Department of Mathematics

National Central University

Probability Qualifying Examination

August 31, 2018

Instructions: This is a closed book exam. There are 10 problems, of which you should turn in solutions for **exactly 6 problems**. **Correct and complete** solutions to 4 problems guarantees a pass. On the first page of your exam sheet, indicate which 6 you have attempted. If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial. Ill-explanation may cause NO points. It is your responsibility to write answers in a proper way.

Problem (1) Suppose the probability space is $([0,1], \mathcal{B}, \lambda)$ where $\mathcal{B}$ is the Borel σ-field on $[0,1]$ and λ is Lebesgue measure.

$$X = \begin{cases} 0, & \text{for } x \in [0,1/2); \\ 1, & \text{for } x \in [1/2,1]. \end{cases} \quad \text{and} \quad Y = \begin{cases} 2, & \text{for } x \in [0,1/4) \bigcup [1/2,3/4); \\ 3, & \text{for } x \in [1/4,1/2) \bigcup [3/4,1]. \end{cases}$$

Are X and Y independent?

Problem (2) Suppose that X_1, X_2, X_3,... are i.i.d. random variables with mean zero and $E[X_1^4] < \infty$. Define $S_n = \sum_{k=1}^n X_k$ for $n \geq 1$. Show that for every ϵ

$$P\{|S_n| > n\epsilon\} \leq \frac{C}{n^2\epsilon^4} \text{ for some } C > 0.$$

Problem (3) Let X_1, X_2, X_3,... be i.i.d. Gaussian random variables with mean zero and variance one. We set

$$S_n := \sum_{k=1}^n X_i \text{ and } M_n := \exp\left(S_n - \frac{n}{2}\right) \text{ for all } n \geq 1.$$

Define $\Lambda(k) := \limsup_{n\to\infty}(\ln E[M_n^k])/n$. Show that

$$\frac{\Lambda(k)}{k} \text{ is strictly increasing in } k \text{ if } k \geq 2$$

and $\lim_{n\to\infty} M_n$ exists and find its limit. (You can use consequences from other problems.)

Problem (4) Let S_n be a symmetric simple random walk starting at 0, i.e., $S_0 = 0$, $S_n := \sum_{k=1}^n X_k$ for $n \geq 1$, where X_1, X_2, X_3,... are i.i.d. random varibales with $P\{X_1 = 1\} = P\{X_1 = -1\} = 1/2$. Define $T = \inf\{n \geq 0 : S_n = a \text{ or } -b\}$, where a and b are positive integers. Find E[T].

Problem (5) Let (X, Y) be an absolutely continuous random variable with continuous density $f(x, y)$ and $E|X| < \infty$. Prove that

$$E[X|Y] = \frac{\int_{-\infty}^{\infty} x f(x, Y) dx}{\int_{-\infty}^{\infty} f(x, Y) dx} a.s.$$

Problem (6) Suppose that X_1, X_2, X_3,... are independent random variables. And for $k \geq 1$, X_k is uniform on $[0, k]$. Define $S_n = \sum_{k=1}^{n} X_k$. Prove that

$$\frac{4S_n - n(n+1)}{n^{3/2}} \text{ converges weakly.}$$

Problem (7) Prove that if $X \in L^p$ for some $p > 0$, then

$$\lim_{t \to \infty} t^p P\{|X| > t\} = 0.$$

Problem (8) Suppose that X and Y are a.s. non-negative random variables such that

$$P\{X > t\} \leq \frac{1}{t} E[Y 1_{\{X \geq t\}}] \text{ for all } t > 0.$$

Prove that if $p > 1$,

$$(E[X^p])^{1/p} \leq \left(\frac{p}{p-1}\right) (E|Y|^p)^{1/p}.$$

Problem (9) If X_n converges weakly to X and the $\{X_n, n \geq 1\}$ are uniformly integrable, show that X is integrable and $E[X_n] \to E[X]$ as $n \to \infty$.

Problem (10) Let X_1, X_2, X_3,... be non-negative and i.i.d. random variables. If $(\sum_{k=1}^{n} X_k)/n$ converges a.s. to a finite limit. Show that $E[X_1]$ is finite and the limit equals $E[X_1]$ a.s.