Analysis

1. (Arithmetic mean $\geq$ geometric mean)

   $x_1,\ x_2\ ,\ \cdots, x_n > 0\ ,\ \alpha_1\ ,\ \alpha_2\ ,\ \cdots,\ \alpha_n > 0,\ \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Show that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \geq x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

   Proof :

   $f(x) = -\ln x$ is convex on $(0,\ \infty)$

   $(\because f'(x) = -\dfrac{1}{x} \Rightarrow f''(x) = \dfrac{1}{x^2} \Rightarrow f(x)$ is convex $)$

   $\Longrightarrow f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$

   i.e. $-\ln x(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \leq \alpha_1(-\ln x_1) + \cdots + \alpha_n(-\ln x_n)$

   $\Longrightarrow -\ln x(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \leq -\ln(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n})$

   $\Longrightarrow \ln x(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \geq \ln(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n})$

   $\Longrightarrow \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \geq x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ □

2. $A(i_1, i_2, \cdots, i_m)$ is a nonnegative integer for $i_1 = 1, 2, \cdots, n$,

   $i_2 = 1, 2, \cdots, n,\ \cdots, i_m = 1, 2, \cdots, n. \quad (m \geq 2)$

   Let

   $$\begin{aligned} S(a_1, a_2, \cdots, a_m) = &\sum_{i_1=1}^{n} A(i_1, a_2, \cdots, a_m) \\ &+ \sum_{i_2=1}^{n} A(a_1, i_2, \cdots, a_m) \\ &\vdots \\ &+ \sum_{i_m=1}^{n} A(a_1, a_2, \cdots, i_m) \end{aligned}$$

   Suppose $S(a_1, a_2, \cdots, a_m) \geq n$ whenever $A(a_1, a_2, \cdots, a_m) = 0$

   Let $S = \displaystyle\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} A(i_1, i_2, \cdots, i_m)$

   Show that $S > \dfrac{n^m}{m+1}$

   Proof :

   For any k, $1 \leq k \leq m$,

   $$\begin{aligned} &\sum_{a_1=1}^{n} \sum_{a_2=1}^{n} \cdots \sum_{a_m=1}^{n} \sum_{i_k=1}^{n} A(a_1, \cdots, i_k, \cdots, a_m) \\ = &\sum_{a_1=1}^{n} \sum_{a_2=1}^{n} \cdots \sum_{a_{k-1}=1}^{n} \sum_{a_k=1}^{n} \sum_{a_{k+1}=1}^{n} \cdots \sum_{a_m=1}^{n} \sum_{i_k=1}^{n} A(a_1, \cdots, i_k, \cdots, a_m) \end{aligned}$$

$$= \sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\cdots\sum_{a_{k-1}=1}^{n} n \left(\sum_{a_{k+1}=1}^{n} \cdots \sum_{a_m=1}^{n}\sum_{i_k=1}^{n} A(a_1, \cdots, i_k, \cdots, a_m) \right)$$

$$= n \sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\cdots\sum_{a_{k-1}=1}^{n}\sum_{a_{k+1}=1}^{n} \cdots \sum_{a_m=1}^{n}\sum_{i_k=1}^{n} A(a_1, \cdots, i_k, \cdots, a_m)$$

$$= nS$$

$$\Rightarrow \qquad \sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\cdots\sum_{a_m=1}^{n} S(a_1, a_2, \cdots, a_m)$$

$$= \sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\cdots\sum_{a_m=1}^{n}\sum_{i_1=1}^{n} A(i_1, a_2, \cdots, a_m)$$

$$+ \sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\cdots\sum_{a_m=1}^{n}\sum_{i_2=1}^{n} A(a_1, i_2, \cdots, a_m)$$

$$+ \quad \cdots$$

$$+ \sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\cdots\sum_{a_m=1}^{n}\sum_{i_m=1}^{n} A(a_1, a_2, \cdots, i_m)$$

$$(\because \text{by def of } S(a_1, a_2, \cdots, a_m))$$

$$= mnS \quad \cdots\cdots (1)$$

Let $Z = |\{(i_1, \cdots, i_m) : \ A(i_1, \cdots, i_m) = 0 \ \}|$

Then $S + Z \geq \sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\cdots\sum_{a_m=1}^{n} 1 \qquad (why?)$

$= n^m$

$\Rightarrow Z \geq n^m - S \quad \cdots\cdots \ (2)$

$$S(a_1, a_2, \cdots, a_m) = \sum_{i_1=1}^{n} A(i_1, a_2, \cdots, a_m)$$

$$+ \sum_{i_2=1}^{n} A(a_1, i_2, \cdots, a_m)$$

$$+ \quad \cdots$$

$$+ \sum_{i_m=1}^{n} A(a_1, a_2, \cdots, i_m)$$

$$\geq \ A(a_1, a_2, \cdots, a_m)$$

$$+ A(a_1, a_2, \cdots, a_m)$$

$$+ \quad \cdots$$

$$+ A(a_1, a_2, \cdots, a_m) \qquad (why?)$$

$$= mA(a_1, a_2, \cdots, a_m) \quad \cdots\cdots (3)$$

$$
\begin{aligned}
\sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\cdots\sum_{a_m=1}^{n} S(a_1,\cdots,a_m) &= \sum_{(a_1,\cdots,a_m)} S(a_1,\cdots,a_m) \\
&= \sum_{A(a_1,\cdots,a_m)=0} S(a_1,\cdots,a_m) + \sum_{A(a_1,\cdots,a_m)\neq 0} S(a_1,\cdots,a_m) \\
&\geq Zn + \sum_{A(a_1,\cdots,a_m)\neq 0} mA(a_1,\cdots,a_m) \\
&\qquad\qquad (\because \textit{assumption and } (3)) \\
&= Zn + m \sum_{A(a_1,\cdots,a_m)\neq 0} A(a_1,\cdots,a_m) \\
&= Zn + m \sum_{(a_1,\cdots,a_m)} A(a_1,\cdots,a_m) \\
&= Zn + mS \\
&\geq (n^m - S)n + mS \quad (\because (2)) \\
&= n^{m+1} + (m-n)S \quad \cdots\cdots (4)
\end{aligned}
$$

$$
\begin{aligned}
(1)\ ,\ (4) &\Rightarrow mnS \geq n^{m+1} + (m-n)S \\
&\Rightarrow (mn+n-m)S \geq n^{m+1} \\
&\Rightarrow\ S \geq \frac{n^{m+1}}{mn+n-m} > \frac{n^{m+1}}{mn+n} = \frac{n^m}{m+1}
\end{aligned}
$$

□

3. Definition: If $r > 0$, $f : R \to R$ and $f(x+r) = f(x)$ for all $x \in R$, then f is function with quasiperiod r.

   Suppose $r > 0$, $g : R \to R$ such that $g(x+r) - g(x)$ is a function with quasiperiod r.

   Assume that g is a bounded function. Show that g is with quasiperiod r.

   Proof :

   Suppose $|g(x)| \leq C$ for all x. $g(x+r) - g(x)$ is a function with quasiperiod r

$$
\begin{aligned}
\Longrightarrow g(x+r) - g(x) &= g(x+2r) - g(x+r) \\
&= g(x+3r) - g(x+2r) \\
&\ \ \vdots \\
&= g(x+(n+1)r) - g(x+nr)
\end{aligned}
$$

$$
\Longrightarrow n(g(x+r) - g(x)) = g(x+(n+1)r) - g(x+r)
$$

$$
\Longrightarrow (g(x+r) - g(x)) = \frac{1}{n}\ (g(x+(n+1)r) - g(x+r))
$$

$$
\begin{aligned}
\Longrightarrow |(g(x+r) - g(x))| &\leq \frac{1}{n}\ |g(x+(n+1)r) - g(x+r)| \\
&\leq \frac{1}{n}\ 2C
\end{aligned}
$$

$\Longrightarrow g(x+r) = g(x)$ for all x. □

4. Let $\alpha > \beta > 0$, $f : R \to R$, and $\dfrac{\alpha}{\beta} = \dfrac{m}{n}$ where $m, n \in \mathbb{N}$.

Suppose that $f(x+\alpha) - f(x)$ is a function with quasiperiod β

Show that

(1) $f(x+n\alpha) - f(x)$ is a function with quasiperiod $\alpha - \beta$

(2) $f(x+mn(\alpha-\beta)) - f(x)$ is a function with quasiperiod $\alpha - \beta$

(3) $f(x)$ is a function with period $mn(\alpha - \beta)$ if f is a bounded function.

Proof :

Note : $f(x+\alpha) - f(x)$ is with quasiperiod β

$\Longrightarrow \quad f(x+\beta) - f(x)$ is quasiperiod α (why ?)

Hence $f(x+\alpha) - f(x)$ is with quasiperiod $m\beta$

$f(x+\beta) - f(x)$ is with quasiperiod $n\alpha$ (why ?)

$\Longrightarrow \quad f(x+\alpha) - f(x) = f(x+\alpha+m\beta) - f(x+m\beta)$

$f(x+\beta) - f(x) = f(x+\beta+n\alpha) - f(x+n\alpha)$

$\Longrightarrow \quad f(x+\alpha) - f(x+\beta) = f(x+\alpha+n\alpha) - f(x+\beta+n\alpha)$

$\Longrightarrow \quad f(x+\alpha-\beta) - f(x) = f(x+\alpha-\beta+n\alpha) - f(x+n\alpha)$($\because$ replace x by $x-\beta$)

$\Longrightarrow \quad f(x+n\alpha) - f(x) = f(x+n\alpha+(\alpha-\beta)) - f(x+(\alpha-\beta))$

$\Longrightarrow \quad f(x+n\alpha) - f(x)$ is a function with quasiperiod $\alpha - \beta$ $\cdots\cdots\cdots\cdots\cdots\cdots$ (1)

$\Longrightarrow \quad f(x+(\alpha-\beta)) - f(x)$ is a function with quasiperiod $n\alpha = m\beta$ ($\because$ Note)

$\Longrightarrow \quad f(x+(\alpha-\beta)) - f(x)$ is with quasiperiod $mn\alpha$

$f(x+(\alpha-\beta)) - f(x)$ is with quasiperiod $mn\beta$

$\Longrightarrow \quad f(x+mn\alpha) - f(x)$ is with quasiperiod $\alpha - \beta$

$f(x+mn\beta) - f(x)$ is with quasiperiod $\alpha - \beta$

$\Longrightarrow \quad f(x+mn\alpha) - f(x+mn\beta)$ is with quasiperiod $\alpha - \beta$

$\Longrightarrow \quad f(x+mn(\alpha-\beta)) - f(x)$ is with quasiperiod $\alpha - \beta$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (2)

$\Longrightarrow \quad f(x+mn(\alpha-\beta)) - f(x)$ is with quasiperiod $mn(\alpha - \beta)$

$\Longrightarrow \quad f(x)$ is is with quasiperiod $mn(\alpha-\beta)$ ($\because$ f is bounded and apply problem 3)
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (3) $\square$

5. Let n and ℓ be integers such that $n \geq 2$ and $\ell \geq n+1$.
Suppose that $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, m_1, m_2, \cdots, m_\ell$ are nonnegative numbers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}$, $m_1 \geq m_2 \geq \cdots \geq m_n \geq \cdots \geq m_\ell$ and

$$\begin{aligned} \sum_{i=1}^{\ell} m_i &= \sum_{i=1}^{n-1} \lambda_i (n-i), \\ \sum_{i=1}^{k} m_i &\leq \sum_{i=1}^{k} \lambda_i (n-i) \qquad \text{for} \quad k = 1, 2, \cdots, n-2. \end{aligned}$$

Let $m'_1 \geq m'_2 \geq \cdots \geq m'_{\ell-1}$ be a rearrangement of $m_1, m_2, \cdots, m_{n-1}, m_n + m_\ell, m_{n+1}, m_{n+2}, \cdots, m_{\ell-1}$. Then

$$\begin{aligned} \sum_{i=1}^{\ell-1} m'_i &= \sum_{i=1}^{n-1} \lambda_i(n-i) \quad \text{and} \\ \sum_{i=1}^{k} m'_i &\leq \sum_{i=1}^{k} \lambda_i(n-i) \qquad \text{for} \quad k = 1, 2, \cdots, n-2. \end{aligned}$$

Proof :

The required equality follows from the fact that $\sum_{i=1}^{\ell-1} m'_i = \sum_{i=1}^{\ell} m_i$. Now we prove the inequalities. Suppose that $m'_t = m_n + m_\ell$ where $t \leq n$. We can see that

$$m'_i = \begin{cases} m_i, & i = 1, 2, \cdots, t-1 \\ m_{i-1}, & i = t+1, t+2, \cdots, n. \end{cases}$$

Suppose, to the contrary of the conclusion, that $\sum_{i=1}^{j} m'_i > \sum_{i=1}^{j} \lambda_i(n-i)$ for some integer j where $t \leq j \leq n-2$. Then

$$\begin{aligned} 2m_n &\geq m_n + m_\ell \\ &= m'_t \\ &= \sum_{i=1}^{j} m'_i - \sum_{i=1}^{j-1} m_i \\ &> \sum_{i=1}^{j} \lambda_i(n-i) - \sum_{i=1}^{j-1} \lambda_i(n-i) \\ &= \lambda_j(n-j). \end{aligned}$$

Hence $\quad m_n > \lambda_j(n-j)/2.$

Then

$$
\begin{aligned}
\sum_{i=1}^{n-1} \lambda_i(n-i) &= \sum_{i=1}^{\ell-1} m_i' \\
&\geq \sum_{i=1}^{j} m_i' + \sum_{i=j+1}^{n} m_i' \\
&= \sum_{i=1}^{j} m_i' + \sum_{i=j}^{n-1} m_i \\
&\geq \sum_{i=1}^{j} m_i' + (n-j)m_n \\
&> \sum_{i=1}^{j} \lambda_i(n-i) + \lambda_j(n-j)^2/2 \\
&\geq \sum_{i=1}^{j} \lambda_i(n-i) + \lambda_j \sum_{i=j+1}^{n-1} (n-i) \\
&\geq \sum_{i=1}^{n-1} \lambda_i(n-i).
\end{aligned}
$$

Thus we obtain $\sum_{i=1}^{n-1} \lambda_i(n-i) > \sum_{i=1}^{n-1} \lambda_i(n-i)$; this absurdity confirms the inequalities.

□

6. An object is moving to the right on the real line. Let $u(s)$ denote the velocity of the object at the position s on the real line. Then the time needed for the object moving from position s_1 to position s_2 $(s_1 < s_2)$ is

$$\int_{s_1}^{s_2} \frac{1}{u(s)}\, ds.$$

Proof Let $s(t)$ and $v(t)$ denote the position and the velocity, respectively of the object at time t. Suppose that $s(t_1) = s_1$, and $s(t_2) = s_2$. First assume that $u(s) \neq 0$ for $s_1 \leq s \leq s_2$. Then

$$
\begin{aligned}
\int_{s_1}^{s_2} \frac{1}{u(s)}\, ds &= \int_{t_1}^{t_2} \frac{1}{u(s(t))}\, ds(t) \\
&= \int_{t_1}^{t_2} \frac{1}{v(t)} s'(t)\, dt \\
&= t_2 - t_1.
\end{aligned}
$$

And $t_2 - t_1$ is just the time needed for the object to move from s_1 to s_2.

Next consider the case that $u(s) = 0$ for some s, $s_1 \leq s \leq s_2$. For example, $u(s_1) = 0$

and $u(s) \neq 0$ for $s_1 < s \leq s_2$. Then

$$\begin{aligned}
\int_{s_1}^{s_2} \frac{1}{u(s)}\, ds &= \lim_{s' \to s_1^+} \int_{s'}^{s_2} \frac{1}{u(s)}\, ds \\
&= \lim_{s' \to s_1^+} t_2 - t' \quad (\text{where } s(t') = s') \\
&= \lim_{t' \to t_1^+} t_2 - t' \\
&= t_2 - t_1.
\end{aligned}$$

This completes the proof. □

7. Two cars left city A at the same time and moved forward along a straight highway which connects city A and city B. These two cars arrived city B at the same time. Show that there must exist someplace (neither city A nor city B) on the highway where two cars had the same velocity.

Proof Let C_1, C_2 denote the two cars. Use the interval $[0, 1]$ to represent the straight highway which connects city A and city B. Let $u_1(s)$ be the velocity of car C_1 at position s $(0 \leq s \leq 1)$, and $u_2(s)$ be that of car C_2. Assume that both $u_1(s)$ and $u_2(s)$ are continuous functions. We need to show that $u_1(s) = u_2(s)$ for some s with $0 < s < 1$. Suppose, to the contrary, that $u_1(s) \neq u_2(s)$ for all s with $0 < s < 1$. Then either $0 \leq u_1(s) < u_2(s)$ for all $0 < s < 1$ or $0 \leq u_2(s) < u_1(s)$ for all $0 < s < 1$, which implies that

either $\int_0^1 \frac{1}{u_2(s)}\, ds < \int_0^1 \frac{1}{u_1(s)}\, ds$ or $\int_0^1 \frac{1}{u_1(s)}\, ds < \int_0^1 \frac{1}{u_2(s)}\, ds$.

Using Problem 6, we see that the time needed for car C_1 to move from city A to city B and that for car C_2 are not equal. This is a contradiction. □