Exercise Problem Sets 2

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Problem 1. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the least upper bound property, and A be a non-empty set of $\mathbb{F}$ which is bounded below. Define the set $-A$ by $-A \equiv \big\{-x \in \mathbb{F} \,\big|\, x \in A\big\}$. Prove that

$$\inf A = -\sup(-A)\,.$$

Proof. Let C be a subset of $\mathbb{F}$. Then

$$\begin{aligned} &b \text{ is a lower bound for a set } C \Leftrightarrow b \leqslant c \text{ for all } c \in C \Leftrightarrow -b \geqslant -c \text{ for all } c \in C \\ &\quad \Leftrightarrow -b \geqslant -c \text{ for all } -c \in -C \Leftrightarrow -b \geqslant c \text{ for all } c \in -C \Leftrightarrow -b \text{ is an upper bound for } -C. \end{aligned}$$

Therefore, we conclude that

$$b \text{ is a lower bound for a set } C \text{ if and only if } -b \text{ is an upper bound for } -C. \tag{$\star$}$$

Now, since A is bounded from below, $-A$ is bounded from above. The least upper bound property then implies that $b = \sup(-A) \in \mathbb{F}$ exists. From $(\star)$, we find that $-b$ is a lower bound for A. Suppose that $-b$ is not the greatest lower bound for A. Then there exists $m > -b$ such that $m \leqslant x$ for all $x \in A$. This implies that m is a lower bound for A; thus $(\star)$ shows that $-m$ is an upper bound for $-A$. By the fact that $-m < b$, we conclude that b is not the least upper bound for $-A$, a contradiction to that b is the least upper bound for $-A$. □

Remark 0.1. Note the Problem 1 in fact shows that if $\mathbb{F}$ satisfies **LUBP**, then $\mathbb{F}$ satisfies **GLBP**.

Problem 2. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the least upper bound property, and A, B be non-empty subsets of $\mathbb{F}$. Define $A + B = \{x + y \,|\, x \in A, y \in B\}$. Justify if the following statements are true or false by providing a proof for the true statement and giving a counter-example for the false ones.

1. $\sup(A + B) = \sup A + \sup B$.
2. $\inf(A + B) = \inf A + \inf B$.
3. $\sup(A \cap B) \leqslant \min\{\sup A, \sup B\}$.
4. $\sup(A \cap B) = \min\{\sup A, \sup B\}$.
5. $\sup(A \cup B) \geqslant \max\{\sup A, \sup B\}$.
6. $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Proof. 1. Let $a = \sup A$, $b = \sup B$, and $\varepsilon > 0$ be given. W.L.O.G. we can assume that $a, b \in \mathbb{F}$ for otherwise $a = \infty$ or $b = \infty$ so that $A + B$ is not bounded from above.

(a) Let $z \in A + B$. Then $z = x + y$ for some $x \in A$ and $y \in B$. By the fact that $x \leqslant a$ and $y \leqslant b$, we find that $z \leqslant a + b$. Therefore, $a + b$ is an upper bound for $A + B$.

(b) There exists $x \in A$ and $y \in B$ such that $x > a - \dfrac{\varepsilon}{2}$ and $y > b - \dfrac{\varepsilon}{2}$; thus there exists $z = x + y \in A + B$ such that

$$z = x + y > a + b - \varepsilon\,.$$

Therefore, $a+b=\sup(A+B)$.

2. By Problem 1,

$$\begin{aligned}\inf(A+B) &= -\sup(-(A+B)) = -\sup(-A+(-B)) = -\sup(-A)-\sup(-B)\\ &= \inf(A)+\inf(B)\,.\end{aligned}$$

3. The desired inequality hold if $A\cap B=\varnothing$ (since then $\sup A\cap B=-\infty$), so we assume that $A\cap B\neq\varnothing$. Then $A\cap B\subseteq A$ and $A\cap B\subseteq B$. Therefore,

$$\sup(A\cap B)\leqslant\sup A \qquad\text{and}\qquad \sup(A\cap B)\leqslant\sup B\,.$$

The inequalities above then implies that $\sup(A\cap B)\leqslant\min\{\sup A,\sup B\}$.

4. If A and B are non-empty bounded sets but $A\cap B=\varnothing$, then $\sup(A\cap B)=-\infty$ but $\sup A,\sup B\in\mathbb{F}$. In such a case $\sup(A\cap B)\neq\min\{\sup A,\sup B\}$.

5. Similar to 3, we have $A\subseteq A\cup B$ and $B\subseteq A\cup B$; thus

$$\sup A\leqslant\sup(A\cup B) \qquad\text{and}\qquad \sup B\leqslant\sup(A\cup B)\,.$$

Therefore, $\max\{\sup A,\sup B\}\leqslant\sup(A\cup B)$.

6. If one of A and B is not bounded from above, then $\sup(A\cup B)=\max\{\sup A,\sup B\}=\infty$. Suppose that A and B are bounded from above. Then $A\cup B$ are bounded from above by $\max\{\sup A,\sup B\}$ since if $x\in A\cup B$, then $x\in A$ or $x\in B$ which implies that $x\leqslant\sup A$ or $x\leqslant\sup B$; thus $x\leqslant\max\{\sup A,\sup B\}$ for all $x\in A\cup B$. This shows that

$$\sup(A\cup B)\leqslant\max\{\sup A,\sup B\}\,.$$

Together with 5, we conclude that $\sup(A\cup B)=\max\{\sup A,\sup B\}$. □

Problem 3. Let $(\mathbb{F},+,\cdot,\leqslant)$ be an ordered field satisfying the least upper bound property, and $S\subseteq\mathbb{F}$ be bounded below and non-empty. Show that

$$\inf S=\sup\big\{x\in\mathbb{F}\,\big|\,x\text{ is a lower bound for }S\big\}$$

and

$$\sup S=\inf\big\{x\in\mathbb{F}\,\big|\,x\text{ is an upper bound for }S\big\}\,.$$

Proof. Define $A=\big\{x\in\mathbb{F}\,\big|\,x\text{ is a lower bound for }S\big\}$. Since S is non-empty, every element in S is an upper bound for A; thus A is bounded from above. By the least upper bound property, $b=\sup A\in\mathbb{F}$ exists. Note that by the definition of A,

$$\text{if } x\in A,\text{ then } x\leqslant s\text{ for all } s\in S. \tag{$\star$}$$

Let $\varepsilon > 0$ be given. Then $b - \varepsilon$ is not an upper bound for A; thus there exists $x \in A$ such that $b - \varepsilon < x$. Then $(\star)$ implies that $b - \varepsilon < s$ for all $s \in S$. Since $\varepsilon > 0$ is given arbitrarily, $b \leqslant s$ for all $s \in S$; thus b is a lower bound for S.

Suppose that b is not the greatest lower bound for S. There exists $m > b$ such that $m \leqslant s$ for all $s \in S$. Therefore, $m \in A$; thus $m \leqslant b = \sup A$, a contradiction. □

Problem 4. Let A, B be two sets, and $f : A \times B \to \mathbb{F}$ be a function, where $(\mathbb{F}, +, \cdot, \leqslant)$ is an ordered field satisfying the least upper bound property. Show that

$$\sup_{(x,y)\in A\times B} f(x,y) = \sup_{y\in B} \big(\sup_{x\in A} f(x,y)\big) = \sup_{x\in A} \big(\sup_{y\in B} f(x,y)\big) .$$

Proof. It suffices to prove the first equality. Note that

$$f(x,y) \leqslant \sup_{(x,y)\in A\times B} f(x,y) \qquad \forall\, (x,y) \in A \times B\,;$$

thus

$$\sup_{x\in A} f(x,y) \leqslant \sup_{(x,y)\in A\times B} f(x,y) \qquad \forall\, y \in B\,.$$

The inequality above further shows that

$$\sup_{y\in B} \Big(\sup_{x\in A} f(x,y)\Big) \leqslant \sup_{(x,y)\in A\times B} f(x,y)\,. \tag{$\star$}$$

Now we show the reverse inequality.

1. Suppose that $\displaystyle\sup_{(x,y)\in A\times B} f(x,y) = M < \infty$. Then for each $k \in \mathbb{N}$, there exists $(x_k, y_k) \in A \times B$ such that
$$f(x_k, y_k) > M - \frac{1}{k}\,.$$
Therefore,
$$M - \frac{1}{k} < f(x_k, y_k) \leqslant \sup_{x\in A} f(x, y_k)$$
which further implies that
$$M - \frac{1}{k} < f(x_k, y_k) \leqslant \sup_{y\in B} \big(\sup_{x\in A} f(x,y)\big)\,.$$
Since the inequality above holds for all $k \in \mathbb{N}$, we find that $\displaystyle\sup_{y\in B} \big(\sup_{x\in A} f(x,y)\big) \geqslant M$.

2. Suppose that $\displaystyle\sup_{(x,y)\in A\times B} f(x,y) = \infty$. Then for each $k \in \mathbb{N}$, there exists $(x_k, y_k) \in A \times B$ such that
$$f(x_k, y_k) > k\,.$$
Therefore,
$$k < f(x_k, y_k) \leqslant \sup_{x\in A} f(x, y_k)$$

which further implies that

$$k < f(x_k, y_k) \leqslant \sup_{y\in B}\big(\sup_{x\in A} f(x,y)\big)\,.$$

Since the inequality above holds for all $k \in \mathbb{N}$, we find that $\sup\limits_{y\in B}\big(\sup\limits_{x\in A} f(x,y)\big) = \infty$.

With the help of ($\star$), we conclude that $\sup\limits_{(x,y)\in A\times B} f(x,y) = \sup\limits_{y\in B}\big(\sup\limits_{x\in A} f(x,y)\big)$. □

Problem 5. Let $(\mathbb{F},+,\cdot,\leqslant)$ be an ordered field satisfying the least upper bound property, and $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{F}^n$. Define

$$\|\boldsymbol{x}\|_1 = \sum_{k=1}^{n} |x_k| \qquad \text{and} \qquad \|\boldsymbol{x}\|_\infty = \max\big\{|x_1|, |x_2|, \cdots, |x_n|\big\}\,.$$

Show that

1. $\|\boldsymbol{x}\|_1 = \sup\Big\{\sum\limits_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{y}\|_\infty = 1\Big\}$.
2. $\|\boldsymbol{y}\|_\infty = \sup\Big\{\sum\limits_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{x}\|_1 = 1\Big\}$.

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^n$ be given. Then

$$\sum_{k=1}^{n} x_k y_k \leqslant \sum_{k=1}^{n} |x_k||y_k| \leqslant \sum_{k=1}^{n} |x_k|\|\boldsymbol{y}\|_\infty = \|\boldsymbol{y}\|_\infty \sum_{k=1}^{n} |x_k| = \|\boldsymbol{y}\|_\infty \|\boldsymbol{x}\|_1\,.$$

Therefore,

$$\sup\Big\{\sum_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{y}\|_\infty = 1\Big\} \leqslant \|\boldsymbol{x}\|_1 \qquad \text{and} \qquad \sup\Big\{\sum_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{x}\|_1 = 1\Big\} \leqslant \|\boldsymbol{y}\|_\infty\,.$$

Next we show that the two inequalities are in fact equalities by showing that the right-hand side of the inequalities belongs to the sets (this is because if $b \in A$ is an upper bound for A, then b is the least upper bound for A).

1. $\sup\Big\{\sum\limits_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{y}\|_\infty = 1\Big\} = \|\boldsymbol{x}\|_1$: W.L.O.G. we can assume that $\boldsymbol{x} \neq \mathbf{0}$. For a given $\boldsymbol{x} \in \mathbb{F}^n$, define $y_k = \text{sgn}(x_k)$, where sgn is the sign function defined by

   $$\text{sgn}(a) = \begin{cases} 1 & \text{if } a > 0\,, \\ -1 & \text{if } a < 0\,, \\ 0 & \text{if } a = 0\,. \end{cases}$$

   Then $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$ satisfies $\|\boldsymbol{y}\|_\infty = 1$ (since at least one component of $\boldsymbol{x}$ is non-zero), and

   $$\sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} x_k \text{sgn}(x_k) = \sum_{k=1}^{n} |x_k| = \|\boldsymbol{x}\|_1\,.$$

2. $\sup\Big\{\sum\limits_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{x}\|_1 = 1\Big\} = \|\boldsymbol{y}\|_\infty$: W.L.O.G. we can assume that $\boldsymbol{y} \neq \mathbf{0}$. Suppose that $\|\boldsymbol{y}\|_\infty = |y_m| \neq 0$ for some $1 \leqslant m \leqslant n$; that is, the maximum of the absolute value of components occurs at the m-th component. Define $x_j = \delta_{jm}\text{sgn}(y_j)$; that is,

   $$x_j = \begin{cases} 0 & \text{if } j \neq m\,, \\ \text{sgn}(y_m) & \text{if } j = m\,. \end{cases}$$

   Then $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ satisfies $\|\boldsymbol{x}\| = 1$ (since only one component of $\boldsymbol{x}$ is 1 or -1), and

$$\sum_{k=1}^{n} x_k y_k = \operatorname{sgn}(y_m) y_m = |y_m| = \|\boldsymbol{y}\|_\infty \,. \qquad \square$$

Problem 6. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the least upper bound property. A set $A \subseteq \mathbb{F}$ is said to be closed if every convergent sequence in A converges to a limit in A. In logic notation,

$$A \subseteq \mathbb{F} \text{ is closed} \quad \Leftrightarrow \quad (\forall \{x_n\}_{n=1}^\infty \subseteq A)\Big(\{x_n\}_{n=1}^\infty \text{ converges} \Rightarrow \lim_{n\to\infty} x_n \in A\Big) \,.$$

1. Show that $\varnothing$ and $\mathbb{F}$ are closed.

2. Show that $[a,b] = \big\{x \in \mathbb{F} \,\big|\, a \leqslant x \leqslant b\big\}$ is closed for all $a, b \in \mathbb{F}$.

3. Show that if $\varnothing \neq A \subseteq \mathbb{F}$ is closed and bounded, then $\sup A \in A$ and $\inf A \in A$.

Proof. 3. Since A is bounded, $a = \inf A$ and $b = \sup A$ exist. For each $n \in \mathbb{N}$, there exists $x_n, y_n \in A$ such that

$$a \leqslant x_n < a + \frac{1}{n} \qquad \text{and} \qquad b - \frac{1}{n} < y_n \leqslant b \,.$$

By the Archimedean property, $\frac{1}{n} \to 0$ as $n \to \infty$; thus the Sandwich Lemma implies that

$$\lim_{n\to\infty} x_n = a \qquad \text{and} \qquad \lim_{n\to\infty} y_n = b \,.$$

Since $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq A$ and A is closed, $a \in A$ and $b \in B$. $\square$

Problem 7. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an Archimedean ordered field, $a, \delta \in \mathbb{F}$ and $\delta > 0$. The δ***-neighborhood*** of a is the set $\mathcal{N}(a,\delta) = \big\{x \in \mathbb{F} \,\big|\, |x-a| < \delta\big\}$. A number $x \in \mathbb{F}$ is called an ***accumulation point*** of a set $A \subseteq \mathbb{F}$ if for all $\delta > 0$, $\mathcal{N}(x,\delta)$ contains at least one point of A distinct from x. In logic notation,

$$x \text{ is an accumulation point of } A \quad \Leftrightarrow \quad (\forall\, \delta > 0)\big(\mathcal{N}(x,\delta) \cap A \supsetneq \{x\}\big) \,.$$

1. Show that if $\{x_n\}_{n=1}^\infty$ is a sequence in $\mathbb{F}$ so that $x_i \neq x_j$ for all $i, j \in \mathbb{N}$ and $A = \big\{x_k \,\big|\, k \in \mathbb{N}\big\}$, then x is an accumulation of A if and only if x is a cluster point of $\{x_n\}_{n=1}^\infty$.

2. How about if the condition $x_i \neq x_j$ for all $i, j \in \mathbb{N}$ is removed? Is the statement in 1 still valid?

Proof. 1. We show that

$$x \text{ is an accumulation point of } A \text{ if and only if } (\forall\, \delta > 0)\big(\#(A \cap (x-\delta, x+\delta)) = \infty\big) \,.$$

The direction "$\Leftarrow$" is trivial since if $\#\big(A \cap (x-\delta, x+\delta)\big) = \infty$, $A \cap (x-\delta, x+\delta)$ contains some point distinct from x.

($\Rightarrow$) Let $\delta_1 = 1$, by the definition of the accumulation points, there exists $x_1 \in A \cap (x-\delta_1, x+\delta_1)$ and $x_1 \neq x$. Define $\delta_2 = \min\big\{|x_1 - x|, \frac{1}{2}\big\}$. Then $\delta_2 > 0$; thus there exists $x_2 \in A \cap (x-\delta_2, x+\delta_2)$ and $x_2 \neq x$. We continue this process and obtain a sequence $\{x_n\}_{n=1}^\infty \subseteq A \backslash \{x\}$ satisfying that

$$x_1 \in A \cap (x-1, x+1), \quad x_n \in A \cap (x-\delta_n, x+\delta_n) \text{ with } \delta_n = \min\big\{|x - x_{n-1}|, \frac{1}{n}\big\} \,.$$

By the Archimedean property, $\{x_n\}_{n=1}^\infty$ converges to x since $|x - x_n| < \delta_n \leqslant \dfrac{1}{n}$. Let $\delta > 0$ be given. There exists $N > 0$ such that $\dfrac{1}{N} < \delta$; thus

$$A \cap (x - \delta, x + \delta) \supseteq A \cap \big(x - \frac{1}{N}, x + \frac{1}{N}\big) \supseteq \{x_N, x_{N+1}, x_{N+2}, \cdots\}\,.$$

Since $x_i \neq x_j$ for all $i, j \in \mathbb{N}$, we must have $\#\big(A \cap (x - \delta, x + \delta)\big) = \infty$. ▫

Problem 8. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathbb{F}$. Show that $\{x_n\}_{n=1}^\infty$ converges if and only if every proper subsequence of $\{x_n\}_{n=1}^\infty$ converges.

Proof. By a Proposition that we have talked about in class, it suffices to prove the direction "$\Leftarrow$". We show that if every proper subsequence of $\{x_n\}_{n=1}^\infty$ converges, then every proper subsequence of $\{x_n\}_{n=1}^\infty$ converges to identical limit. Suppose the contrary that there exist two subsequence $\{x_{n_k}\}_{k=1}^\infty$ and $\{x_{m_j}\}_{j=1}^\infty$ that converge to a and b and $a \neq b$, respectively. We construct a new subsequence $\{y_\ell\}_{\ell=1}^\infty$ of $\{x_n\}_{n=1}^\infty$, as follows. Let $k_1 = 1$ and $y_1 = x_{n_{k_1}}$. Let j_1 be the smallest integer so that $m_{j_1} > n_{k_1}$, and define $y_2 = x_{m_{j_1}}$. Let k_2 be the smallest integer so that $n_{k_2} > m_{j_1}$, and define $y_3 = x_{n_{\ell_2}}$. We continue this process and obtain a sequence $\{y_\ell\}_{\ell=1}^\infty$ satisfying that

$$y_\ell = \begin{cases} y_{n_{k_{\frac{\ell+1}{2}}}} & \ell \text{ is odd}\,, \\ y_{m_{j_{\frac{\ell}{2}}}} & \ell \text{ is even}\,, \end{cases}$$

where $k_1, k_2, \cdots$ and $j_1, j_2, \cdots$ satisfy that $k_1 = 1$,

$$j_r = \min\big\{j \in \mathbb{N} \,\big|\, m_j > k_r\big\} \quad \text{and} \quad k_{r+1} = \min\big\{k \in \mathbb{N} \,\big|\, n_k > m_{j_r}\big\} \qquad \forall\, r \in \mathbb{N}\,.$$

Then $\{y_{2\ell-1}\}_{\ell=1}^\infty$, the collection of odd terms of $\{y_\ell\}_{\ell=1}^\infty$, is a subsequence of $\{x_{n_k}\}_{k=1}^\infty$ and $\{y_{2\ell}\}_{\ell=1}^\infty$, the collection of even terms of $\{y_\ell\}_{\ell=1}^\infty$, is a subsequence of $\{x_{m_j}\}_{j=1}^\infty$, and $\{y_{2\ell-1}\}_{\ell=1}^\infty$ converges to a while $\{y_{2\ell}\}_{\ell=1}^\infty$ converges to b, and $a \neq b$. By a Proposition we talked about in class, $\{y_\ell\}_{\ell=1}^\infty$ does not converges, a contradiction. ▫