Probability Qualifying Examination

January 24, 2014

This is a closed book exam. There are 10 problems, of which you should turn in solutions for **exactly** 6 problems. **Correct and complete** solutions to 4 problems guarantees a pass. On the first page of your exam sheet, indicate which 6 you have attempted.

"$\Rightarrow$" means convergence in distribution throughout.

Problem (1) Let $\Omega = [0,1]$ and $\mathcal{F} := \sigma\{[0,1/4),[1/4,3/4),[1/2,1]\}$. Define $X(x) := x^2$ on $[0,1]$, find $E[X|\mathcal{F}]$.

Problem (2) Let $\{\xi_n : n \geq 1\}$ be a sequence of i.i.d. random variables with mean zero and variance one. Define $S_n := \sum_{j=1}^n \xi_j$. Does there exist a unique increasing process A_n with $A_0 = 0$ such that $S_n^2 - A_n$ is a martingale? Find A_n if it exists.

Problem (3) Suppose that $X_n \Rightarrow X$ and $\sup_{n\geq 1} E[X_n^2] < \infty$. Prove that $E|X| < \infty$ and $\lim_{n\to\infty} E[X_n] = E[X]$.

Problem (4) Suppose that $\{X_n : n \geq 1\}$ is a sequence of independent random variables with distribution:

$$P\{X_n = 1\} = P\{X_n = -1\} = \frac{1}{2}\left(1 - \frac{1}{n^2}\right)$$

and

$$P\{X_n = n^2\} = P\{X_n = -n^2\} = \frac{1}{2n^2}.$$

Define $S_n := \sum_{j=1}^n X_n$. Prove that S_n/n converges to a constant C and find it.

Problem (5)
(a) Suppose that X is a random variable with standard normal distribution (i.e. $P\{X \in B\} = \int_B n(x)dx$, where $n(x) = \sqrt{2\pi}^{-1}\exp(-x^2/2)$), then prove

$$\lim_{x\to\infty} \frac{P\{X_n > x\}}{n(x)/x} = 1.$$

(b) Suppose that $\{X_n : n \geq 1\}$ are i.i.d. random variables with standard normal distribution. Show that

$$P\left\{\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{\log n}} = \sqrt{2}\right\} = 1.$$

Problem (6) Construct three random variables X, Y and Z such that any two of X, Y and Z are independent, but X, Y and Z are not independent.

Problem (7) Let $\{L_j^{(n)} : j \geq 1, n \geq 1\}$ be i.i.d. random variables taking values on the set $\{0, 1, 2, 3...\}$ with $m := EL^1_{(1)} < 1$. Define the branching process

$$Z_0 = 1 \text{ and } Z_n := \sum_{j=1}^{Z_{n-1}} L_j^{(n)},\ n \geq 1.$$

Conventionally, $\sum_1^0 a_j := 0$. Prove that $Z_n/(m^n)$ is martingale and find the limit of Z_n if it exists.

Problem (8) (Wald's Identity) Let $\{X_n : n \geq 1\}$ be a sequence of random variables with finite mean. Define the random walk $S_n := \sum_{j=1}^n X_j$. Let T be a stopping time with respect to the filtration $\mathcal{F}_n := \sigma\{X_1, ..., X_n\}$, $n \geq 1$ and $E[T] < \infty$. Show that $E[S_T] = E[X_1]E[T]$. (**Hint**: Start with the martingale $S_n - nE[X_1]$, w.r.t. $\mathcal{F}_n$.)

Problem (9) Suppose that $\{X_n : n \geq 0\}$ is a sequence of Poisson distributed random variables so that for $n \geq 0$ there exist constants λ_n and

$$P\{X_n = k\} = \frac{e^{-\lambda_n}\lambda_n^k}{k!},\ k \geq 0.$$

Give necessary and sufficient conditions for $X_n \Rightarrow X_0$.

Problem (10) Let X be a random variable. Show that $(E[|X|^p])^{1/p}$ is increasing in $p \in [1, \infty)$.