國立中央大學數學系
博士班資格考試
＜機率論＞試題
2002 年 9 月

1. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $\mathcal{N}_0$ be the set of all null sets in $(\Omega, \mathcal{F}, \mathcal{P})$. Show that for any Borel subfield $\mathcal{F}_1$ of $\mathcal{F}$, there exists a minimal Borel field $\mathcal{F}_2$ satisfing $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$ and $\mathcal{N}_0 \subset \mathcal{F}_2$.
   (A set E belongs to $\mathcal{F}_2$ if and only if there exists a set F in $\mathcal{F}$, such that $E \triangle F \in \mathcal{N}_0$.)
   (10%)

2. Let Y be a nonnegative random variable and $k > 0$. Show that $E(Y^k) = \int_0^\infty k y^{k-1} \mathcal{P}(Y > y)dy$. *(10%)*

3. (a) What is the (1st) Borel-Cantelli Lemma? Prove or disprove the converse of the Borel-Cantelli lemma.
   (b) Show that $X_n \to X$ in probability if and only if for every subsequence $X_{n(m)}$ there is a further subsequence $X_{n(m_k)}$ that converges almost surely to X.
   (15%)

4. (a) What is the Kolmogorov's Three Series Theorem?
   (b) Show that if $X_1, X_2, \cdots$ are independent with $EX_n = 0$ and E $\sum_{n=1}^{\infty}\left(X_n^2 I_{(|X_n| \le 1)} + |X_n| I_{(|X_n|>1)}\right) < \infty$ then $\sum_i X_n$ converges.
   (10%)

5. (a) Let $X_1, X_2, \cdots$ be independent and let $\alpha_n = (Var\, S_n)^{1/2}$, where $S_n = X_1 + \cdots + X_n$. If there is a $\delta > 0$ so that $\lim_{n\to\infty} \alpha_n^{-(2+\delta)} \sum_{m=1}^{n} E(|X_m - EX_m|^{2+\delta}) = 0$, show that $(S_n - ES_n)/\alpha_n$ converges in distribution to the unit normal.