynomials of degree 1 in variables $x = (x_1, x_2)$. What is the dimension of $\Pi_1(K)$? Please give one set of basis functions $\lambda_i(x)$ for $\Pi_1(K)$.

For a given triangle K, let h be the longest side of K, ρ be the diameter of the circle inscribed in K. For a given continuously differentiable function v on K, let the interpolant $\hat{v} \in \Pi_1(K)$ be defined by

$$\hat{v}(a^i) = v(a^i), \quad i = 1, 2, 3.$$

Show that

$$\|v - \hat{v}\|_\infty \leq 2h^2 \max_{|\alpha|=2} \|D^\alpha v\|_\infty \tag{3.1}$$

and

$$\max_{|\alpha|=1} \|D^\alpha (v - \hat{v})\|_\infty \leq 6\frac{h^2}{\rho} \max_{|\alpha|=2} \|D^\alpha v\|_\infty \tag{3.2}$$

where α is a multi-index (α_1, α_2) with whole number elements and $|\alpha| = \alpha_1 + \alpha_2$ and

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}}.$$

In the above, $\|\cdot\|_\infty$ is an abbreviation of

$$\|v\|_{L^\infty(K)} = \max_{x \in K} |v(x)|.$$

4. Parabolic Problems

Consider the model parabolic problem

$$\begin{cases} \dot{u} - \Delta u = f & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \Gamma \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases} \tag{4.1}$$

where $u = u(x, t)$ for $x \in \Omega$ and $t \in (0, T)$, $\dot{u} = \partial u / \partial t$ and Δu is the Laplacian of u in space.

The *semi-discretization* (in space) for (4.1) is based on the variational formulation:

$$\text{Find } u(t) \in H^1_0(\Omega) \text{ such that } \begin{cases} (\dot{u}(t), v) + a(u(t), v) = (f(t), v) \\ u(0) = u_0 \end{cases} \quad \forall v \in H^1_0(\Omega) \tag{4.2}$$

What is the proper meaning of $a(\cdot, \cdot)$ in (4.2)?

Let V_h be a finite element space in $H_0^1(\Omega)$, describe the semi-discretization of the problem (4.1). And, let $u_h(t) \in V_h$ be the finite element solution of your semi-discretization problem, show the stability property that

$$\|u_h(t)\| \leq \|u_0\|, \quad \text{for } t \in (0,T).$$

5. Hyperbolic Problems

Consider the *reduced model* problem

$$\begin{cases} u_\beta + u = f & \text{in } \Omega \\ u = g & \text{on } \Gamma_- \end{cases} \tag{5.1}$$

It comes from setting $\epsilon = 0$ in the stationary part

$$\operatorname{div}(\beta u) + \sigma u - \epsilon \Delta u = f \quad \text{in } \Omega$$

of the *convection-diffusion* equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(\beta u) + \sigma u - \epsilon \Delta u = 0 \quad \text{in } \Omega \times (0,T).$$

Here Γ_- is the inflow part of the boundary. Let β be a unit constant vector and $v_\beta = \beta \cdot \nabla v$ denote the derivative of v in the direction of β. Consider β as the direction of flow, then

$$\Gamma_- = \{x \in \Gamma \mid n(x) \cdot \beta < 0\}$$

where $n(x)$ is the outward unit normal vector at $x \in \Gamma$.

Let $\Gamma_+ = \Gamma / \Gamma_-$, define

$$\langle v, w \rangle = \int_\Gamma vw(\beta \cdot n)\, ds, \quad \langle v, w \rangle_- = \int_{\Gamma_-} vw(\beta \cdot n)\, ds \ \text{ and } \ \langle v, w \rangle_+ = \int_{\Gamma_+} vw(\beta \cdot n)\, ds,$$

and let $|v|^2 = \langle v, v \rangle$. With the notation

$$\begin{aligned} b(w,v) &= (w_\beta + w, v) - \langle w, v \rangle_- \\ \ell(v) &= (f, v) - \langle g, v \rangle_- \end{aligned}$$

we can formulate the standard Galerkin method with weakly imposed boundary conditions:

$$\text{Find } u^h \in V_h \text{ such that } b(u^h, v) = \ell(v), \quad \forall v \in V_h. \tag{5.2}$$

Show the stability property of the problem: For any $v \in H^1(\Omega)$, we have

$$b(v,v) = \|v\|^2 + \frac{1}{2}|v|^2.$$

And argue that the problem (5.2) has a unique solution.