NCU Ph. D Qualification Exam. Probability

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共 10 題，每題十分。

1. Let $\{X_n, n \ge 1\}$ be a sequence of independent, identically distributed random variables with distribution function $F(x) = 1 - e/(x \log x)$ for $x \ge e$. Prove that there is a sequence of constants $\mu_n \to \infty$ so that $S_n / n - \mu_n \to 0$ in probability .
2. Let $\{X_n, n \ge 1\}$ be a sequence of independent Poisson random variables with mean λ_n for each n. Prove that if $\sum_{n=1}^{\infty} \lambda_n = \infty$ then $S_n / ES_n \to 1$ a.s.
3. Let $\{X_n, n \ge 1\}$ be a sequence of independent, identically distributed random variables with normal distribution with mean 0 and variance 1. Prove that, for t.>0,

   $P(S_n \ge nt) \le \exp(-nt^2/2)$.
4. Let P be a probability measure on $(R, \beta(R))$ with characteristic function φ.

   Prove that $P\{x : |x| > 2/\delta\} \le (\int_{-\delta}^{\delta} (1 - \varphi(t)dt)) / \delta$ for $\delta > 0$.
5. Let $\{f, f_n, n \ge 1\}$ be a sequence probability densities such that $f_n \to f$ pointwise as $n \to \infty$.

   Prove that $| \int_B f_n(x)dx - \int_B f(x)dx | \to 0$ for all Borel sets B.
6. Let B(n,p) be a Binomial random variable with parameters n and p.

   Prove that $P(B(n,p)) \le tn) \le ((\frac{p}{t})^t (\frac{1-p}{1-t})^{1-t})^n$ for 0<t<p.
7. Let X be a random variable. Denote by $m(X)$ a median of X , $\mu(X)$ the mean of X and $\sigma^2(X)$ the variance of X. Prove that $| m(X) - \mu(X) | \le \sqrt{2\sigma^2(X)}$.
8. Let $\{X_{nj,} 1 \le j \le k_n\}$ be an array of random variables, where $k_n \to \infty$, as $n \to \infty$,

   such that $\lim_{n\to\infty} \max_{1 \le j \le k_n} P(|X_{nj}| > \varepsilon) = 0$ for every ε>0. Prove that

   $\lim_{n\to\infty} \max_{1 \le j \le k_n} |\varphi_{nj}(t) - 1| = 0$ for $t \in R$, where $\varphi_{nj}(t)$ is the chracteristic function of X_{nj}.
9. A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after two hours of travel. The second door leads to a tunnel that returns him to his miner after five hours of travel. The third door leads to a tunnel that returns him to his miner after three hours of travel. Assuming that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until the miner reaches safety?

10. Let $\{X_n, n \geq 1\}$ be a sequence of independent, identically distributed random variables with probability as following:

$P(X_n = \pm 1) = \frac{1}{2n}$ and $P(X_n = 0) = 1 - \frac{1}{n}$ for each n>0.

Denote by $T_0 = 0$ and $T_n = \{\begin{matrix} X_n \text{ if } T_{n-1} = 0 \\ nT_{n-1} |X_n| \text{ if } T_{n-1} \neq 0 \end{matrix}$ for each n>0.

Show that $\{T_n, n \geq 1\}$ is a martingale and that $T_n \to 0$ in probability but $T_n \to 0$ almost surely does not hold.