(b) If $X_1, X_2, \cdots$ are independent with $|X_i| \le M$ and $\sum_{n=1}^{\infty} Var(X_n) = \infty$, show that there are constants $a_n, b_n$ so that $(S_n - b_n)/a_n$ converges to the normal distribution.

(15%)

6. Show that

(a) $\mu(\{a\}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\varphi(t)dt$, where $\varphi(t)$ is the characteristic function of the propability measure $\mu(\cdot)$.

(b) $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2 dt = \sum_{X \in R} \mu(\{X\})^2$.

(Hint:you may use (a) and consider the random variable $X - Y$, where $X, Y$ are iid.) (15%)

7. Suppose that $X$ is an integrable random variable on $(\Omega, \mathcal{F}, \mathcal{P})$ and that $\mathcal{G}$ is sub-$\sigma$-field in $\mathcal{F}$. Show that (15%)

(i) there is a random variable $E_{\mathcal{G}}(X)$, called the conditional expectation of $X$ given $\mathcal{G}$, which has the following two properties:

(a) $E_{\mathcal{G}}(X)$ is integrable w.r.t. $\mathcal{G}$, and

(b) $\int_G E_{\mathcal{G}}(X) d\mathcal{P} = \int_G X d\mathcal{P}$ for all $G \in \mathcal{G}$.

(ii) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are $\sigma$-fields such that $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$, then $E_{\mathcal{G}_1}[E_{\mathcal{G}_2}(X)] = E_{\mathcal{G}_1}(X)$.

(iii) If $EX^2 < \infty$, then $E_{\mathcal{G}}(X)$ is the projection of $X$ from $L^2(\mathcal{F})$ onto to $L^2(\mathcal{G})$, where $L^2(\mathcal{F}) = \{Y \in \mathcal{F} : EY^2 < \infty\}$.

8. (a) What is the Martingale Convergence Theorem?

(b) Show that if $X_n \ge 0$, $n \ge 0$, is a supermartingale, then $X_n \to X$ a.s. and $EX \le EX_0$.

(10%)