Exercises within a Section: 1.1-2 means Section 1.1 problem #2 etc...

☞**Watch out! To save paper and spaces, some solutions may not be in the proper order. You should be able to find them.**（解答裡題目的順序可能不會照書上的順序，請大家在「該出現的地方」找不到時往後翻一下）

☞**You are required to reproduce or to paraphrase of the "Solution" (NOT the "Sketch") to a problem.** （考試寫證明的時候，是要寫 Solution 的部份，而不是 Sketch 的部份！Sketch 的部份是告訴大家證明的 idea 是什麼。）

☞**Do not skip over problems that you think are complicated. We can still ask you part of the steps in a test.**（不要跳過你覺得證明太複雜的問題，我們考試時仍可能會考你！）

◇ **3.1-4.** Let $x_k \to x$ be a convergent sequence in a metric space and let $A = \{x_1, x_2, \dots\} \cup \{x\}$.

(a) Show that A is compact.

(b) Verify that every open cover of A has a finite subcover.

Suggestion. Parts (a) and (b) might as well be done together since (b) is the definition of compactness. ◇

Solution. (a) Since the assertion in part (b) is actually the definition of compactness, we could do both parts of the problem together by doing (b). For interest, we present an argument for part (a) using Theorem 3.1.5. When we are all done, the reader may agree that checking the abstract definition seems easier.

We know that the set A is closed from previous work. It is also totally bounded. If $\varepsilon > 0$, then There is an N such that $d(x_k, x) < \varepsilon$ whenever $k \geq N$. So $x_k \in D(x, \varepsilon)$ for such n. Since x is also in $D(x, \varepsilon)$, and $x_j \in D(xj, \varepsilon)$, we have $A \subseteq D(x, \varepsilon) \cup \bigcup_{j=1}^{N-1} D(x_j, \varepsilon)$. Thus A is totally bounded. So A is a closed, totally bounded subset of the metric space M. If we knew that M was complete, this would show that A is compact.

But unfortunately we do not know that. Thus to pursue this method we need to show that A is complete even though M might not be.

Let $\langle y_n \rangle_1^\infty$ be a Cauchy sequence in A. So each of the points y_n is either equal to x or to one of the x_k. We need to show that this sequence converges to some point in A. Let $\varepsilon > 0$. Then there are indices K and N such that $d(y_{n+p}, y_n) < \varepsilon/2$ and $d(x_k, x) < \varepsilon/2$ whenever $n \geq N$, $k \geq K$, and $p > 0$. We distinguish three cases.

CASE ONE: If y_n is eventually equal to x_{k_0} for some index k_0, then the sequence $\langle y_n \rangle_1^\infty$ certainly converges to x_{k_0}.

CASE TWO: If all but finitely many of the y_n are among the finitely many points $\{x_1, x_2, \ldots x_K\}$, then there are only finitely many different distances involved among them. If they are all 0, then we are in CASE ONE. If all but finitely many are 0, we are still in CASE ONE. If not, then one of the finitely many nonzero distances must occur infinitely often and the sequence could not have been a Cauchy sequence.

CASE THREE: If we are not in CASE ONE or CASE TWO, then there are infinitely many indices n for which y_n is not in the set $\{x_1, x_2, \ldots x_K\}$. We can pick an index $n_1 > N$ such that $y_{n_1} = x_{k_1}$ for an index $k_1 > K$. If $n \geq n_1$, we have

$$d(y_n, x) \leq d(y_n, y_{n_1}) + d(y_{n_1}, x) = d(y_n, y_{n_1}) + d(x_{k_1}, x) \\ \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So, in this case, $y_n \to x$. Every Cauchy sequence in A converges to a point in A. So A is complete. The set A is a complete, totally bounded subset of the metric space M, so it is a complete, totally bounded metric space in its own right and is compact by Theorem 3.1.5.

(b) This assertion is the definition of compactness. Suppose $\{U_\beta\}_{\beta \in B}$ is any collection of open subsets of M with $A \subseteq \bigcup_{\beta \in B} U_\beta$. (The set B is just any convenient set of indices for labeling the sets.) Then there is at least one index β_0 such that $x \in U_{\beta_0}$. Since U_{β_0} is open and $x_k \to x$, there is an index K such that $x_k \in U_{\beta_0}$ whenever $k \geq K$. For the finitely many points $x_1, x_2, \ldots, x_{K-1}$, there are indices $U_{\beta_1}, U_{\beta_2}, \ldots, U_{\beta_{K-1}}$ such that $x_k \in U_{\beta_k}$. So

$$A = \{x_1, x_2, \ldots\} \cup \{x\} \subseteq U_{\beta_0} \cup U_{\beta_1} \cup U_{\beta_2} \cup \cdots \cup U_{\beta_{K-1}}.$$

Thus every open cover of A has a finite subcover, so A is compact. ♦

◇ **3.1-5.** Let M be a set with the discrete metric. Show that any infinite subset of M is noncompact. Why does this not contradict the statement in Exercise 3.1-4?

Sketch. A sequence converges if and only if it is eventually constant. This does not contradict Exercise 3.1-4 since the entries in such a convergent sequence form a finite set. ♢

Solution. Let A be an infinite subset of M. Since we are using the discrete metric, single point sets are open. In fact, $\{x\} = D(x, 1/2)$ We certainly have $A = \bigcup_{x \in A} \{x\} = \bigcup_{x \in A} D(x, 1/2)$. This is an open cover of the set A which uses infinitely many sets, and, since they are pairwise disjoint, we cannot omit any of them. There can be no finite subcover. Thus A is not compact.

If $\langle x_k \rangle_1^\infty$ is a sequence in A converging to x, then there must be an index K such that $d(x, x_k) < 1/2$ whenever $k \geq K$. But since we are using the discrete metric which only has the values 0 and 1, this forces $d(x, x_k) = 0$ and so $x_k = x$. Thus $x_k = x$ for all $k \geq K$ and there are only finitely many points in the set $\{x_1, x_2 \ldots\} \cup \{x\}$. Such a finite set of points is compact, and the statement of Exercise 3.1-4 is not contradicted. ♦

♢ **3.2-1.** Which of the following sets are compact?

(a) $\{x \in \mathbb{R} \mid 0 \leq x \leq 1 \text{ and } x \text{ is irrational}\}$
(b) $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$
(c) $\{(x, y) \in \mathbb{R}^2 \mid xy \geq 1\} \cap \{(x, y) \mid x^2 + y^2 < 5\}$

Answer. None of them. ♢

Solution. (a) No. The point $1/2$ is not in the set $A = \{x \in \mathbb{R} \mid 0 \leq x \leq 1 \text{ and } x \text{ is irrational }\}$, but every interval around it contains irrational numbers which are in A. The complement of A is not open. The set A is not closed. It cannot be compact.

(b) No. The set $B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$ is not bounded. It is an infinite "vertical" strip containing the points $(1/2, y)$ for arbitrarily large y. Since B is not bounded, it cannot be compact.

(c) No. The set $C = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 1\} \cap \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 5\}$ is bounded, but it is not closed. The point $(\sqrt{5/2}, \sqrt{5/2})$ is on the boundary of C but not in C. Since C is not closed it cannot be compact. ♦

♢ **3.3-4.** Let $x_k \to x$ be a convergent sequence in a metric space. Let $\mathcal{A}$ be a family of closed sets with the property that for each $A \in \mathcal{A}$, there is an N such that $k \geq N$ implies $x_k \in A$. Prove that $x \in \cap\mathcal{A}$.

Solution. Let A be one of the sets in the collection $\mathcal{A}$. Then there is an N such that $x_N, x_{N+1}, x_{N+2}, x_{N+3}, \ldots$ are all in A. Since $x_k \to x$ as $k \to \infty$, this sequence of points in A converge to x. Since A is closed, we have $x \in A$. Since this is true for every set A in the collection $\mathcal{A}$, we have $x \in \bigcap \mathcal{A}$ as claimed. ♦

◇ **3.2-5.** Let A be an infinite set in $\mathbb{R}$ with a single accumulation point in A. Must A be compact?

Answer. No. ◇

Solution. The set might not be bounded. Consider the example

$$A = \{0, 1, 2, 1/2, 3, 1/3, 4, 1/4, 5, \dots\}.$$

There is one accumulation point, 0, which is in A. But A is not bounded, so it is not compact.

The wording of the question is a bit ambiguous. It could be taken to allow the possibility that there was another accumulation point which was not in A. That would also prevent the set from being compact. That route to noncompactness would have been eliminated by a wording something like: "Let A be an infinite set in $\mathbb{R}^n$ with a single accumulation point which is in A. Must A be compact?" ♦

◇ **3.3-2.** Is the nested set property true if "compact nonempty" is replaced by "open bounded nonempty"?

Answer. No. ◇

Solution. The words "compact nonempty" may not be replaced by "open bounded nonempty" in the nested set property and still retain a valid assertion. The modified statement would be:

Conjecture. *Let F_k be a sequence of nonempty open bounded subsets of a metric space M such that $F_{k+1} \subseteq F_k$ for each k. Then there is at least one point in $\bigcap_{k=1}^{\infty} F_k$.*

This assertion is false. For example, we could use the sequence of open intervals $F_k =]0, 1/k[= \{x \in \mathbb{R} \mid 0 < x < 1/k\}$ for $k = 1, 2, 3, \dots$. Each of these is a nonempty open interval, and

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots \supseteq F_k \supseteq F_{k+1} \supseteq \dots.$$

But there is no real number which is in all of these intervals (Archimedean Property). So the intersection is empty. ♦

◇ **3.4-1.** Determine which of the following sets are path-connected:

(a) $\{x \in [0,1] \mid x \text{ is rational}\}$
(b) $\{(x,y) \in \mathbb{R}^2 \mid xy \geq 1 \text{ and } x > 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid xy \leq 1 \text{ and } x \leq 1\}$
(c) $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq z\} \cup \{(x,y,z) \mid x^2+y^2+z^2 > 3\}$
(d) $\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x < 1\} \cup \{(x,0) \mid 1 < x < 2\}$

Sketch. (a) Not path-connected. Any path between two rationals must contain an irrational.

(b) Path-connected.

(c) Path-connected.

(d) Not path-connected. If the point $(0,1)$ were added, it would be path-connected. ◇

Solution. (a) The set $A = \{x \in [0,1] \mid x \in \mathbb{Q}\}$ is not path-connected. This is probably intuitively clear since any path from 0 to 1 would have to cross through irrational points such as $1/\sqrt{2}$ which are not in A. Actually proving this is somewhat delicate. See Exercise 3.4-2.

(b) The set $B = \{(x,y) \in \mathbb{R}^2 \mid xy \geq 1 \text{ and } x > 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid xy \leq 1 \text{ and } x \leq 1\}$ is path-connected. It consists of two solid regions linked at the point $(1,1)$. Sketch it.

(c) The set $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq z\} \cup \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 > 3\}$ is path-connected. It is the set of points inside the paraboloid of revolution $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq z\}$ lying outside the sphere $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 3\}$. Since the paraboloid extends only above the xy-plane, $z \geq 0$, this solid set has only one piece and is path-connected.

(d) The set $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x < 1\} \cup \{(x,0) \in \mathbb{R}^2 \mid 1 < x < 2\}$ is not path-connected. The first term of the union is a vertical strip not containing any of its right boundary line, $\{(x,y) \mid x = 1\}$. The second term is the line segment along the x–axis from $(1,0)$ to $(2,0)$ but not including either end. The point $(1,0)$ is not in the union, but any plausible continuous path from a point in the strip to a point in the segment would have to go through it. ♦

◇ **3.5-2.** Is $\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1\} \cup \{(x,0) \mid 1 < x < 2\}$ connected? Prove or disprove.

Answer. Connected. ◇

Solution. Let $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1\}$ and $B = \{(x,0) \mid 1 < x < 2\}$. Set $C = A \cup B$. The set B is certainly path-connected, hence connected. If U and V were any pair of open sets which proposed to disconnect C, then B would have to be entirely contained within one of them or they would disconnect B. Say $B \subseteq V$. Now the set A is also path-connected. We can get from any point (x,y) in A to any other point (c,d) in A by proceeding horizontally from (x,y) to (x,d) and then vertically from (x,d) to (c,d). So A is also connected and must be completely contained within one of the sets U or V. If $A \subseteq V$, then we would have $C = A \cup B \subseteq V$, so the pair $\{U,V\}$ would not disconnect C. On the other hand, if $A \subseteq U$, then the point $(1,0)$ would be in U since it is in A. But then $(x,0)$ would be in U for x slightly larger than 1 since U is open. But these points are in B, so they are also in V. That is, for x slightly larger than 1 we would have $(x,0) \in U \cap V \cap C$. Once again, the sets U and V fail to disconnect C. We conclude that C must be connected. ◆

◇ **3.5-4.** Discuss the components of

(a) $[0,1] \cup [2,3] \subset \mathbb{R}$.
(b) $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} \subset \mathbb{R}$.
(c) $\{x \in [0,1] \mid x \text{ is rational}\} \subset \mathbb{R}$.

Answer. (a) The connected components are the sets $[0,1]$ and $[2,3]$.
(b) The connected components are the single point sets $\{n\}$ such that $n \in \mathbb{Z}$.
(c) The connected components are the single point sets $\{r\}$ such that $r \in \mathbb{Q} \cap [0,1]$. ◇

Solution. (a) Let $A = [0,1] \cup [2,3]$. Each of the intervals $[0,1]$ and $[2,3]$ is connected. But any larger subset of A cannot be connected. For example, if $[0,1] \subseteq D \subseteq A$, and there is a point $x \in D \cap [2,3]$, then the open sets $U = \{x \in \mathbb{R} \mid x < 3/2\}$ and $V = \{x \in \mathbb{R} \mid x > 3/2\}$ would disconnect D. Thus $[0,1]$ is a maximal connected subset of A and is one of its connected components. Similarly, if $[2,3] \subseteq D \subseteq A$ and there is a point $x \in D \cap [0,1]$, then the same two sets would disconnect D. So $[2,3]$ is a maximal connected subset of A and is a connected component of A. Thus the connected components of A are the intervals $[0,1]$ and $[2,3]$.

(b) Let $B = \mathbb{Z} \subseteq \mathbb{R}$. The single point sets $\{n\}$ with $n \in \mathbb{Z}$ are certainly connected. But, if $D \subseteq B$ has two different integers $n < m$ in it, then the open sets $U = \{x \in \mathbb{R} \mid x < n+(1/2)\}$ and $V = \{x \in \mathbb{R} \mid x > n+(1/2)\}$ would disconnect D. So no connected subset of B can contain more than one point. The connected components are the single point sets $\{n\}$ such that $n \in \mathbb{Z}$.

(c) Let $C = \{x \in [0,1] \mid x \in \mathbb{Q}\} \subseteq \mathbb{R}$. The single point sets $\{r\}$ with $r \in \mathbb{Q} \cap [0,1]$ are certainly connected. But, if $D \subseteq C$ has two different rational numbers $r < s$ in it, then there is an irrational number z with $r < z < s$. ($z = r + (s-r)/\sqrt{2}$ will do.) The open sets $U = \{x \in \mathbb{R} \mid x < z\}$ and $V = \{x \in \mathbb{R} \mid x > z\}$ would disconnect D. So no connected subset of C can contain more than one point. The connected components are the single point sets $\{r\}$ such that $r \in \mathbb{Q}$. ♦

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◇ **3E-1.** Which of the following sets are compact? Which are connected?

(a) $\{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \le 1\}$
(b) $\{x \in \mathbb{R}^n \mid ||x|| \le 10\}$
(c) $\{x \in \mathbb{R}^n \mid 1 \le ||x|| \le 2\}$
(d) $\mathbb{Z} = \{\text{integers in } \mathbb{R}\}$
(e) A finite set in $\mathbb{R}$
(f) $\{x \in \mathbb{R}^n \mid ||x|| = 1\}$ (distinguish between the cases $n = 1$ and $n \ge 2$)
(g) Perimeter of the unit square in $\mathbb{R}^2$
(h) The boundary of a bounded set in $\mathbb{R}$
(i) The rationals in $[0,1]$
(j) A closed set in $[0,1]$

Answer. (a) Connected, not compact.

(b) Compact and connected.
(c) $n = 1$: compact and not connected. $n \ge 2$: compact and connected.
(d) Neither compact nor connected.
(e) Compact, but not connected if it contains more than one point.
(f) $n = 1$: compact and not connected. $n \ge 2$: compact and connected.
(g) Compact and connected.
(h) Compact. Not necessarily connected.
(i) Neither compact nor connected.
(j) Compact. Not necessarily connected.

Solution. (a) The set $A = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq 1\}$ is an infinitely long vertical strip. It is closed since it contains its boundary lines, the verticals $x_1 = 1$ and $x_1 = -1$. It is not compact since it is not bounded. It contains the entire vertical x_2-axis, all points $(0, x_2)$ and the norm of such a point is $|x_2|$. This can be arbitrarily large.

(b) The set $B = \{x \in \mathbb{R}^n \mid \| x \| \leq 10\}$ is connected since it is path-connected. An easy way to establish this is to go from one point to another first by going in to the origin along a radius and then out to the second point along another radial path. It is bounded since its points have norm no larger than 10, and it is closed since it includes the boundary sphere where $\| x \| = 10$. It is a closed and bounded subset of $\mathbb{R}^n$, so it is compact by the Heine-Borel theorem.

(c) The set $C = \{x \in \mathbb{R}^n \mid 1 \leq \| x \| \leq 2\}$ is closed since it contains both boundary spheres, the points where $\| x \| = 1$ and those where $\| x \| = 2$. It is bounded since all of its points have norm no more than 2. It is a closed bounded subset of $\mathbb{R}^n$ and so is compact by the Heine-Borel Theorem. If $n \geq 2$, then it is path-connected. This is fairly obvious but not quite so easily implemented as in part (b). We cannot go all the way in to the origin. If x and y are in C, we can proceed from x in to the sphere of radius 1.5 along a radial line. Then we can go along a great circle on that sphere cut by the plane containing x, y, and the origin to the point on the same ray as y, then along that radial path to y. Thus C is path-connected and so connected if $n \geq 2$. If $n = 1$ it is not connected since it is the union of two intervals, $[-2, -1] \cup [1, 2]$.

(d) The set $D = \mathbb{Z} = \{\text{integers in } \mathbb{R}\}$ is not bounded (Archimedean Principle) so it is not compact. It is certainly not connected. The open sets $U = \{x \in \mathbb{R} \mid x < 1/2\}$ and $V = \{x \in \mathbb{R} \mid x > 1/2\}$ disconnect it.

(e) Suppose E is a finite set in $\mathbb{R}$. Say $E = \{x_1, x_2, \ldots, x_n\}$. If $\{U_\alpha\}_{\alpha \in A}$ is any open cover of E, then for each k there is an index α_k such that $x_k \in U_{\alpha_k}$. So $E \subseteq \bigcup_{k=1}^n U_{\alpha_k}$. Every open cover of E has a finite subcover. So E is compact. If E has only one point then it is certainly connected. If it has more than one, then the open sets $U = \{x \in \mathbb{R} \mid x < (x_1 + x_2)/2\}$ and $V = \{x \in \mathbb{R} \mid x > (x_1 + x_2)/2\}$ disconnect it.

(f) The set $F = \{x \in \mathbb{R}^n \mid \| x \| = 1\}$ is closed and bounded in $\mathbb{R}^n$ and so compact by the Heine-Borel theorem. If $n = 1$, then $F = \{-1, 1\}$ and is not connected. If $n \geq 2$, then we can proceed from any point x in F to any other point y in F along the great circle cut in F by the plane containing x, y and the origin. So, if $n \geq 2$, then F is path-connected and so connected.

(g) Let G be the perimeter of the unit square in $\mathbb{R}^2$. Then G is a closed and bounded subset of $\mathbb{R}^2$ and so is compact by the Heine-Borel Theorem. It is certainly path-connected. It is built as the union of four continuous straight line paths which intersect at the corners. So it is connected.

(h) Let S be a bounded set in $\mathbb{R}$ and let $H = \text{bd}(S)$. Then H is a closed set since it is the intersection of the closure of S with the closure of its complement. Since S is bounded, its closure is also bounded. (There is a constant r such that $|x| \leq r$ for all x in S. If $y \in \text{cl}(S)$, then there is an x in S with $|x - y| < 1$. Then $|y| \leq |y - x| + |x| < 1 + r$. So $|y| \leq r + 1$ for all y in $\text{cl}(S)$.) Since $H \subseteq \text{cl}(S)$, it is also bounded. Since H is a closed bounded subset of $\mathbb{R}^1$, it is compact by the Heine-Borel Theorem. The set $H = \text{bd}(S)$ is most likely not connected. For example, if $S = [0, 1]$, then H is the two point set $\{0, 1\}$, which is not connected. (Can you figure out for which sets S we do have $\text{bd}(S)$ connected?)

(i) Let $I = \mathbb{Q} \cap [0, 1]$. Then I is not closed. Its closure is $[0, 1]$. So it is not compact. It is not connected since the open sets $U = \{x \in \mathbb{R} \mid x < 1/\sqrt{2}\}$ and $V = \{x \in \mathbb{R} \mid x > 1/\sqrt{2}\}$ disconnect it.

(j) Let J be a closed set in $[0, 1]$. Then J is given as closed, and it is certainly bounded since $|x| \leq 1$ for every x in J. Since it is a closed bounded set in $\mathbb{R}^1$, J is compact by the Heine-Borel Theorem. It will be connected if and only if it is a closed interval. If x and y are in J and there is a point z with $x < z < y$, then the open sets $U = \{x \in \mathbb{R} \mid x < z\}$ and $V = \{x \in \mathbb{R} \mid x > z\}$ disconnect J. ♦

◇ **3E-2.** Prove that a set $A \subset \mathbb{R}^n$ is not connected iff we can write $A \subset F_1 \cup F_2$, where F_1, F_2 are closed, $A \cap F_1 \cap F_2 = \varnothing$, $F_1 \cap A \neq \varnothing$, $F_2 \cap A \neq \varnothing$.

Suggestion. Suppose U_1 and U_2 disconnect A and consider the sets $F_1 = M \setminus U_1$ and $F_2 = M \setminus U_2$. ◇

Solution. By definition, A is not connected if and only if there are open sets U_1 and U_2 such that

1. $U_1 \cap U_2 \cap A = \emptyset$

2. $U_1 \cap A$ is not empty
3. $U_2 \cap A$ is not empty
4. $A \subseteq U_1 \cup U_2$.

But U_1 and U_2 are open if and only if their complements $F_1 = M \setminus U_1$ and $F_2 = \setminus U_2$ are closed. Using DeMorgan's Laws, our four conditions translate to

1. $F_1 \cup F_2 \cup (M \setminus A) = M$
2. $F_1 \cup A$ is not all of M
3. $F_2 \cup A$ is not all of M
4. $M \setminus A \supseteq F_1 \cap F_2$.

Since none of the points in A are in $M \setminus A$, condition 1 is equivalent to $A \subseteq F_1 \cup F_2$. Condition 4 is equivalent to $A \cap F_1 \cap F_2 = \emptyset$. So, with a bit more manipulations, our conditions become

1. $A \subseteq F_1 \cup F_2$
2. $F_1 \cap A = (M \setminus U_1) \cap A = (M \setminus U_1) \cap (M \setminus (M \setminus A)) = M \setminus (U_1 \cup (M \setminus A))$. This is not empty since there are points in $U_2 \cap A$ and these cannot be in U_1.
3. $F_2 \cap A = (M \setminus U_2) \cap A = (M \setminus U_2) \cap (M \setminus (M \setminus A)) = M \setminus (U_2 \cup (M \setminus A))$. This is not empty since there are points in $U_1 \cap A$ and these cannot be in U_2.
4. $A \cap F_1 \cap F_2 = \emptyset$.

These are exactly the conditions required of the closed sets F_1 and F_2 in the problem.

In the converse direction, if we have closed sets F_1 and F_2 satisfying

1. $A \subset F_1 \cup F_2$
2. $F_1 \cap A$ not empty
3. $F_2 \cap A$ not empty
4. $A \cap F_1 \cap F_2 = \emptyset$,

we can consider the open sets $U_1 = M \setminus F_1$ and $U_2 = M \setminus F_2$ and work backwards through these manipulations to obtain

1. $U_1 \cap U_2 \cap A = \emptyset$
2. $U_1 \cap A$ is not empty
3. $U_2 \cap A$ is not empty
4. $A \subseteq U_1 \cup U_2$,

so that A is not connected. ♦

◇ **3E-5.** Show that the following sets are not compact, by exhibiting an open cover with no finite subcover.

(a) $\{x \in \mathbb{R}^n \mid ||x|| < 1\}$
(b) $\mathbb{Z}$, the integers in $\mathbb{R}$

Solution. (a) Suppose $A = \{x \in \mathbb{R}^n \mid \| x \| < 1\}$. This set is not closed, so it should not be compact. For each integer $k > 0$, let $U_k = \{x \in \mathbb{R}^2 \mid \| x \| < k/(k+1)\}$. Then the sets U_k are open balls and are contained in A. Furthermore, $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots \subseteq U_k \subseteq U_{k+1} \subseteq \ldots$. If $x \in A$, then there is an integer k such that $0 \le \| x \| < k/(k+1) < 1$. So $x \in U_k$. Thus $A = \bigcup_{k=1}^{\infty} U_k$. However, the union of any finite subcollection of this open cover is contained in $\bigcup_{k=1}^{N} U_k = U_N$ for some N. If we put $r = (N/(N+1)) + (1 - (N/(N+1)))/2$, then $N/(N+1) < r < 1$. The point $x = (r, 0, \ldots, 0) \in A \setminus U_N$. So no finite subcollection can cover A. This open cover has no finite subcover. The set A is not compact.

(b) Let $B = \mathbb{Z}$, the integers in $\mathbb{R}$. For each integer k, let U_k be the open interval $=]k - (1/3), k + (1/3)[$. Then each U_k is an open subset of $\mathbb{R}$, and $B \cap U_k = \{k\}$. So the infinite collection $\{U_k\}_{k\in\mathbb{Z}}$ is an open cover of $B = \mathbb{Z} \subseteq \mathbb{R}$. There can be no finite subcover since if U_n is deleted from the collection, then the integer n is no longer included in the union. ♦

◇ **3E-7.** Let x_k be a sequence in $\mathbb{R}^n$ that converges to x and let $A_k = \{x_k, x_{k+1}, \ldots\}$. Show that $\{x\} = \cap_{k=1}^{\infty} \mathrm{cl}(A_k)$. Is this true in *any* metric space?

Sketch. Start with $\mathrm{cl}(A_k) = \{x\} \cup \{x_k, x_{k+1}, \ldots\}$. Remember to show that if y and x are different, then $y \notin \mathrm{cl}(A_k)$ for large k. (No such x_k can be close to y since they are close to x.) Give detail. ◇

Solution. One can proceed directly as in the sketch or one can employ the ideas of Exercise 3E-6. The basic idea is essentially the same. Let $F_k = \mathrm{cl}(A_k)$. From Proposition 2.7.6(ii), we know that $x \in \mathrm{cl}(A_k)$. From Exercise 2.7-2 we know that $\{x\} \cup A_k$ is closed. So $F_k = \mathrm{cl}(A_k) = \{x\} \cup A_k$. From Exercise 3.1-4 (or directly) we know that $F_k = \{x\} \cup A_k$ is compact.

Since $F_{k+1} = \{x\} \cup \{x_{k+1}, x_{k+2}, \ldots\} \subseteq \{x\} \cup \{x_k, x_{k+1}, x_{k+2}, \ldots\} = F_k$, they are nested. From the nested set property, there must be at least one point in their intersection. But we already know that since x is certainly in the intersection.

Let $\varepsilon > 0$. Then there is a K such that $d(x, x_k) < \varepsilon/2$ whenever $k \ge K$. So, if z and w are in F_k for such a k, each of them must be equal either to x or do some x_j with $j \ge K$. So $d(z, w) \le d(z, x) + d(x, w) < \varepsilon$. So diameter$(F_k) \to 0$ as $k \to \infty$. As in Exercise 3E-6, this implies that there can be no more than one point in the intersection. (The distance between two points in the intersection would have to be 0.) So $\{x\} = \bigcap_{k=1}^{\infty} \mathrm{cl}(A_k)$ as claimed. All the steps work in any metric space. So the assertion is true in every metric space. ♦

◇ **3E-9.** Determine (by proof or counterexample) the truth or falsity of the following statements:

(a) (A is compact in $\mathbb{R}^n$) $\Rightarrow$ ($\mathbb{R}^n \backslash A$ is connected).
(b) (A is connected in $\mathbb{R}^n$) $\Rightarrow$ ($\mathbb{R}^n \backslash A$ is connected).

(c) (A is connected in $\mathbb{R}^n$) $\Rightarrow$ (A is open or closed).

(d) ($A = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$) $\Rightarrow$ ($\mathbb{R}^n \backslash A$ is connected). [Hint: Check the cases $n = 1$ and $n \geq 2$.]

Answer. (a) False; $[0, 1]$ is compact, but $\mathbb{R} \setminus [0, 1]$ is not connected. In $\mathbb{R}^n$, $A = \{x \in \mathbb{R}^n \mid 1 \leq \| x \| \leq 2\}$ is compact, but $\mathbb{R}^n \setminus A$ is not connected.

(b) False; same examples as in (a).

(c) False; $]a, b]$ is connected but is neither open nor closed.

(d) False for $n = 1$, true for $n \geq 2$. ($\mathbb{R}^n \setminus A$ is path-connected if $n \geq 2$.) ♢

Solution. (a) Given above.

(b) Given above.

(c) Given above.

(d) If $n = 1$, then A is the closed interval $[-1, 1]$, and its complement $\mathbb{R} \setminus A$, is the union of two disjoint open sets, $]-\infty, -1[\cup]1, \infty[$ and is not connected. In more than one dimension, A is the exterior of the unit ball and is path-connected. Connect each point to the point on the same ray from the origin on the sphere of radius 1. Then connect any two points on this sphere by following a great circle route (a circle containing those two points and the origin). ♦

◇ **3E-17.** Let K be a nonempty closed set in $\mathbb{R}^n$ and $x \in \mathbb{R}^n \backslash K$. Prove that there is a $y \in K$ such that $d(x, y) = \inf\{d(x, z) \mid z \in K\}$. Is this true for open sets? Is it true in general metric spaces?

Sketch. As in Worked Example 1WE-2, get $z_k \in K$ with $d(x, z_k) \to d(z, K) = \inf\{d(x, z) \mid z \in K\}$. For large k they are all in the closed ball of radius $1 + d(z, K)$ around x. Use compactness to get a subsequence converging to some z. Then $z \in K$ (why?) and $d(z_{n(j)}, x) \to d(z, x)$. (Why?) So $d(x, z) = d(x, K)$. (Why?) This does not work for open sets. The proof does not work unless closed balls are compact. ♢

Solution. Let $x \in \mathbb{R}^n \setminus K$ and $S = \{d(x,z) \in \mathbb{R} \mid z \in K\}$. Since K is not empty, neither is S, and S is certainly bounded below by 0. So $a = \inf S$ exists as a nonnegative real number. There must be a sequence $\langle t_k \rangle_1^\infty$ of points in S with $t_k \to a$, and thus a sequence $\langle z_k \rangle_1^\infty$ of points in K with $d(x, z_k) = t_k \to a$. There is an N such that $d(x, z_k) < a + 1$ whenever $k \geq N$. So for these k we have $z_k \in F = \{z \mid d(x,z) \leq a+1\}$. The set F is a closed bounded set in $\mathbb{R}^n$ and so is compact. The sequence $z_K, z_{K+1}, z_{K+2}, \dots$ in F must have a subsequence converging to a point in F. Thus there are indices $k(1) < k(2) < k(3) < \dots$ and a point $z \in F$ with $z_{k(j)} \to z$ as $j \to \infty$. Since each z_k is in K and K is closed, we have $z \in K$.

From the triangle inequality we know that $d(z_{k(j)}, x) \leq d(z_{k(j)}, z) + d(z,x)$ and that $d(z,x) \leq d(z, z_{k(j)}) + d(z_{k(j)}, x)$. So

$$d(z,x) - d(z, z_{k(j)}) \leq d(z_{k(j)}, x) \leq d(z_{k(j)}, z) + d(z,x).$$

Since $d(z, z_{k(j)}) \to 0$, we conclude (Sandwich Lemma) that $d(z_{k(j)}, x) \to d(z,x)$. But, we know that $d(z_{k(j)}, x) \to a$. Since limits are unique, we must have $d(x,z) = a = d(x,K)$ as desired.

This certainly does not work for open sets. If $K = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ is the open unit disk in $\mathbb{R}^2$ and $v = (1,0)$, then $\inf\{d(v,w) \mid w \in K\} = 0$, but there certainly is no point w in K with $d(v,w) = 0$ since v is not in K.

The proof just given used the fact that closed bounded sets in $\mathbb{R}^n$ are sequentially compact. This is not true in every metric space, and, in fact, there are complete metric spaces in which the assertion is false.

We will see in Chapter 5 that the space $\mathcal{V} = \mathcal{C}([0,1], \mathbb{R})$ of all continuous real valued functions on the closed unit interval with the norm $\| f \| = \sup\{|f(x)| \mid x \in [0,1]\}$ is a complete metric space. For $k = 1, 2, 3, \dots$, let f_k be the function whose graph is sketched in Figure 3-2. Then $\| f_k \| = 1 + (1/k)$, and, if n and k are different, then each is 0 wherever the other is nonzero. So $\| f_n - f_k \| > 1$. So the f_k are all smaller than 2 in norm and form a bounded set. Furthermore the set $K = \{f_1, f_2, f_3 \dots\}$ can have no accumulation points and so is a closed bounded set. $\| f_k - 0 \| \to 1$ as $k \to \infty$, but the distance is not equal to 1 for any function in the set.

♦

◇ **3E-19.** Let $V_n \subset M$ be open sets such that $\mathrm{cl}(V_n)$ is compact, $V_n \neq \varnothing$, and $\mathrm{cl}(V_n) \subset V_{n-1}$. Prove $\cap_{n=1}^\infty V_n \neq \varnothing$.

Sketch. $\mathrm{cl}(V_{n+1}) \subseteq V_n \subseteq \mathrm{cl}(V_n)$. Use the nested set property. ◇

Solution. Let $K_n = \mathrm{cl}(V_n)$, we have assumed that the sets K_n are compact, not empty, and that $\mathrm{cl}(V_k) \subseteq V_{k-1}$ for each k. Applying this with $k = n+1$ gives

$$K_{n+1} = \mathrm{cl}(V_{n+1}) \subseteq V_n \subseteq \mathrm{cl}(V_n) = K_n.$$

So we have a nested sequence of nonempty compact sets. By the nested set property, there must be at least one point x_0 in the intersection $\bigcap_1^\infty K_n$. For each n we have $x_0 \in K_n - 1 \subseteq V_n$. Thus $x_0 \in \bigcap_1^\infty V_n$, and this intersection is not empty. ♦

◇ **3E-20.** Prove that a compact subset of a metric space must be closed as follows: Let x be in the complement of A. For each $y \in A$, choose disjoint neighborhoods U_y of y and V_y of x. Consider the open cover $\{U_y\}_{y \in A}$ of A to show the complement of A is open.

Solution. Suppose A is a compact subset of a metric space M, and suppose $x \in M \setminus A$. If $y \in A$, then let $r = d(x, y)/2 > 0$, and set $U_y = D(y, r)$ and $V_y = D(x, r)$. Then U_y and V_y are disjoint open sets with $y \in U_y$ and $x \in V_y$. We certainly have $A \subseteq \bigcup_{y \in A} U_y$. Since A is compact, this open cover must have a finite subcover:

$$A \subseteq U_{y_1} \cup U_{y_2} \cup \cdots \cup U_{y_N}.$$

If $\rho = \min\{d(x, y_k) \mid 1 \le k \le N\}$, then

$$D(x, \rho) \cap A \subseteq D(x, \rho) \cap (U_{y_1} \cup U_{y_2} \cup \cdots \cup U_{y_N}) = \bigcup_{k=1}^{N} (D(x, \rho) \cap U_{y_k}) = \emptyset.$$

Thus $D(x, \rho) \subseteq M \setminus A$. This shows that $M \setminus A$ is open. so A is closed as claimed. ♦

◇ **3E-23.** Let $\mathbb{Q}$ denote the rationals in $\mathbb{R}$. Show that both $\mathbb{Q}$ and the irrationals $\mathbb{R} \backslash \mathbb{Q}$ are not connected.

Sketch. $\mathbb{Q} \subset]-\infty, \sqrt{2}[\cup [\sqrt{2}, \infty[$; both intervals are open, they are disjoint. They disconnect $\mathbb{Q}$. Similarly $\mathbb{R} \backslash \mathbb{Q} \subset]-\infty, 0[\cup]0, \infty[$ disconnects $\mathbb{R} \setminus \mathbb{Q}$. ◊

Solution. To show that $\mathbb{Q} \subseteq \mathbb{R}$ is not connected, recall that $\sqrt{2}$ is not rational. The two open half lines $U = \{x \in \mathbb{R} \mid x < \sqrt{2}\}$ and $V = \{x \in \mathbb{R} \mid x > \sqrt{2}\}$ are disjoint. Each intersects $\mathbb{Q}$ since $0 \in U$ and $3 \in V$. Their union is $\mathbb{R} \setminus \{\sqrt{2}\}$ which contains $\mathbb{Q}$. Thus U and V disconnect $\mathbb{Q}$.

To show that $\mathbb{R} \setminus \mathbb{Q}$ is not connected, we do essentially the same thing but use a rational point such as 0 as the separation point. Let $U = \{x \in \mathbb{R} \mid x < 0\}$ and $V = \{x \in \mathbb{R} \mid x > 0\}$. Then U and V are disjoint open half lines. Each intersects $\mathbb{R} \setminus \mathbb{Q}$ since $-\sqrt{2} \in U$ and $\sqrt{2} \in V$. Their union is $\mathbb{R} \setminus \{0\}$ which contains $\mathbb{R} \setminus \mathbb{Q}$. Thus U and V disconnect $\mathbb{R} \setminus \mathbb{Q}$. ♦

◇ **3E-24.** Prove that a set $A \subset M$ is not connected if we can write A as the disjoint union of two sets B and C such that $B \cap A \neq \varnothing$, $C \cap A \neq \varnothing$, and neither of the sets B or C has a point of accumulation belonging to the other set.

Suggestion. Show $\text{cl}(B) \cap C$ and $\text{cl}(C) \cap B$ are empty. Show that the complements of these closures disconnect A. ◊

Solution. Suppose A is a subset of a metric space M with nonempty disjoint subsets B and C such that neither B nor C has an accumulation point in the other. Then $\text{cl}(B) \cap C = \emptyset$ since a point is in the closure of B if and only if it is either in B or is an accumulation point of B. None of these are in C. Similarly, $\text{cl}(C) \cap B = \emptyset$. Let $U = M \setminus \text{cl}(B)$ and $V = M \setminus \text{cl}(C)$. Then U and V are open since they are the complements of the closed sets $\mathrm{cl}(B)$ and $\mathrm{cl}(C)$. Since $B \cap \mathrm{cl}(C) = \emptyset$, we have $B \subseteq V$, and since $C \cap \mathrm{cl}(B) = \emptyset$, we have $C \subseteq U$. In fact:

(1) $U \cap V \cap A = \emptyset$: To see this compute

$$\begin{aligned} U \cap V &= (M \setminus \mathrm{cl}(B)) \cap (M \setminus \mathrm{cl}(C)) = M \setminus (\mathrm{cl}(B) \cup \mathrm{cl}(C)) \\ &\subseteq M \setminus (B \cup C) = M \setminus A. \end{aligned}$$

(2) $B \subseteq V$, so $A \cap V$ is not empty.
(3) $C \subseteq U$, so $A \cap U$ is not empty.
(4) $A = B \cup C \subseteq U \cup V$.

So the open sets U and V disconnect A.
For a related result, see Exercise 3E-21. ♦

◇ **3E-27.** Let $A \subset \mathbb{R}$ be a bounded set. Show that A is closed iff for every sequence $x_n \in A$, $\limsup x_n \in A$ and $\liminf x_n \in A$.

Sketch. If $x \in \mathrm{cl}(A)$, there is a sequence $\langle x_n \rangle_1^\infty$ in A converging to x. If the condition holds, then $x \in A$. (Why?) So $\mathrm{cl}(A) \subseteq A$, and A is closed. For the converse, if A is closed and bounded, the lim inf and lim sup of a sequence in A are the limits of subsequences, so they are in A. ◇

Solution. First suppose A is a bounded set in $\mathbb{R}$ and that $\limsup x_n$ and $\liminf x_n$ are in A for every sequence $\langle x_n \rangle_1^\infty$ of points in A. If $x \in \mathrm{cl}(A)$, then there is a sequence $\langle x_n \rangle_1^\infty$ of points in A converging to x. Since the limit exists, we have

$$x = \lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

By hypothesis, this is in A. This is true for every x in $\mathrm{cl}(A)$. So $\mathrm{cl}(A) \subseteq A$, and A is closed.

For the converse, suppose A is a closed, bounded set in $\mathbb{R}$ and that $\langle x_n \rangle_1^\infty$ is a sequence in A. Since A is a bounded set, the sequence is bounded and $a = \liminf x_n$ and $b = \limsup x_n$ exist as finite real numbers. Furthermore, there are subsequences $x_{n(j)}$ and $x_{k(j)}$ converging to a and b respectively. Since A is closed, the limits of these subsequences must be in A. ♦

◇ **3E-30.** Let U_k be a sequence of open bounded sets in $\mathbb{R}^n$. Prove or disprove:

(a) $\bigcup_{k=1}^{\infty} U_k$ is open.
(b) $\bigcap_{k=1}^{\infty} U_k$ is open.
(c) $\bigcap_{k=1}^{\infty} (\mathbb{R}^n \backslash U_k)$ is closed.
(d) $\bigcap_{k=1}^{\infty} (\mathbb{R}^n \backslash U_k)$ is compact.

Answer. (a) Yes.
(b) Not necessarily.
(c) Yes.
(d) Not necessarily. ◇

Solution. (a) The union of any collection of open subsets of $\mathbb{R}^n$ is open, so in particular $\bigcup_{k=1}^{\infty} U_k$ is open.

(b) The intersection of an infinite collection of open sets need not be open. Boundedness does not help. Let $U_k = \{v \in \mathbb{R}^n \mid \| v \| < 1/k\}$. Each of the sets U_k is open and bounded. But $\bigcap_{k=1}^{\infty} U_k = \{0\}$ which is not open.

(c) The sets $\mathbb{R}^n \backslash U_k$ are closed since the U_k are open. The intersection of any family of closed subsets of $\mathbb{R}^n$ is closed, so in particular $\bigcap_{k=1}^{\infty} (\mathbb{R}^n \setminus U_k)$ is closed.

(d) As in part (c) the intersection is closed, but it need not be bounded. Let U_k be the sets defined in part (b). Then $U_1 \supseteq U_2 \supseteq U_3 \supseteq U_4 \supseteq \ldots$. So

$$(\mathbb{R}^n \setminus U_1) \subseteq (\mathbb{R}^n \setminus U_2) \subseteq (\mathbb{R}^n \setminus U_3) \subseteq \ldots.$$

Thus

$$\bigcap_{k=1}^{\infty} (\mathbb{R}^n \setminus U_k) = \mathbb{R}^n \setminus U_1 = \{v \in \mathbb{R}^2 \mid \| v \| \geq 1\}.$$

This set is not bounded and so is not compact. ♦

◇ **3E-35.** Let $a \in \mathbb{R}$ and define the sequence $a_1, a_2, \ldots$ in $\mathbb{R}$ by $a_1 = a$, and $a_n = a_{n-1}^2 - a_{n-1} + 1$ if $n > 1$. For what $a \in \mathbb{R}$ is the sequence

(a) Monotone?
(b) Bounded?
(c) Convergent?

Compute the limit in the cases of convergence.

Answer. (a) All a. If $a = 0$ or 1, the sequence is constant.

(b) $0 \le a \le 1$.

(c) $0 \le a \le 1$. ◊

Solution. Let $f(x) = x^2 - x + 1$. Our sequence is defined by $a_1 = a$ and $a_{n+1} = f(a_n)$ for $n = 1, 2, 3, ...$ The graph of $y = f(x)$ is a parabola opening upward. Its vertex is at the point $x = 1/2$, $y = 3/4$.

(a) For each $n \ge 1$ we have

$$a_{n+1} - a_n = a_n^2 - 2a_n + 1 = (a_n - 1)^2 \ge 0.$$

So the sequence is monotonically increasing (or at least nondecreasing), no matter what the starting point is. If $a = 1$, then the differences are always 0 and the sequence is constant. If $a = 0$, then $a_1 = 1$, and the sequence is constant beyond that point.

$a_{n+1} = a_n$ if and only if $a_n = 1$, so the sequence is strictly increasing unless there is some n with $a_n = 1$. But $a_n = 1$ if and only if a_{n-1} is either 1 or 0, and 0 has no possible predecessor since the equation $x^2 - x + 1 = 0$ has no real root. Thus the sequence is increasing for all a and strictly increasing unless $a = 0$ or $a = 1$.

(b) The function $f(x) = x^2 - x + 1$ defining our sequence has an absolute minimum value of 3/4 occurring at $x = 1/2$. Also $f(0) = f(1) = 1$, and if $0 \le x \le 1$, then $3/4 \le f(x) \le 1$. If $0 \le a_k \le 1$, this shows that $0 \le f(a_k) = a_{k+1} \le 1$. If $0 \le a = a_1 \le 1$, then it follows by induction that $0 \le a_n \le 1$ for all n, and the sequence is bounded.

If $a_k > 1$, then $a_m > a_k > 1$ for all $m > k$ by part (a). Furthermore,

$$\begin{aligned}
(a_{m+2} - a_{m+1}) - (a_{m+1} - a_m) &= (a_{m+1} - 1)^2 - (a_m - 1)^2 \\
&= (a_{m+1}^2 - 2a_{m+1} + 1) - (a_m - 2a_m + 1) \\
&= a_{m+1}^2 - a_m^2 - 2(a_{m+1} - a_m) \\
&= (a_{m+1} + a_m - 2)(a_{m+1} - a_m) \\
&\ge 0.
\end{aligned}$$

So the differences between successive terms is increasing. The terms must diverge to $+\infty$.

If $a = a_1 > 1$, then this analysis applies directly and the sequence must diverge to $+\infty$.

If $a = a_1 < 0$, then $a_2 = f(a_1) = f(a) = a^2 - a + 1 = a^2 + |a| + 1 > 1$. So the analysis still applies and the sequence must diverge to $+\infty$.

Combining these observations, we see that the sequence is bounded if and only if $0 \le a \le 1$.

(c) From parts (a) and (b), we know that if $0 \le a \le 1$, then the sequence is a bounded monotone sequence in $\mathbb{R}$ and must converge to some $\lambda \in \mathbb{R}$. From part (b), we know that if a is not in this interval, then the sequence diverges to $+\infty$. Thus the sequence converges if and only if $0 \le a \le 1$. If the limit λ exists, then $a_{n+1} \to \lambda$ and $a_n^2 - a_n + 1 \to \lambda^2 - \lambda + 1$. Since $a_{n+1} = a_n^2 - a_n + 1$ and limits are unique, we must have $\lambda = \lambda^2 - \lambda - 1$. So $0 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. Thus we must have $\lim_{n\to\infty} a_n = \lambda = 1$.

The action of f on a to produce the sequence $\langle a_n \rangle_1^\infty$ can be very effectively illustrated in terms of the graph of the function $y = f(x)$. Starting with a_k on the x-axis, the point $(a_k, f(a_k))$ is located on the graph of f. The point a_{k+1} is then located on the x-axis by moving horizontally to the line $y = x$, and then vertically to the x-axis. The fixed point 1 occurs when the graph of f touches the line $y = x$. Repetition produces a visual representation of the behavior of the sequence. See the figure.

(Sorry, no figure is provided. See if you can reproduce it on your own).

The iteration of the function f to produce the sequence $\langle a_n \rangle_1^\infty$ from the starting point a is an example of a *discrete dynamical system*. The sequence obtained is called the *orbit* of the point. The study of the behavior of sequences obtained from the iteration of functions is a rich field with many applications in mathematics and other fields. It can lead to nice clean behavior such as we have seen here or to much more complicated behavior now characterized by the term "chaos". A couple of nice references are:

An Introduction to Chaotic Dynamical Systems, Robert L. Devaney, Addison-Wesley Publishing Company.

Encounters with Chaos, Denny Gulick, McGraw-Hill, Inc. ♦

◇ **3E-37.** Let $A, B \subset M$ with A compact, B closed, and $A \cap B = \varnothing$.

(a) Show that there is an $\varepsilon > 0$ such that $d(x, y) > \varepsilon$ for all $x \in A$ and $y \in B$.

(b) Is (a) true if A, B are merely closed?

Sketch. (a) For each $x \in A$ there is a δ_x such that $D(x, \delta_x) \subseteq M \setminus B$; apply compactness to the covering $\{D(x, \delta_x/2) \mid x \in A\}$.

(b) No; let A be the y-axis and B be the graph of $y = 1/x$. ◇

Solution. (a) If either A or B is empty, then the assertion is vacuously true for any $\varepsilon > 0$. Otherwise for $x \in A$, let $\Delta_x = \inf\{d(x, y) \mid y \in B\}$. Then there is a sequence of points $y_1, y_2, y_3, \ldots$ in B with $d(x, y_k) \to \Delta_x$. If $\Delta_x = 0$, this would say that $y_k \to x$. Since B is closed, this would say that $x \in B$. But it is not since $A \cap B = \emptyset$. Thus $\Delta_x > 0$. We have

$d(y,x) \geq \Delta_x$ for every $y \in B$. If we pick any δ_x with $0 < \delta_x < \Delta_x$, then we have $d(y,x) > \delta_x$ for every $y \in B$ and $x \in D(x, \delta_x) \subseteq M \setminus B$. So

$$A \subseteq \bigcup_{x \in A} D\left(x, \frac{\delta_x}{2}\right) \subseteq M \setminus B.$$

This open cover of the compact set A must have a finite subcover, so there are points x_1, x_2, x_3, … a_N in A such that

$$A \subseteq D\left(x_1, \frac{\delta_{x_1}}{2}\right) \cup D\left(x_1, \frac{\delta_{x_1}}{2}\right) \cup \cdots \cup D\left(x_N, \frac{\delta_{x_N}}{2}\right) \subseteq M \setminus B.$$

Let

$$\varepsilon = \min\left\{\frac{\delta_1}{2}, \frac{\delta_2}{2}, \ldots, \frac{\delta_N}{2}\right\}$$

and suppose $x \in A$ and $y \in B$. Then there is an index j with $1 \leq j \leq N$ and $x \in D(x_j, \delta_{x_j}/2) \subseteq M \setminus B$. So $d(x_j, y) \leq d(x_j, x) + d(x, y)$. This gives

$$d(x,y) \geq d(x_j,y) - d(x_j,x) > \delta_j - d(x_j,x) > \delta_j - \frac{\delta_j}{2} = \frac{\delta_j}{2} > \varepsilon.$$

So $d(x,y) > \varepsilon$ for all x in A and y in B as required.

(b) This is not true if we only assume that both A and B are closed. Let $A = \{(0,y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$ and $B = \{(x, 1/x) \in \mathbb{R}^2 \mid x > 0\}$. Then A is a straight line (the y-axis), and B is one branch of a hyperbola (the graph of $y = 1/x$ for $x > 0$). Each of these sets is closed and their intersection is empty. But $d((0,1/x),(x,1/x)) = x \to 0$ as $x \to 0$. So no such ε can exist for these sets. ♦