2003.9.

Probability Qualify Examination

1. Let $\mathcal{F}$ be a σ-algebra on Ω and let δ be a π-system with $\sigma(\delta)=\mathcal{F}$. If P and Q are probabilities on $\mathcal{F}$, such that P=Q on δ, show that P=Q on $\mathcal{F}$.

2. Let X and Y be random variables and let A be an event. Prove that the function $Z = XI_A + YI_{A^C}$ is a random variable.

3. Prove that for $X \geq 0$, $\sum_{k=1}^{\infty} P\{X \geq k\} \leq E(X) \leq \sum_{k=0}^{\infty} P(X \geq k)$.

4. Let X be a P(λ) (Poisson distribution with parameter λ), show that $P(X \geq 2\lambda) \leq (e/4)^{\lambda}$.

5. Let $X_1, X_2, \ldots$ be pairwise uncorrelated with mean 0 and partial sums $S_n = \sum_{k=1}^{n} X_k$. Prove that if there is a constant c such that $Var(X_k) \leq c$ for every k, then $\frac{S_n}{n^{\alpha}} \xrightarrow{q.m.} 0$ for all $\alpha > 1/2$.

6. Let $X_1, X_2, \ldots$ be i.i.d. and integer-valued, with partial sums $S_n = \sum_{i=1}^{n} X_i$. Prove that for each n and k, $P(S_n = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\phi_X(t)^n \, dt$, where $\phi_X(t)$ is the characteristic function of X.

7. Let $X_1, X_2, \ldots$ be independent random variables, $\sigma_n^2 = \mathrm{var}(X_n)$.

   If there is a constant c such that $P\{|X_k| \leq c\} = 1$ for each k and if $\sigma_n^2 \to \infty$ as $n \to \infty$. Show that the Lindeberg condition is satisfied.