Probability Qualifying Examination
January 2013

There are two groups and 6 problems in each. You choose exact 3 problems in each group and turn in solutions. Indicate the problems you have chosen on the front page of your answer book. Each problem is worth 10 points.

Group A

Choose three and only three problems from this group and write down the numbers you choose in your answer book.

A1 Let X_n be independent Poisson distributed random variables with mean λ_n. Assume

$$\sum_n \lambda_n = \infty.$$

Denote

$$S_n = X_1 + X_2 + \cdots + X_n, \ \mu_n = \sum_{j=1}^n \lambda_j, \ \sigma_n = \sqrt{\mu_n}.$$

Prove as $n \to \infty$, $\frac{S_n - \mu_n}{\sigma_n}$ converges to standard normal (normal distribution with mean 0 and variance 1) in distribution.

A2 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables. The distribution function of X_i is given by F,

$$F(x) = P(X_1 \leq x), \ x \in R.$$

The empirical distribution function is defined by

$$F_n(x, \omega) = \frac{1}{n} \sum_{j=1}^n \mathbf{1}_{X_j(\omega) \leq x},$$

where $\mathbf{1}_{X_j(\omega) \leq x} = 1$ if $X_j(\omega) \leq x$ and 0, otherwise. Define

$$D_n(\omega) = \sup_x |F_n(x, \omega) - F(x)|. \ n \to \infty$$

Prove $D_n \to 0$ almost surely as $n \to \infty$.

A3 Let $\{X_n, n \geq 1\}$ be independent random variables with

$$E[X_n] = 0, \ E[X_n^2] = \sigma_n^2.$$

Define

$$S_n = \sum_{j=1}^n X_j.$$

(a) Prove $\{S_n, n \geq 1\}$ is L^2-convergence as $n \to \infty$ if and only if

$$\sum_{j=1}^{\infty} \sigma_j^2 < \infty.$$

(b) If

$$\sum_{j=1}^{\infty} \sigma_j^2 < \infty,$$

then $\{S_n, n \geq 1\}$ is almost surely convergence as $n \to \infty$.

A4 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables with exponential distribution

$$P(X_n \geq x) = \exp(-x), x \geq 0.$$

Define

$$M_n = \max\{X_1, X_2, \cdots, X_n\}.$$

Show $M_n - \log n$ converges in distribution to a random variable Y, where

$$P(Y \leq x) = \exp(-e^{-x}).$$

(**Hint:** Calculate $P(M_n - \log n \leq x)$)

A5 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables with finite second moment. Denote

$$E[X_1] = \mu, \;\; E[(X_1 - \mu)^2] = \sigma^2.$$

Assume the law of large number and also the central limit theorem are known. Set

$$\bar{X}_n = \frac{1}{n} \sum_{j=1}^{n} X_j,$$

$\bar{X}_n$ is the emperical sample mean. Denote Y a standard normal random variable. Prove

$$\sqrt{n}((\bar{X}_n)^2 - \mu^2)$$

converges in distribution to $2\mu\sigma Y$.

A6 Suppose the probability space $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the collection of Borel subsets of $[0,1]$, λ is the Lebesgue measure on $[0,1]$. For each n, $\mathcal{B}_n$ is the σ-field on $[0,1]$ generated by $B_k^{(n)}, k = 0, 1, \cdots, 2^n - 1$. For $k = 0, 1, \cdots, 2^n - 2$,

$$B_k^{(n)} = [\frac{k}{2^n}, \frac{k+1}{2^n}),$$

and $B_{2^n-1}^{(n)} = [\frac{2^n-1}{2^n}, 1]$. Let f be an integrable function defined on $[0,1]$. Show for each n, $f_n = E[f|\mathcal{B}_n]$ is a step function on $[0,1]$ and $f_1, f_2, \cdots$ is a martingale with respect to $\mathcal{B}_1, \mathcal{B}_2, \cdots$. State the martingale convergence theorem and the martingale inequality for $\{f_n, n \geq n\}$.

Group B

Choose three and only three problems from this group and write down the numbers you choose in the answer book.

B1 Let $X_n,\ n = 1, 2, \cdots$ be iid integrable. Denote

$$S_n = X_1 + X_2 + \cdots X_n.$$

Prove $\{S_n/n; n \geq 1\}$ is uniformly integrable.

B2 Suppose $\{X_n\}_{n=1}^{\infty}$ is a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$ and define the induced random walk by

$$S_0 = 0, S_1 = X_1, S_2 = X_1 + X_2, \cdots, S_n = \sum_{i=1}^{n} X_i, n = 1, 2, \cdots.$$

Let

$$\tau = \inf\{n > 0; S_n > 0\}$$

be the first upgoing ladder time. Assume we know $\tau(\omega) < \infty$ for all $\omega \in \Omega$. (a) Prove τ is a random variable. (b) Prove S_τ is a random variable.

B3 Suppose $\{A_n\}$ are independent events satisfying $P(A_n) < 1$ for all n. Show

$$P(\cup_{n=1}^{\infty} A_n) = 1 \quad \text{iff } \mathrm{P}(A_n \text{ i.o})=1 .$$

B4 Let ϕ be the characteristic function of a real valued random variable. (a) Prove ϕ must be uniformly continuous. (b) Assume the random variable is integrable. Prove ϕ is continuous differentiable. (c) Assume as $t \to 0$,

$$\frac{\phi(t) + \phi(-t) - 2}{t^2}$$

has finite limit. Prove the random variable is in L^2.

B5 Let $X_n,\ n = 1, 2, \cdots$ be iid with uniform distribution on $[0, 1]$. That is,

$$P(X_n \leq x) = x,\ 0 < x < 1.$$

Show

$$\max(X_1, X_2, \cdots, X_n)$$

converges to 1 almost surely. Prove also

$$\min(X_1, X_2, \cdots, X_n)$$

converges to 0 almost surely as $n \to \infty$.

B6 Suppose X_1, X_2 are iid unit exponential random variables. That is,

$$P(X_1 > x) = P(X_2 > x) = \exp(-x).$$

Calculate (a) $P(X_1 < 3|X_1 + X_2)$ and (b) $E[X_1| \min(X_1, 1)]$.