國立中央大學數學系博士班資格考試

〈機率〉試題，2007 年 2 月

1. (12 points) Let $\mathcal{E}$ be a class of subsets of a sample space Ω, let $B \subseteq \Omega$, and define

$$\mathcal{E} \cap B = \{A \cap B : A \in \mathcal{E}\}.$$

   Show that $\sigma(\mathcal{E} \cap B) = \sigma(\mathcal{E}) \cap B$.

2. (16 points) Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed random variables. Show that for any fixed $\varepsilon > 0$

$$\mathbb{E}|X_1| < \infty \qquad \Longleftrightarrow \qquad \sum_{n=1}^{\infty} \mathbb{P}(|X_1| > \varepsilon \cdot n) < \infty \qquad \Longrightarrow \qquad \frac{X_n}{n} \longrightarrow 0 \quad \mathbb{P}\text{-a.s.}$$

3. (16 points) Let (X_n) be a sequence of independent random variables with $\mathbb{E}[X_n] = 0$. If

$$\sum_n \mathbb{E}\left(\frac{X_n^2}{1 + |X_n|}\right) < \infty,$$

   show that the series $\sum X_n$ converges $\mathbb{P}$-a.s.

4. (26 points) For a distribution function F and $h \geq 0$, define

$$Q_F(h) = \sup_x [F(x+h) - F(x-)]. \tag{1}$$

   (a) (6 points) Suppose $F \equiv \Phi$, the standard normal distribution function, find $Q_\Phi(h)$.

   (b) (8 points) Prove that the sup in (1) is attained for the general distribution function F.

   (c) (12 points) If G is also a distribution function, for every $h > 0$, prove that

$$Q_{F*G}(h) \leq Q_F(h) \wedge Q_G(h).$$

5. (20 points)

   (a) (4 points) Please state the Lindeberg's condition.

   (b) (4 points) Let (X_n) be defined as follows for some $\alpha > 1$:

$$X_n = \begin{cases} \pm n^\alpha, & \text{with probability } \dfrac{1}{6n^{2(\alpha-1)}} \text{ each;} \\ \\ 0, & \text{with probability } 1 - \dfrac{1}{3n^{2(\alpha-1)}}. \end{cases}$$

   Find the necessary and sufficient conditions on α such that the Lindeberg's condition is satisfied.

   (c) (12 points) Justify your answer in (b).

6. (10 points) Let $X_1, X_2, ..., X_n$ be independent uniformly distributed random variables on the probability space $([0,1], \mathcal{B}, m)$ and let $S = \min\{X_1, ..., X_n\}$. Compute $\mathbb{E}[X_1|S]$.