Department of Mathematics, National Central University, Qualifying Exam for Numerical Analysis

Fall 2012

Instructions: Do all 3 problems. Show all your work.

1. Consider the following boundary value problem defined in the unit interval $I = (0, 1)$.

$$(D)\begin{cases} u'' = 0 & \text{in } I, \\ u(0) = 0 \\ u'(1) = g \end{cases}$$

   (a) (10pts) Set up a variational formulation (V) for (D) and write it as the following abstract variational formulation

$$(V)\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = L(v)\ \forall v \in V, \end{cases}$$

   give the explicit formulations for V, $a(u,v)$, and $L(v)$.

   (b) (15pts) Do the four conditions, (i) $a(\cdot,\cdot)$ is symmetric, (ii) $a(\cdot,\cdot)$ is continuous,

$$|a(v,w)| \le \gamma \|v\|_V \|w\|_V\ \forall v, w \in V$$

   (iii) $a(\cdot,\cdot)$ is V-elliptic,

$$a(v,v) \ge \alpha \|v\|_V^2\ \forall v \in V.$$

   (iv) L is continuous,

$$|L(v)| \le \Lambda \|v\|_V\ \forall v \in V.$$

   hold for this problem? If yes, find all constants in these four conditions. 15pts

   (c) (10pts) Show the variational formulation (V) has a unique solution.

2. (Continuation) Let us now partition the unit interval uniformly N elements, $I = \bigcup K$, and consider the usual continuous piecewise **quadratic** functions as finite element subspace $V_h \subset V$.

   (a) (10pts) Find the quadratic basis functions $N_i(\xi), i = 1, 2, 3$ for a reference element $\hat{K}$ defined in $[-1, 1]$.

   (b) (15pts) Show that the corresponding element stiffness matrices for each element are the same and are given by

$$\frac{1}{3h}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix},$$

   where $h = 1/N$ is the element size. **Hint:** the change of intervals is

$$\int_K f(x)\,dx = \int_{\hat{K}} f(x(\xi))x'(\xi)\,d\xi.$$

   Here, for a given element $K = [x_i, x_{i+1}]$, the mapping $x(\xi)$ is defined as $x(\xi) = x_i N_1(\xi) + \frac{1}{2}(x_i + x_{i+1})N_2(\xi) + x_{i+1}N(\xi)$.

(c) (5pts) Use the following connectivity array of dimension 3×4.

a/e	1	2	3	4
1	1	3	5	7
2	2	4	6	8
3	3	5	7	9

to assembly the element stiffness matrices for the case of $N = 4$ to obtain the global stiffness matrix. Note that e on the first column is the element number, a on the first row is the local node number, the elements in the connectivity array of dimension 3×4 are the global node numbers. **Make sure to impose some proper boundary conditions.**

3. Consider the following initial boundary value problem.

$$\begin{cases} \text{PDE } u_t = u_{xx} \quad 0 < x < 1 \\ \text{BCs } u(0,t) = u(1,t) = 0 \quad 0 < t < \infty \\ \text{IC } u(x,0) = u_0 \quad 0 \leq x \leq 1 \end{cases}$$

By employing the linear Galerkin finite elements with a uniform partition, we obtain the following semi-discretization formulation as

$$\begin{cases} B\dot{X}(t) + AX(t) = 0 \\ BX(0) = U_0, \end{cases}$$

where B is called the mass matrix given by

$$B = \frac{h}{6}\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix}$$

A is called the stiffness matrix given by

$$A = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

and $X(t)$ be a vector of n consisting of the nodal values of the basis functions. h is the element size.

(a) (5pts) Write an explicit and an implicit schemes based on forward and backward Euler's formulations for the temporal discretization, respectively and Galerkin finite element method for spatial discretization.

(b) (5pts) Show that the eigenvalues of A are given by

$$\lambda_j = 2 - 2\cos(j\theta), j = 1, ..., n,$$

and the eigenvector associated with each λ_j is given by

$$V_j = [\sin(j\theta), \sin(2j\theta), ..., \sin(nj\theta)]^T,$$

where $\theta = \frac{\pi}{n+1}$.

(c) (5pts) Show that the the condition number of A is $O(h^{-2})$.

(d) (5pts) Find the Cholesky decomposition of $B = E^T E$ for the case of 4 elements, $h = 1/4$.

(e) (5pts) We can rewrite the as

$$\begin{cases} \dot{Y}(t) + \hat{A}Y(t) = 0 \\ Y(0) = \hat{U}_0, \end{cases}$$

where $\hat{A} = E^{-T} A E^{-1}$, $Y = EX$ and $\hat{U}_0 = E^{-T} U_0$. Show that A and $\hat{A}$ are similar, i.e., these two matrices have the same eigenvalues.

(f) (10pts) Discuss the stability of these two schemes, the explicit and implicit schemes you derived in Problem 3, Part (a).