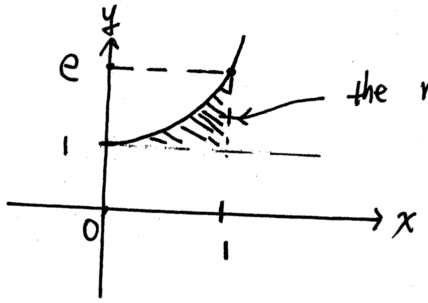
Exercise 9.2.

\# 2.



the region corresponding to the integral $\int_0^1\int_0^{e^x}(x+y)\,dy\,dx$

$$\int_0^1\int_1^{e^x}(x+y)\,dy\,dx = \int_0^1\left(\int_1^{e^x}(x+y)\,dy\right)dx$$

$$= \int_0^1 \left[\, xy + \frac{y^2}{2}\,\right]_1^{e^x} dx$$

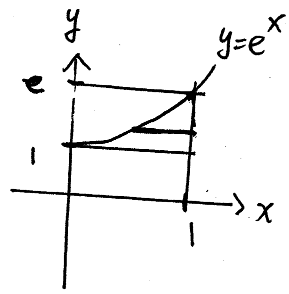
$$= \int_0^1 xe^x + \frac{1}{2}e^{2x} - x - \frac{1}{2}\,dx$$

$$= \int_0^1 xe^x dx + \frac{1}{2}\int_0^1 e^{2x}dx - 1$$

$$= 1 + \frac{1}{4}(e^2-1) - 1$$

$$= \frac{1}{4}(e^2-1) \quad ※$$

\# 3.



$$\int_0^1\int_1^{e^x}(x+y)\,dy\,dx \underset{\text{Fubini}}{=} \int_1^e\int_{\log y}^1 (x+y)\,dx\,dy$$

$$= \int_1^e\left(\int_{\log y}^1 x+y\,dx\right)dy$$

$$= \int_1^e \frac{x^2}{2} + xy\,\Big|_{\log y}^1\,dy$$

$$= \int_1^e \frac{1}{2} + y - \frac{1}{2}(\log y)^2 - y\log y\,dy$$

$$= \frac{e}{2} + \frac{e^2}{2} - 1 - \frac{1}{2}\int_1^e(\log y)^2dy - \int_1^e y\log y\,dy$$

$$= \cancel{\frac{e}{2}} + \frac{e^2}{2} - \cancel{1} - \cancel{\frac{e}{2}} + \cancel{1} - \frac{e^2}{4} - \frac{1}{4}$$

$$= \frac{1}{4}(e^2-1) \quad ※$$

Exercises for § 9.7

#1. a. Let $t>-1$. Show that $\int_0^1 x^t \log x\,dx$ exists.

Pf: Let $y=\log x \Rightarrow e^y = x \Rightarrow dx = e^y dy$

also $x \in (0,1) \Rightarrow y \in (-\infty, 0)$

$$\Rightarrow \int_0^1 x^t \log x\,dx = \int_{-\infty}^0 e^{ty}\cdot y \cdot e^y dy$$

$$= \int_{-\infty}^0 y\, e^{(t+1)y} dy$$

Let $a=t+1$, Since $t>-1 \Rightarrow a>0$.

$$\Rightarrow \int_{-\infty}^0 y e^{ay} dy = \int_{-\infty}^0 \frac{1}{a} y\, de^{ay}$$

$$= \cancel{\frac{y}{a} e^{ay} \Big|_{-\infty}^0}^{=0} - \int_{-\infty}^0 \frac{1}{a} e^{ay} dy$$

$$= -\int_{-\infty}^0 \frac{1}{a^2} de^{ay}$$

$$= -\frac{1}{a^2} e^{ay} \Big|_{-\infty}^0 = -\frac{1}{a^2}$$

$$= -\frac{1}{(t+1)^2}$$

Thus $\int_0^1 x^t \log x\,dx = -\frac{1}{(t+1)^2}$ ✱

b. $$\frac{d}{dt}\int_0^1 x^t dx = \int_0^1 \frac{d}{dt} x^t dx$$

$$= \int_0^1 x^t \log x\,dx. \quad \text{—— ①}$$

Since $\frac{d}{dt}\left(\int_0^1 x^t dx\right) = \frac{d}{dt}\left(\frac{1}{t+1}\right) = -\frac{1}{(t+1)^2}$,

thus by ①, we have $\int_0^1 x^t \log x\,dx = -\frac{1}{(t+1)^2}$ ✱

Exercise for § 9.7

1. c.

$$\frac{d}{dt}x^t = x^t \cdot \log x$$

$$\frac{d^2}{dt^2}x^t = \frac{d}{dt}(x^t \cdot \log x) = x^t \cdot (\log x)^2$$

$$\vdots$$

$$\frac{d^n}{dt^n}x^t = x^t(\log x)^n$$

$$\frac{d^n}{dt^n}\int_0^1 x^t dx = \int_0^1 \frac{d^n}{dt^n}x^t dx \quad \text{for } t>-1.$$

$$\therefore \frac{d^n}{dt^n}\int_0^1 x^t dx = \frac{d^n}{dt^n}\left(\frac{1}{t+1}\right) = (-1)^n \frac{n!}{(t+1)^{n+1}},$$

and

$$\int_0^1 \frac{d^n}{dt^n}x^t dx = \int_0^1 x^t(\log x)^n dx$$

$$\therefore \int_0^1 x^t(\log x)^n dx = (-1)^n \frac{n!}{(t+1)^{n+1}} \quad \#$$

Exercise for § 9.7.

#2. $\because$ $\cos tx$ and $\frac{d}{dt}\cos(tx) = -x\sin tx$ are continuous on $[0,2\pi]$.

$$\therefore \quad \frac{d}{dt}\int_0^{2\pi}\cos tx\,dx = \int_0^{2\pi}\frac{d}{dt}\cos tx\,dx = -\int_0^{2\pi} x\sin tx\,dx.$$

$$\frac{d}{dt}\int_0^{2\pi}\cos tx\,dx = \frac{d}{dt}\left[\frac{1}{t}\sin tx\Big|_0^{2\pi}\right]$$

$$= \frac{d}{dt}\,\frac{1}{t}\sin 2\pi t$$

$$= -\frac{1}{t^2}\sin 2\pi t + \frac{2\pi}{t}\cos 2\pi t$$

$$= \frac{-\sin 2\pi t + 2\pi t\cos 2\pi t}{t^2}.$$

Thus $\displaystyle\int_0^{2\pi} x\sin tx\,dx = \frac{\sin 2\pi t - 2\pi t\cos 2\pi t}{t^2}$ ※

#5. Let $f(x,a) = e^{-ax}$, $a>0$

$$\left.\begin{aligned} &\frac{\partial f}{\partial a}(x,a) = -xe^{-ax} \\ &\frac{\partial^n f}{\partial a^n}(x,a) = (-1)^n x^n e^{-ax}, \quad n\in\mathbb{N} \end{aligned}\right\}\ \text{continuous.}$$

$\because$ $\int_0^\infty e^{-ax}dx = \frac{1}{a} < \infty$ and $\int_0^\infty -xe^{-ax}dx < \infty$ $\forall a>0$

also $\int_0^\infty (-1)^n x^n e^{-ax}dx$ converges uniformly $\forall a>0$.

$\therefore$ By Prop 9.7.5

$$\frac{d^n}{da^n}\int_0^\infty e^{-ax}dx = \int_0^\infty \frac{d^n}{da^n}e^{-ax}dx = \int_0^\infty (-1)^n x^n e^{-ax}dx$$

$$\frac{d^n}{da^n}\frac{1}{a} = (-1)^n\frac{n!}{a^{n+1}}.$$ Thus $\displaystyle\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$ $\forall a>0$.

Chapter 9.

#4. $\because$ $f, f_1, f_2, f_3 \cdots$ are continuous on $[0,1]$

and $|f|, |f_1|, |f_2|, |f_3| \cdots$ are also continuous on $[0,1]$.

$\therefore$ $|f|, |f_1|, |f_2|, |f_3| \cdots\cdots$ are integrable on $[0,1]$.

$\because$ $f_n \to f$ uniformly

$\therefore$ $|f_n| \to |f|$ also uniformly.

Thus $\int_0^1 |f_n| \to \int_0^1 |f|$ as $n \to \infty$ #.

#11. $\because$ $\int_0^1 1_S \, dy = \begin{cases} 0 & \text{if } x \in Q \\ 0 & \text{if } x \in Q^c \end{cases}$

$\therefore$ $\int_0^1 \left(\int_0^1 1_S \, dy \right) dx = \int_0^1 0 \, dx = 0.$

$\because$ 1_S is discontinuous on $(0,1) \times (0,1)$

and the measure of $(0,1)\times(0,1)$ does not equal to zero

$\therefore$ By Lebesgue's Thm, $\int_{[0,1]\times[0,1]} 1_S$ does not exist.

14.

(F) a. Let $A=[0,1]$ and $f(x)=1\ \forall x\in[0,1]$

$\Rightarrow f$ is integrable on $[0,1]$.

Let $g(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\notin\mathbb{Q}\end{cases}$ on $[0,1]$

then $g\le f$ on $[0,1]$

but g is not integrable on $[0,1]$.

(T) b. 1_A is integrable and $D=\{x\in\mathbb{R}^n \mid 1_A$ is discontinuous at $x\}$ has measure zero.

Thus if f is continuous on A

$B=\{x\in\mathbb{R}^n \mid f\cdot 1_A$ is discontinuous at $x\}\subset D$

$\Rightarrow$ B has measure zero.

$\Rightarrow \int_{\mathbb{R}^n} f\cdot 1_A = \int_A f$ exists.

(F) c. Let $A=[0,1]$. Then A has volume.

Let $f(x)=\begin{cases}0, & 0\le x<\frac{1}{2}\\ 1, & \frac{1}{2}\le x\le 1.\end{cases}$

$\Rightarrow \int_A f=\int_0^1 f(x)dx=\int_{\frac{1}{2}}^1 dx=\frac{1}{2}$ exists.

But f is not continuous on $[0,1]$ #

14.

(T) d. Let E be bounded in $\mathbb{R}^2$

$\Rightarrow \exists\ [a,b]\times[c,d]\subset\mathbb{R}^2$ s.t. $[a,b]\times[c,d]\supset E$, $-\infty<a<b<\infty$, $-\infty<c<d<\infty$

$\forall \varepsilon>0$

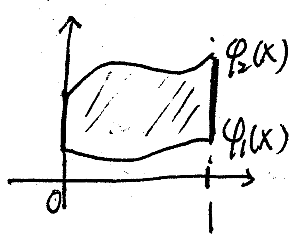
$\mathbb{R}^3 \supset [a,b]\times[c,d]\times[-\varepsilon,\varepsilon]\supset E$.

and $V([a,b]\times[c,d]\times[-\varepsilon,\varepsilon]) = 2\varepsilon(b-a)\times(d-c)$

$\Rightarrow V(E) \leq 2\varepsilon(b-a)\times(d-c) \longrightarrow 0$ as $\varepsilon\to 0$

$\Rightarrow V(E)=0$.

(T) e.



$S=\{(x,y)\mid x\in I,\ \varphi_1(x)<y<\varphi_2(x)\}$.

The boundary of $S=\{(x,y)\mid x\in I,\ \varphi_1(x)<y<\varphi_2(x)\}$

is $\partial S=\{(x,y)\mid x=0,\ \varphi_1(0)\leq y\leq\varphi_2(0)\}\cup\{(x,y)\mid x\in I,\ y=\varphi_1(x)\}$
$\cup\{(x,y)\mid x=1,\ \varphi_1(1)\leq y\leq\varphi_2(1)\}\cup\{(x,y)\mid x\in I,\ y=\varphi_2(x)\}$.

Since ∂S has measure zero, by corollary 8.3.2 S has volume. #

#15. A is bounded set with volume.

$\because$ A_i is a sequence of sets with volume

s.t. $A_{i+1} \supset A_i$ and $A_1 \cup A_2 \cup \cdots = A$

$\therefore$ $1_{A_{i+1}} \geq 1_{A_i} \quad \forall i$

and $1_{A_i} \longrightarrow 1_A$

$\because 0 \leq 1_{A_i} \leq 1_{A_{i+1}} \leq 1_{A_{i+2}} \leq \cdots \leq 1_A$

and $1_{A_i} \longrightarrow 1_A$

$\therefore$ By Lebegue's Monotone Convergence Theorem.

$$\begin{array}{ccc} \int 1_{A_i} & \longrightarrow & \int 1_A \quad \text{as } i \to \infty. \\ \| & & \| \\ v(A_i) & & v(A) \end{array}$$

i.e. $v(A_i) \longrightarrow v(A)$ as $i \longrightarrow \infty$ ※

#17. Let $f_M(x) = \begin{cases} f(x) & \text{if } f(x) \leq M \\ 0 & \text{if } f(x) > M. \end{cases}$ $\Rightarrow \int_0^1 f_M = 0 \because f_M \neq 0$ at only finitely many points.

for $M_1 \leq M_2 \leq M_3 \leq \cdots$

$0 \leq f_{M_1}(x) \leq f_{M_2} \leq f_{M_3} \leq \cdots$

and $f_M(x) \longrightarrow f(x)$ on $[0,1]$

By Lebesgue Monotone Convergence Theorem

$$\int_0^1 f(x)dx = \lim_{M \to \infty} \int_0^1 f_M(x)dx = 0 \quad ※$$

\# 18. Let $f_n(x) = \sum_{m=1}^{n} \left(\frac{1}{2^m}\right) \sin mx$ be defined for all $x \in \mathbb{R}$.

a. $\because$ for $i > j$

$$|f_i(x) - f_j(x)| = \left|\sum_{m=j+1}^{i} \left(\frac{1}{2^m}\right) \sin mx\right|$$
$$\leq \sum_{m=j+1}^{i} \frac{1}{2^m}$$

$\therefore$ $\forall \varepsilon > 0$ we can choose $N \in \mathbb{N}$ s.t. $\forall i > j \geq N$.

$$|f_i(x) - f_j(x)| \leq \sum_{m=N+1}^{i} \frac{1}{2^m} \leq \frac{1}{2^N} < \varepsilon$$

That is $\{f_n(x)\}$ is a cauchy sequence in $\mathbb{R}$

$\Rightarrow$ $\lim_{n\to\infty} f_n(x)$ exists.

b. Since $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall i > j \geq N$

$$|f_i(x) - f_j(x)| < \varepsilon \quad \forall x \in \mathbb{R}$$

$\Rightarrow$ $\{f_n\}$ is uniformly Cauchy in $\mathbb{R}$.

$\Rightarrow$ f_n converges uniformly on $\mathbb{R}$.

c. $\because$ $\{f_n\}$ converges uniformly on $\mathbb{R}$

$\therefore$ $\{f_n\}$ converges uniformly on $[0, 2\pi]$.

$$\Rightarrow \int_0^{2\pi} \lim_{n\to\infty} f_n(t)\,dt = \lim_{n\to\infty} \underbrace{\int_0^{2\pi} f_n(t)\,dt}_{=0} = 0$$

19.

Since f, g are integrable, $F(x,y) = f(x) + g(y)$ is integrable on $A \times B$.

By Fubini's theorem,

$$\int_{A\times B} F(x,y)\,dx\,dy = \int_{A\times B} (f(x) + g(y))\,dx\,dy$$

$$= \int_{A\times B} f(x)\,dx\,dy + \int_{A\times B} g(y)\,dx\,dy$$

Fubini

$$= \int_A \left(\int_B f(x)\,dy\right)dx + \int_B\left(\int_A g(y)\,dx\right)dy$$

$$= \int_A f(x)\left(\int_B dy\right)dx + \int_B g(y)\left(\int_A dx\right)dy$$

$$= \int_A f(x)\,v(B)\,dx + \int_B g(y)\,v(A)\,dy$$

$$= \left(\int_A f\right)v(B) + \left(\int_B g\right)v(A)$$

※.

20.

(a). Let $f: [0,1] \to \mathbb{R}$ be defined by

$$f(x) = x^{\underset{>0}{k}} \Rightarrow \int_0^1 x^k\,dx = \frac{1}{k+1}$$

Also we have $\lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\frac{i}{n}\right)^k \cdot \frac{1}{n}\right) = \int_0^1 x^k\,dx = \frac{1}{k+1}$.

$$\Rightarrow \lim_{n\to\infty} (1 + 2^k + \cdots + n^k)/n^{k+1} = \frac{1}{k+1}$$

(b) Let $f: [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{1+x} \Rightarrow \int_0^1 \frac{1}{1+x}\,dx = \log 2$$

$$\because \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}}\cdot\frac{1}{n}\right) = \int_0^1\frac{1}{1+x}\,dx = \log 2$$

$$\therefore \lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \log 2 \quad ※.$$