Ph.D. Qualifying Exam. , Probability Theory, Fall 2007

1.(20 points) Let $(X_n, n \geq 1)$ and $(Y_n, n \geq 1)$ be two sequences of r.v.s which take values in $\{0,1\}$. Assume that all the variables $(X_n, Y_m, n, m \geq 1)$ are independent and that, for any $n \geq 1$, one has $P(X_n = 1) = p$ and $P(Y_n = 1) = q$ where $0 < p, q < 1$.

a. Show that the r.v.s $Z_n = X_n Y_n, n \geq 1$ are independent and identically distributed. Compute their common distribution.

b. Define $T_n = \sum_{m=1}^{n} Z_m$ and $\tau(\omega) = \inf\{n \geq 1 : T_n(\omega) = 1\}$. What are the laws of T_n and τ?

2.(30 points) Let λ_1 and λ_2 be two positive reals such that $0 < \lambda_1 < \lambda_2 < \infty$. Let T_1 and T_2 be two independent r.v..s such that for every $t \geq 0$, $P(T_i > t) = e^{-\lambda_i t}(i = 1, 2)$. Consider a third variable T_3, which is independent of the pair (T_1, T_2), and for every $t \geq 0$, $P(T_3 \geq t) = e^{-(\lambda_2 - \lambda_1)t}$.

a. Show that there exist two constants α and β such that for every Borel set B in $R_+ = [0, \infty)$,
$$P(T_1 + T_2 \in B) = \alpha P(T_1 \in B) + \beta P(T_2 \in B).$$
Compute explicitly α and β.

b. Compute $P(T_1 > T_3)$.

c. Compare the law of the pair (T_1, T_3), conditionally on $(T_1 > T_3)$, and the law of the pair $(T_1 + T_2, T_2)$.

3.(20 points) Let $(X_n, n \leq 1)$ be a sequence of r.v.s which take values in $[0, 1]$.

a. Prove that, if for every positive integer k, $E(X_n^k) \to \frac{1}{k+1}$ as $n \to \infty$, then the sequence (X_n) converges in law; identify the limit law.

b. Let $a > 0$. Solve the same question in (a) when $\frac{1}{k+1}$ is replaced by $\frac{a}{k+a}$.

4.(10 points) Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i^+ = \infty$ and $EX_i^- < \infty$. If $S_n = X_1 + X_2 + \ldots + X_n$, then show that $S_n/n \to \infty$ a.s.

5.(20 points) If X_n is a martingale such that the differences $Y_n = X_n - X_{n-1}$ are all square-integrable, show that for $n \neq m$, $E(Y_n Y_m) = 0$. Therefore,
$$E(X_n^2) = E(X_0^2) + \sum_{j=1}^{n} E(Y_j^2).$$
If, in addition, $\sup_n E(X_n^2) < \infty$, then show there is a random variable X such that
$$\lim_{n \to \infty} E(|X_n - X|^2) = 0.$$