# Probability Qualifying Examination

August 29, 2014

This is a closed book exam. There are 10 problems, of which you should turn in solutions for **exactly** 6 problems. **Correct and complete** solutions to 4 problems guarantees a pass. On the first page of your exam sheet, indicate which 6 you have attempted.

**Problem (1)** Let $X$ and $Y$ be random variables with joint density $f(x, y)$ and $g$ be a measurable function from $\mathbb{R}$ to $\mathbb{R}$ such that $E[g(X)] < \infty$, find $E[g(X)|Y = y]$.

**Problem (2)** Prove the Paley–Zygmund inequality: Suppose that $Z \geq 0$ be a random variable with finite variance. If $0 < \theta < 1$, then

$$P[Z \geq \theta E[Z]] \geq (1-\theta)^2 \frac{(E[Z])^2}{E[Z^2]}.$$

**Problem (3)** Suppose $\{X_n, n \geq 1\}$ are i.i.d. and Poisson distributed with parameter $\lambda$, i.e., $P\{X_1 = n\} = e^{-\lambda}\lambda^n/n!$ for $n \geq 0$. Prove

$$\frac{\lambda^n}{n!} e^{-\lambda} \leq P[X_1 \geq n] \leq \frac{\lambda^n}{n!},$$

and therefore

$$P\left[\limsup_{n\to\infty} \frac{X_n}{\log n / \log\log n} = 1\right] = 1.$$

**Problem (4)** Suppose that $\{X_n : n \geq 1\}$ is a sequence of independent random variables with distribution:

$$P\{X_n = 1\} = P\{X_n = -1\} = \frac{1}{2}\left(1 - \frac{1}{n^{\frac{3}{2}}}\right)$$

and

$$P\{X_n = n^2\} = P\{X_n = -n^2\} = \frac{1}{2n^{\frac{3}{2}}}.$$

Define $S_n := \sum_{j=1}^n X_j$. Prove that $S_n/\sqrt{n}$ converges to a normal distribution with mean $m$ and variance $\sigma^2$, and find $m$ and $\sigma^2$.

**Problem (5)** Let $X$ and $Y$ be two random variables, both of which are defined on a common probability space $(\Omega, \mathcal{F}, P)$. Define

$$X_n = \sum_{j=-\infty}^{\infty} \left(\frac{j}{2^n}\right) 1_{\{X \in [j2^{-n}, (j+1)2^{-n})\}}$$

Prove that for any $Y \in L^1(P)$, $\lim_{n\to\infty} E[Y|X_n] = E[Y|X]$ a.s. and in $L^1(P)$.

**Problem (6)** Consider three types of convergence: convergence a.s., in probability and in $L^p$. Indicate which of these types of convergence **can** or **cannot** imply which; if it fails to imply, give an example.

**Problem (7)** Suppose $F_n$ has density

$$f_n(x) = \begin{cases} 1 - \cos(2n\pi x), & \text{if } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $F_n$ converges weakly to the unifrom distribution on $[0, 1]$ but that the densities $f_n$ do not converge.

**Problem (8)** Suppose that $\{X_n, n \ge 1\}$ are i.i.d. with a symmetric distribution. Then $\sum_{n=1}^{\infty} X_n/n$ converges almost surely iff $E[|X_1|] < \infty$.

**Problem (9)** Let $X$ be a random variable with $P\{X = n\} = p_n$ $n \ge 1$ where $\sum_{n=1}^{\infty} p_n = 1$. Express the $\sum_{n=1}^{\infty} p_n^2$ in terms of the characteristic function of $X$, i.e., $E[e^{i\xi X}]$.

**Problem (10)** Let $\{X_j, j \ge 1\}$ be a sequence of i.i.d. random variables with $P\{X_j = 1\} = P\{X_j = -1\} = 1/2$. Define a random walk $S_n := \sum_{j=1}^{n} X_j$, $n \ge 1$ and $S_0 = 0$. Consider the first time that the random walk $S_n$ hits $m > 0$, i.e., $\tau_m := \inf\{n \ge 0 : S_n = m\}$. Compute $P\{\tau_m < \tau_{-2m}\}$.