Exercises for §7.1

3 Let $f(x) = x + 2x^2\sin(\frac{1}{x})$, $x \neq 0$, $f(0) = 0$. Show that $f'(0) \neq 0$ but that f is not locally invertible near 0. Why does this not contradict the inverse function theorem?

Pf: $\frac{df}{dx} = 1 + 4x\sin(\frac{1}{x}) + 2x^2\cos\frac{1}{x}(-1)\frac{1}{x^2}$

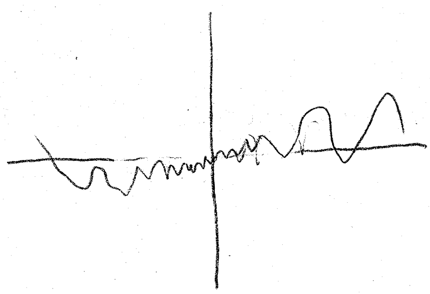
$= 1 + 4x\sin(\frac{1}{x}) - 2\cos\frac{1}{x}$

$\lim_{x\to 0}\frac{df}{dx}$ 不存在

$$\frac{df}{dx}(0) = \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0} 1 + 2x\sin(\tfrac{1}{x}) = 1$$

$\Rightarrow f'$ is not continuous at 0

f is not locally invertible near 0 since it is not one to one in any neighborhood of 0



4 Let $L: \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism and $f(x) = L(x) + g(x)$, where $\|g(x)\| \leq M\|x\|^2$ and f is C^1. Show f is locally invertible near 0

Pf: By exercises for §6.2

$Df(0) = L$ $\because$ L be a linear isomorphism

$\det(L) \neq 0$

By inverse function theorem f is locally invertible near 0

Exercises for §7.2

3 In the system

$$3x+2y+z^2+u+v^2=0$$
$$4x+3y+z+u^2+v+w+2=0$$
$$x+z+w+u^2+2=0$$

discuss the solvability for $u.v.w$ in terms of $x.y.z$ near $x=y=z=0$ $u=v=0$ $w=-2$

Ans: Let $F_1=3x+2y+z^2+u+v^2$

$F_2=4x+3y+z+v+w+2$

$F_3=x+z+w+u^2+2$

$$\left.\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}\right|_{(0.0.0.0.0,-2)}=\left[\begin{matrix}\frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial w}\\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial w}\\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} & \frac{\partial F_3}{\partial w}\end{matrix}\right]_{(0.0.0.0.0-2)}$$

$$=\left[\begin{matrix}1 & 2v & 0\\ 0 & 1 & 1\\ 2u & 0 & 1\end{matrix}\right]$$

$$\det\left[\begin{matrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{matrix}\right]=1\neq 0$$

So $u.v.w$ can be expressed in terms of $x.y.z$ for $(x.y.z)$ is some neighborhood of $(0.0.0)$

5 Discuss the solvability of

$$y+x+uv=0$$
$$uxy+v=0$$

for u, v in terms of $x\ y$ near $x=y=u=v=0$, and check directly

Ans Let $F_1 = y+x+uv$

$F_2 = uxy+v$

$$\begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{bmatrix}_{(0,0,0,0)} = \begin{bmatrix} v & u \\ xy & 1 \end{bmatrix}_{(0,0,0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

The implicit function theorem does not guarantee local solvability