Probability Qualifying Examination
February 2018

There are problems given in Group A and Group B. Indicate the problems you have chosen to solve on the front page of your answer book.

Group A

Choose four and only four problems from this group and write down the numbers you choose in your answer book. Each with complete answer receives 10 points.

A1 Assume P_1, P_2 are two probability measures defined on $(R, \mathcal{B}(R))$. F_1, F_2 are the distributions for P_1 and P_2 respectively. Assume

$$F_1(x) = F_2(x), \ x \in R.$$

Prove $P_1 = P_2$. That is, $P_1(B) = P_2(B)$ for any $B \in \mathcal{B}(R)$.

A2 Let X be a random variable such that the distribution of X given by F is continuous,

$$F(x) = P(X \leq x).$$

Prove $F(X)$ is a random variable. Prove also $Y = F(X)$ has uniform distribution. That is,

$$P(Y \leq y) = y, \ y \in [0, 1].$$

A3 Assume $X_1, X_2, \cdots$ is a sequence of independent random variables. Prove the following are equivalent.

(a) $P(\sup_n X_n < \infty) = 1$.

(b) $\sum_{n=1}^{\infty} P(X_n > M) < \infty$ for some constant M.

A4 Let X be a random variable defined on a probability space $(\Omega, \mathcal{B}, P)$. P_X is defined on $(R, \mathcal{B}(R))$ given by

$$P_X(B) = P(X \in B), \ B \in \mathcal{B}(R).$$

(a) Prove P_X is a probability measure.

(b) Let f be a bounded Borel measurable function,

$$f : R \to R.$$

Prove $f(X)$ is a bounded random variable on $(\Omega, \mathcal{B}, P)$.

(c) In (b), prove

$$E^P[f(X)] = E^{P_X}[f],$$

where $E^P[\cdot]$ is the expectation with respect to P and $E^{P_X}[\cdot]$ is the expectation with respect to P_X.

A5 Let X be a random variable with $E[X^2] < \infty$ on a probability space $(\Omega, \mathcal{B}, P)$. $\mathcal{G}$ is a sub-σ field of $\mathcal{B}$. Denote $Y = E[X|\mathcal{G}]$, the conditional expectation of X given $\mathcal{G}$. Prove

$$E[|X - Y|^2] = \min\{E[|X - Z|^2];\ Z \text{ is } \mathcal{G} \text{ measurable and } E[Z^2] < \infty\ \}.$$

A6 Levy metric. Let F, G be distribution functions. Define

$$d(F, G) = \inf\{\delta > 0;\ \text{for any } x \text{ we have } F(x - \delta) - \delta \leq G(x) \leq F(x + \delta) + \delta\ \}.$$

Assume $F_1, F_2, \cdots, F_0$ are distribution functions. Prove $d(F_n, F_0) \to 0$ as $n \to \infty$ if and only if F_n converges to F_0 in dsitribution as $n \to \infty$.

Group B

Choose three and only three problems from this group and write down the numbers you choose in the answer book. Each with complete answer receives 20 points.

B1 Let $X_n, n = 1, 2, \cdots$ be independent random variables with exponential distributions,

$$P(X_n > x) = \exp(-\lambda_n x), x \geq 0.$$

Prove the following.

(a) $P(\sum_{n=1}^{\infty} X_n < \infty)$ is 1 or 0.

(b) The answer in (a) is according to $\sum_{n=1}^{\infty} \frac{1}{\lambda_n}$ is finite or not. That is, the probability in (a) is 1, if

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty.$$

Otherwise, the probability in (a) is 0.

B2 Calculate the limits. You need to give proper explanation of your answer to receive points.

(a)

$$\lim_{n\to\infty} \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} (\frac{x_1 + x_2 + \cdots + x_n}{n})^2 \exp(-(x_1 + x_2 + \cdots + x_n)) dx_1 dx_2 \cdots dx_n.$$

(b) For any positive integer k, find the limit,

$$\lim_{n\to\infty} \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} (\frac{x_1 + x_2 + \cdots + x_n}{n})^k \exp(-(x_1 + x_2 + \cdots + x_n)) dx_1 dx_2 \cdots dx_n.$$

(c) f is a bounded continuous function defined on R.

$$\lim_{n\to\infty} \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} f(\frac{x_1 + x_2 + \cdots + x_n}{n}) \exp(-(x_1 + x_2 + \cdots + x_n)) dx_1 dx_2 \cdots dx_n.$$

B3 Let $x \in (0,1)$ have binary expansion

$$x = \sum_{n=1}^{\infty} \frac{d_n}{2^n},$$

with $d_n = 0$ or 1. Define

$$f_n(x) = 2 \text{ if } d_n = 0;\ f_n(x) = 0 \text{ if } d_n = 1.$$

Prove the following.

(a) $\int_0^1 f_n(x)dx = 1$ for all n.

(b) f_n converges only on a set of Lebesgue measure 0.

(c) Assume X_n is a random variable with density f_n. Then X_n converges in distribution to the uniform distribution on $[0,1]$.

B4 Let X be an integrable random variable and $\mathcal{B}_n, n = 1, 2, \cdots$ be a filtration on a probability space $(\Omega, \mathcal{B}, P)$ such that for each n, the σ-field $\mathcal{B}_n \subset \mathcal{B}$. Define

$$X_n = E[X|\mathcal{B}_n].$$

Prove the following.

(a) $\{X_n, n = 1, 2, \cdots\}$ is uniformly integrable.

(b) $\{X_n, \mathcal{B}_n\}$ is a martingale.

(c) X_n converges to X almost surely and also in L^1.