國立中央大學數學系
博士班資格考試〈統計推論〉試題

Qualifying Exam (Mathematical Statistics)

September 5, 2002

1. Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ random variables, where $\mu > 0$ and $\sigma^2 > 0$ are both unknown parameters. Find the UMVUE for $\theta = \log \sigma^2$. [10%]

2. Let X_1, X_2 be iid truncated Poisson random variables with pmf

$$p(x; \lambda) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!\,(1-e^{-\lambda})} & x = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Find the UMP $\alpha = 0.05$ test of $H_0 : \lambda = 1$ versus $H_1 : \lambda > 1$. [10%]

3. Let $X_1, X_2, \ldots, X_n$ be iid $N(\theta, 1)$ random variables, and let a be a fixed constant, $-\infty < a < \infty$.

a) Find the UMVUE of $p(\theta) = P_\theta(X_i \le a)$ in terms of $\Phi(\cdot)$, the cdf of the standard normal distribution. [10%]

b) Let $\hat{\theta}_1$=(number of $X_i's \le a)/n$. Find the asymptotic relative efficiency of $\hat{\theta}_1$ with respect to the UMVUE obtained in a). [10%]

4. Let $X_1, X_2, \ldots, X_n$ be a random sample with pdf $f(x)$. For $H_0 : f(x) = I_{(0,1)}(x)$ versus $H_1 : f(x) = 2xI_{(0,1)}$, find the test which minimizes $\alpha + 2\beta$, where α and β are the Type I and Type II error probabilities, respectively. [10%]

5. Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ random variables. Given $X_1, X_2, \ldots, X_n$, it is known that $Y_1, Y_2, \ldots, Y_n$ are independent Poisson random variables with $E(Y_i|X_i) = e^{X_i}$, $i = 1, \ldots, n$. Find consistent estimators of μ and σ^2 based only on the $Y_i's$. [10%]

6. Let $X_1, X_2, \ldots, X_{10}$ be iid Bernoulli(p) random variables. Find a $1-\alpha$ lower confidence bound for p. [10%]

7. Let X_i have a binomial distribution with parameters n and p_i, $i = 1, 2$, and assume that X_1 and X_2 are independent. The value of n is known, but p_1 and p_2 are unknown. However, it is known that $0 \le p_2 \le p_1 \le 1$.

a) Derive the maximum likelihood estimators $\hat{p}_1$ and $\hat{p}_2$ of p_1 and p_2. [10%]

b) If $p_1 > p_2$, determine the asymptotic ($n \to \infty$) joint distribution of

$n^{1/2}(\hat{p}_1 - p_1, \hat{p}_2 - p_2)$. [10%]

8. Let $X_1, X_2, \ldots, X_n$ be a random sample with pdf $f(x; \theta)$. Suppose $\hat{\theta}$ is the MLE of θ, and under regularity conditions, $\sqrt{n}(\hat{\theta} - \theta_0)$ is approximately $N(0, I(\theta_0)^{-1})$, where $I(\theta_0)$ is the Fisher information based on $f(x; \theta)$ evaluated at the true parameter value θ_0. Prove or disprove that $E(\hat{\theta}) = \theta_0$ and $Var(\hat{\theta}) = I(\theta_0)^{-1}/n$. [10%]