2003.9.

8. Let $X_1, X_2, \ldots$ be i.i.d. with $E(X_1)=\mu, \mathrm{var}(X_1)=\sigma^2$ and finite fourth central moment $\mu_4 = E[(X_1-\mu)^4]$. Define the sample variance $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean. Prove that

$$\frac{\sqrt{n}(S_n^2-\sigma^2)}{\sqrt{\mu_4-\sigma^4}} \xrightarrow{d} N(0,1).$$

9. Let $X, Y \in L^2$ be such that $E(X \mid Y) = Y$ and $E(Y \mid X) = X$.

Prove that $X \overset{a.s.}{=} Y$.

10. Let X be a submartingale with respect to Y. Prove that for each n and each $a > 0$,

$$P\{\max_{1\le k\le n} X_k \ge a\} \le \frac{1}{a} E(X_n; \{\max_{1\le k\le n} X_k \ge a\}) \le \frac{E(X_n^+)}{a}.$$