Exercise Problem Sets 6

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Problem 1. Show that $(\mathbb{C}, |\cdot|)$ is complete.

Proof. Let $\{z_n\}_{n=1}^\infty$ be a Cauchy sequence in $\mathbb{C}$. Write $z_n = x_n + iy_n$, where x_n and y_n are real numbers. Then

$$|x_n - x_m| \leqslant |z_n - z_m| \quad \text{and} \quad |y_n - y_m| \leqslant |z_n - z_m| \qquad \forall\, n, m \in \mathbb{N}\,.$$

Therefore, $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are Cauchy sequences in $\mathbb{R}$; thus by the completeness of $\mathbb{R}$, $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$ for some $x, y \in \mathbb{R}$. Let $z = x + yi$. Then

$$|z_n - z| = \sqrt{(x_n - x)^2 + (y_n - y)^2} \to 0 \qquad \text{as} \quad n \to \infty\,;$$

thus we establish that every Cauchy sequence in $\mathbb{C}$ converges to a point in $\mathbb{C}$. This implies that $(\mathbb{C}, |\cdot|)$ is complete. □

Problem 2. Let (M, d) be a metric space. Two Cauchy sequences $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ in M are said to be equivalent, denoted by $\{p_n\}_{n=1}^\infty \sim \{q_n\}_{n=1}^\infty$, if $\lim\limits_{n\to\infty} d(p_n, q_n) = 0$.

1. Prove that $\sim$ is an equivalence relation; that is, show that
   (a) $\{p_n\}_{n=1}^\infty \sim \{p_n\}_{n=1}^\infty$.
   (b) If $\{p_n\}_{n=1}^\infty \sim \{q_n\}_{n=1}^\infty$, then $\{q_n\}_{n=1}^\infty \sim \{p_n\}_{n=1}^\infty$.
   (c) If $\{p_n\}_{n=1}^\infty \sim \{q_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty \sim \{r_n\}_{n=1}^\infty$, then $\{p_n\}_{n=1}^\infty \sim \{r_n\}_{n=1}^\infty$.

2. Let $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ be two Cauchy sequences. Show that the sequence $\big\{d(p_n, q_n)\big\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$; thus is convergent.

3. Let M^* be the set of all equivalence classes. If $P, Q \in M^*$, we define
$$d^*(P, Q) = \lim_{n\to\infty} d(p_n, q_n)\,,$$
where $\{p_n\}_{n=1}^\infty \in P$ and $\{q_n\}_{n=1}^\infty \in Q$. Show that the definition above is well-defined; that is, show that if $\{p'_n\}_{n=1}^\infty \in P$ and $\{q'_n\}_{n=1}^\infty \in Q$ are another two Cauchy sequences, then $\lim\limits_{n\to\infty} d(p_n, q_n) = \lim\limits_{n\to\infty} d(p'_n, q'_n)$.

4. Define $\varphi : M \to M^*$ as follows: for each $x \in M$, $\{x_n\}_{n=1}^\infty$, where $x_n \equiv x$ for all $n \in \mathbb{N}$, is a Cauchy sequence in M. Then $\{x_n\}_{n=1}^\infty \in \varphi(x)$ for one particular $\varphi(x) \in M^*$. In other words, $\varphi(x)$ is the equivalence class where $\{x_n\}_{n=1}^\infty$ belongs to. Show that
$$d^*\big(\varphi(x), \varphi(y)\big) = d(x, y) \qquad \forall\, x, y \in M.$$

5. Show that $\varphi(M)$ is dense in M^*; that is, for each $x \in M^*$ there exists a sequence $\{x_k\}_{k=1}^{\infty} \subseteq \varphi(M)$ such that $x_k \to x$ as $k \to \infty$.

6. Show that (M^*, d^*) is a complete metric space. The metric space (M^*, d^*) is called the completion of (M, d).

Proof. 請自行搜尋 "completion of metric space"。 □

Problem 3. Consider the function $f(x) = \sum\limits_{k=1}^{\infty} \dfrac{\sin(kx)}{k}$.

1. Find the domain of f.

2. Show that for each $\varepsilon > 0$ and $0 < \delta < \pi$, there exists $N > 0$ and N depends only on ε and δ but is independent of x, such that

$$\Big|\sum_{k=n}^{n+p} \frac{\sin(kx)}{k}\Big| < \varepsilon \qquad \forall\, n \geqslant N, p \geqslant 0 \text{ and } x \in [\delta, 2\pi - \delta]\,.$$

Proof. Let $S_n(x) = \sum\limits_{k=1}^{n} \sin(kx)$.

1. (a) If $x = 2n\pi$ for some $n \in \mathbb{Z}$ (or $x = 0 \pmod{2\pi}$), then $S_n(x) = 0$ for all $n \in \mathbb{N}$; thus for each $x = 0 \pmod{2\pi}$, $\{S_n(x)\}_{n=1}^{\infty}$ is bounded by 1.

   (b) If $x \neq 2n\pi$ for all $n \in \mathbb{Z}$ (or $x \neq 0 \pmod{2\pi}$), then

$$2\sin\frac{x}{2} S_n(x) = \sum_{k=1}^{n} 2\sin\frac{x}{2}\sin(kx) = \sum_{k=1}^{n} \cos\big(k - \frac{1}{2}\big)x - \cos\big(k + \frac{1}{2}\big)x = \cos\frac{x}{2} - \cos\big(n + \frac{1}{2}\big)x$$

   which implies that

$$\big|S_n(x)\big| \leqslant \Big|\frac{\cos\frac{x}{2} - \cos\big(n + \frac{1}{2}\big)x}{2\sin\frac{x}{2}}\Big| \leqslant \frac{1}{\big|\sin\frac{x}{2}\big|} \qquad \forall\, x \neq 0 \pmod{2\pi}\,.$$

   In either cases, for each $x \in \mathbb{R}$ there exists $M = M(x) \in \mathbb{R}$ such that $|S_n(x)| \leqslant M$. Therefore, the Dirichlet test $\big($with $a_k = \sin(kx)$ and $p_k = \dfrac{1}{k}\big)$ implies that f is defined everywhere; thus the domain of f is $\mathbb{R}$.

2. We mimic the proof of the Dirichlet test. Let $\varepsilon > 0$ and $\delta \in (0, \pi)$ be given. Then $\csc\dfrac{\delta}{2} > 0$; thus the Archimedean property of $\mathbb{R}$ implies that there exists $N > \dfrac{2}{\varepsilon}\csc\dfrac{\delta}{2}$. If $n \geqslant N$, $p \geqslant 0$ and $x \in [\delta, 2\pi - \delta]$ (thus $x \neq 0 \pmod{2\pi}$), then

$$\begin{aligned}
\Big|\sum_{k=n}^{n+p} \frac{\sin(kx)}{k}\Big| &= \Big|\sum_{k=n}^{n+p} \big[S_{k+1}(x) - S_k(x)\big]\frac{1}{k}\Big| \\
&= \Big|-S_n(x)\frac{1}{n} + S_{n+1}(x)\big(\frac{1}{n} - \frac{1}{n+1}\big) + \cdots + S_{n+p}(x)\big(\frac{1}{n+p-1} - \frac{1}{n+p}\big) + S_{n+p+1}(x)\frac{1}{n+p}\Big| \\
&\leqslant \frac{1}{\big|\sin\frac{x}{2}\big|}\Big[\frac{1}{n} + \big(\frac{1}{n} - \frac{1}{n+1}\big) + \cdots + \big(\frac{1}{n+p-1} - \frac{1}{n+p}\big) + \frac{1}{n+p}\Big] = \frac{2}{n\big|\sin\frac{x}{2}\big|} < \frac{\sin\frac{\delta}{2}}{\big|\sin\frac{x}{2}\big|}\varepsilon\,.
\end{aligned}$$

Since $x \in [\delta, 2\pi - \delta]$, $\sin \frac{x}{2}$ attains its minimum at $x = \delta$ or $2\pi - \delta$; thus

$$0 < \sin \frac{\delta}{2} \leqslant \sin \frac{x}{2} \qquad \forall\, x \in [\delta, 2\pi - \delta]\,.$$

Therefore,

$$\Big|\sum_{k=n}^{n+p} \frac{\sin(kx)}{k}\Big| < \varepsilon \quad \text{ whenever } \quad n \geqslant N, p \geqslant 0 \text{ and } x \in [\delta, 2\pi - \delta]\,. \qquad \square$$

Problem 4. Let $\{a_n\}_{n=1}^\infty \subseteq \mathbb{R}$ ba a sequence. A series $\sum\limits_{n=1}^\infty b_n$ is said to be a rearrangement of the series $\sum\limits_{n=1}^\infty a_n$ if there exists a rearrangement π of $\mathbb{N}$; that is, $\pi : \mathbb{N} \to \mathbb{N}$ is bijective, such that $b_n = a_{\pi(n)}$. Show that if $\sum\limits_{n=1}^\infty a_n$ converges absolutely, then any rearrangement of the series $\sum\limits_{n=1}^\infty a_n$ converges and has the value $\sum\limits_{n=1}^\infty a_n$.

Proof. Suppose that $\sum\limits_{n=1}^\infty a_n$ is an absolutely convergent series with limit a, and $\pi : \mathbb{N} \to \mathbb{N}$ is a rearrangement of $\mathbb{N}$. Let $\varepsilon > 0$ be given. Then there exists $N > 0$ such that

$$\Big|\sum_{k=1}^{n} a_k - a\Big| < \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{k=n+1}^{\infty} |a_k| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n \geqslant N\,.$$

Choose $K > 0$ such that $\pi(n) > N$ if $n \geqslant K$. In fact, $K = \max\big\{\pi^{-1}(1), \cdots, \pi^{-1}(N)\big\} + 1$ suffices the purpose. Then $K \geqslant N$ and if $n \geqslant K$, $\pi\big(\{1, 2, \cdots, n\}\big) \supseteq \{1, 2, \cdots, N\}$. Therefore,

1. if $n \geqslant K$ and $p \geqslant 0$, we have

$$\sum_{k=n}^{n+p} \big|a_{\pi(k)}\big| \leqslant \sum_{k=N+1}^{\infty} |a_k| < \frac{\varepsilon}{2} < \varepsilon$$

which, by Cauchy's criterion, shows that $\sum\limits_{k=1}^\infty a_{\pi(k)}$ converges absolutely.

2. If $n \geqslant K$,

$$\Big|\sum_{k=1}^{n} a_{\pi(k)} - a\Big| \leqslant \Big|\sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{N} a_k\Big| + \Big|\sum_{k=1}^{N} a_k - a\Big| \leqslant \sum_{k=N+1}^{\infty} |a_k| + \frac{\varepsilon}{2} < \varepsilon$$

which implies that $\sum\limits_{n=1}^\infty a_{\pi(n)} = a$. $\square$

Problem 5. Determine whether the following series converge or not. Also test for their absolute convergence.

1. $\sum\limits_{n=1}^\infty \sin(n^{-\alpha}),\ \alpha > 0$; 2. $\sum\limits_{n=1}^\infty \dfrac{\log(n+1) - \log n}{\arctan \frac{2}{n}}$; 3. $\sum\limits_{n=2}^\infty \dfrac{(-1)^n}{\sqrt{n} + (-1)^n}$;

4. $\sum\limits_{n=1}^\infty \dfrac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{1 \cdot 2 \cdots n \cdot c(c+1)\cdots(c+n-1)}$; 5. $\sum\limits_{n=1}^\infty \dfrac{(-1)^n}{n+1}\Big(1 + \dfrac{1}{3} + \cdots + \dfrac{1}{2n+1}\Big)$.

The a, b, c in (4) are real numbers except negative integers.

Problem 6. Let (M, d) be a metric space.

1. Let A be a non-empty subset of M, and $d(:, A) : M \to \mathbb{R}$ be defined by

$$d(x, A) = \inf \{d(x, y) \,|\, y \in A\}$$

Prove or disprove the following inequality

$$d(x, y) \leqslant d(x, A) + d(y, A) \qquad \forall\, x, y \in M\,, A \subseteq M\,.$$

2. For non-empty subsets A, B of M, define $d(A, B) = \inf \{d(a, b) \,|\, a \in A, b \in B\}$. Show that

$$d(A, B) \leqslant d(x, A) + d(x, B) \qquad \forall\, x \in M\,.$$

Note that for subsets A, B, C in M, we do **NOT** have $d(A, B) \leqslant d(A, C) + d(C, B)$ in general. Find a counter-example for this inequality.

In Problem 7 through 12, we first introduce the concepts of accumulation points, isolated points and derived set of a set as follows. **Definition.** Let (M, d) be a normed vector space, and A be a subset of M.

1. A point $x \in M$ is called an ***accumulation point*** of A if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $A\backslash\{x\}$ such that $\{x_n\}_{n=1}^{\infty}$ converges to x.

2. A point $x \in A$ is called an ***isolated point*** (孤立點) (of A) if there exists no sequence in $A\backslash\{x\}$ that converges to x.

3. The ***derived set*** of A is the collection of all accumulation points of A, and is denoted by A'.

Problem 7. Let (M, d) be a metric space, and A be a subset of M.

1. Show that the collection of all isolated points of A is $A\backslash A'$.

2. Show that $A' = \bar{A}\backslash(A\backslash A')$. In other words, the derived set consists of all limit points that are not isolated points. Also show that $\bar{A}\backslash A' = A\backslash A'$.

Proof. 1. By the definition of isolated points of sets,

$$\begin{aligned} x \in A\backslash A' &\Leftrightarrow x \in A \text{ and } x \text{ is not an accumulation point of } A \\ &\Leftrightarrow x \in A \text{ and } \exists\, \varepsilon > 0 \ni B(x, \varepsilon) \cap A\backslash\{x\} = \varnothing \\ &\Leftrightarrow x \in A \text{ and } \exists\, \varepsilon > 0 \ni B(x, \varepsilon) \cap A \subseteq \{x\} \\ &\Leftrightarrow \exists\, \varepsilon > 0 \ni B(x, \varepsilon) \cap A = \{x\}\,; \end{aligned}$$

thus x is an isolated point of A if and only if $x \in A\backslash A'$.

2. First we show that $\bar{A} = A \cup A'$. To see this, let $x \in \bar{A} \backslash A$. By the fact that $A = A \backslash \{x\}$, there exists $\{x_n\}_{n=1}^{\infty} \subseteq A \backslash \{x\}$ such that $\lim_{n \to \infty} x_n = x$. Therefore, $x \in A'$ which implies that

$$\bar{A} \backslash A \subseteq A' \subseteq \bar{A},$$

where we use the fact that $\bar{A} \supseteq A'$ to conclude the last inclusion. The inclusion relation above then shows that

$$\bar{A} = A \cup \bar{A} = A \cup (\bar{A} \backslash A) \subseteq A \cup A' \subseteq A \cup \bar{A} = \bar{A};$$

thus we establish that $\bar{A} = A \cup A'$. This identity further shows that

$$\bar{A} \cap A^{\complement} = (A \cup A') \cap A^{\complement} = A' \cap A^{\complement} \subseteq A.$$

Now, using the identity $A \backslash B = A \cap B^{\complement}$ we find that

$$\begin{aligned} \bar{A} \backslash (A \backslash A') &= \bar{A} \cap \big(A \cap (A')^{\complement}\big)^{\complement} = \bar{A} \cap (A^{\complement} \cup A') = (\bar{A} \cap A^{\complement}) \cup (\bar{A} \cap A') \\ &= (\bar{A} \cap A^{\complement}) \cup A' = A'. \end{aligned}$$

Moreover, using $\bar{A} = A \cup A'$ we also have

$$\bar{A} \backslash A' = (A \cup A') \cap (A')^{\complement} = A \cap (A')^{\complement} = A \backslash A'.$$

□

Problem 8. Let A and B be subsets of a metric space (M, d). Show that

1. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
2. $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
3. $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(B)$. Find examples of that $\mathrm{cl}(A \cap B) \subsetneq \mathrm{cl}(A) \cap \mathrm{cl}(B)$.

Proof. 1. Since $\mathrm{cl}(A)$ is closed, by the definition of closed set we have $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

2. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we have $\mathrm{cl}(A) \subseteq \mathrm{cl}(A \cup B)$ and $\mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$; thus $\mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$. On the other hand, if $x \in \mathrm{cl}(A \cup B)$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $A \cup B$ such that $\lim_{n \to \infty} x_n = x$. Since $A \cup B$ contains infinitely many terms of $\{x_n\}_{n=1}^{\infty}$, at least one of A and B contains infinitely many terms of $\{x_n\}_{n=1}^{\infty}$. W.L.O.G., suppose that $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in A\big\} = \infty$. Let

$$\big\{n \in \mathbb{N} \,\big|\, x_n \in A\big\} = \big\{n_k \in \mathbb{N} \,\big|\, n_k < n_{k+1}\big\}.$$

Then $\{x_{n_k}\}_{k=1}^{\infty} \in A$. Since $x_n \to x$ as $n \to \infty$, we must have $x_{n_k} \to x$ as $k \to \infty$; thus $x \in \mathrm{cl}(A)$. Therefore, $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

3. Let $x \in \mathrm{cl}(A \cap B)$. Then

$$(\forall\, \varepsilon > 0)(B(x, \varepsilon) \cap (A \cap B) \neq \varnothing).$$

Therefore, by the fact that $B(x, \varepsilon) \cap A \subseteq B(x, \varepsilon) \cap (A \cap B)$ and $B(x, \varepsilon) \cap B \subseteq B(x, \varepsilon) \cap (A \cap B)$, we have

$$(\forall\, \varepsilon > 0)(B(x, \varepsilon) \cap A \neq \varnothing) \qquad \text{and} \qquad (\forall\, \varepsilon > 0)(B(x, \varepsilon) \cap B \neq \varnothing).$$

This implies that $x \in \bar{A} \cap \bar{B}$. Note that if $A = \mathbb{Q}$ and $B = \mathbb{Q}^{\complement}$, then $\mathrm{cl}(A \cap B) = \varnothing$ while $\bar{A} = \bar{B} = \mathbb{R}$ which provides an example of $\mathrm{cl}(A \cap B) \subsetneq \bar{A} \cap \bar{B}$. ▫

Problem 9. Let A and B be subsets of a metric space (M, d). Show that

1. $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.
2. $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$.
3. $\mathrm{int}(A \cup B) \supseteq \mathrm{int}(A) \cup \mathrm{int}(B)$. Find examples of that $\mathrm{int}(A \cup B) \supsetneq \mathrm{int}(A) \cup \mathrm{int}(B)$.

Proof. 1. Since $\mathrm{int}(A)$ is open, by the definition of open sets we have $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.

2. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we have $\mathrm{int}(A \cap B) \subseteq \mathrm{int}(A)$ and $\mathrm{int}(A \cap B) \subseteq \mathrm{int}(B)$; thus $\mathrm{int}(A \cap B) \subseteq \mathrm{int}(A) \cap \mathrm{int}(B)$. On the other hand, let $x \in \mathrm{int}(A) \cap \mathrm{int}(B)$. Then $x \in \mathrm{int}(A)$ and $x \in \mathrm{int}(B)$; thus there exist $r_1, r_0 > 0$ such that

$$B(x, r_1) \subseteq A \qquad \text{and} \qquad B(x, r_) \subseteq B\,.$$

Let $r = \min\{r_1, r_2\}$. Then $r > 0$, and $B(x, r) \subseteq B(x, r_1)$ and $B(x, r) \subseteq B(x, r_2)$. Therefore, $B(x, r) \subseteq A$ and $B(x, r) \subseteq B$ which further implies that $B(x, r) \subseteq A \cap B$; thus $x \in \mathrm{int}(A \cap B)$.

3. Let $x \in \mathring{A} \cup \mathring{B}$. Then $x \in \mathring{A}$ or $x \in \mathring{B}$; thus there exists $r > 0$ such that $B(x, r) \subseteq A$ or $B(x, r) \subseteq B$. Therefore, there exists $r > 0$ such that $B(x, r) \subseteq A \cup B$ which shows that $\mathrm{int}(A \cup B) \supseteq \mathrm{int}(A) \cup \mathrm{int}(B)$. Note that if $A = \mathbb{Q}$ and $B = \mathbb{Q}^{\complement}$, then $\mathrm{int}(A \cup B) = \mathbb{R}$ while $\mathrm{int}(A) = \mathrm{int}(B) = \varnothing$; thus we obtain an example of $\mathrm{int}(A \cup B) \supsetneq \mathrm{int}(A) \cup \mathrm{int}(B)$. ▫

Problem 10. Let (M, d) be a metric space, and A be a subset of M. Show that

$$\partial A = \big(A \cap \mathrm{cl}(M \backslash A)\big) \cup \big(\mathrm{cl}(A) \backslash A\big).$$

Proof. By the definition of the boundary, $\partial A = \bar{A} \cap \overline{A^{\complement}}$; thus

$$\begin{aligned}
\big(A \cap \mathrm{cl}(M \backslash A)\big) \cup \big(\mathrm{cl}(A) \backslash A\big) &= \big(A \cap \overline{A^{\complement}}\big) \cup \big(\bar{A} \cap A^{\complement}\big) \\
&= \big[A \cup \big(\bar{A} \cap A^{\complement}\big)\big] \cap \big[\overline{A^{\complement}} \cup \big(\bar{A} \cap A^{\complement}\big)\big] = \bar{A} \cap \big[\big(\overline{A^{\complement}} \cup \bar{A}\big) \cap \big(\overline{A^{\complement}} \cup A^{\complement}\big)\big] \\
&= \bar{A} \cap \big[\big(\overline{A^{\complement}} \cup \bar{A}\big) \cap \overline{A^{\complement}}\big] = \partial A \cap \big(\overline{A^{\complement}} \cup \bar{A}\big) = \partial A\,,
\end{aligned}$$

where the last equality follows from that $\partial A \subseteq \bar{A}$ and $\partial A \subseteq \overline{A^{\complement}}$. ▫

Problem 11. Recall that in a metric space (M, d), a subset A is said to be dense in S if subsets satisfy $A \subseteq S \subseteq \mathrm{cl}(A)$. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$.

1. Show that if A is dense in S and if S is dense in T, then A is dense in T.
2. Show that if A is dense in S and $B \subseteq S$ is open, then $B \subseteq \mathrm{cl}(A \cap B)$.

Proof. 1. If A is dense in S and if S is dense in T, then $A \subseteq S \subseteq \bar{A}$ and $S \subseteq T \subseteq \bar{S}$. Since $S \subseteq \bar{A}$, we must have $\bar{S} \subseteq \bar{A}$; thus

$$A \subseteq S \subseteq T \subseteq \bar{S} \subseteq \bar{A}$$

which shows that A is dense in T.

2. Let $x \in B$. Since B is open, there exists $\varepsilon_0 > 0$ such that $B(x, \varepsilon_0) \subseteq B \subseteq S$. On the other hand, $x \in S$ since B is a subset of S; thus the denseness of A in S implies that

$$(\forall\, \varepsilon > 0)(B(x, \varepsilon) \cap A \neq \varnothing)\,.$$

Therefore, for a given $\varepsilon > 0$, if $\varepsilon \geqslant \varepsilon_0$, then

$$B(x, \varepsilon) \cap (A \cap B) \supseteq B(x, \varepsilon_0) \cap (A \cap B) = B(x, \varepsilon_0) \cap A \neq \varnothing)$$

while if $\varepsilon < \varepsilon_0$, then

$$B(x, \varepsilon) \cap (A \cap B) = B(x, \varepsilon) \cap A \neq \varnothing\,.$$

This implies that

$$(\forall\, \varepsilon > 0)(B(x, \varepsilon) \cap (A \cap B) \neq \varnothing)\,;$$

thus $x \in \mathrm{cl}(A \cap B)$. □

Problem 12. Let A and B be subsets of a metric space (M, d). Show that

1. $\partial(\partial A) \subseteq \partial(A)$. Find examples of that $\partial(\partial A) \subsetneq \partial A$. Also show that $\partial(\partial A) = \partial A$ if A is closed.
2. $\partial(A \cup B) \subseteq \partial A \cup \partial B \subseteq \partial(A \cup B) \cup A \cup B$. Find examples of that equalities do not hold.
3. If $\mathrm{cl}(A) \cap \mathrm{cl}(B) = \varnothing$, then $\partial(A \cup B) = \partial A \cup \partial B$.
4. $\partial(A \cap B) \subseteq \partial A \cup \partial B$. Find examples of the equalities do not hold.
5. $\partial(\partial(\partial A)) = \partial(\partial A)$.

Proof. 1. We note that if F is closed, then

$$\partial F = \bar{F} \cap \overline{F^{\complement}} = F \cap \overline{F^{\complement}} \subseteq F\,. \tag{$\diamond$}$$

Since ∂F is closed, we must have $\partial(\partial A) \subseteq \partial A$. Note that if $A = \mathbb{Q} \cap [0, 1]$, then $\partial A = [0, 1]$; thus $\partial(\partial A) = \{0, 1\} \subsetneq \partial A$. Finally we show that $\partial(\partial A) = \partial A$ if A is closed. Using ($\diamond$), it suffices to show that $\partial A \subseteq \partial(\partial A)$. Using 2 of Problem 8,

$$\begin{aligned}\partial(\partial A) &= \partial A \cap \mathrm{cl}((\partial A)^{\complement}) = \partial A \cap \mathrm{cl}(A^{\complement} \cup \overline{A^{\complement}}^{\complement}) = \partial A \cap \big(\overline{A^{\complement}} \cup \mathrm{cl}(\overline{A^{\complement}}^{\complement}\big) \\ &= \big(\partial A \cap \overline{A^{\complement}}\big) \cup \big(\partial A \cap \mathrm{cl}(\overline{A^{\complement}}^{\complement})\big) \supseteq \big(\partial A \cap \overline{A^{\complement}}\big) = \partial A\,.\end{aligned}$$

2. Using 2 and 3 of Problem 8,

$$\begin{aligned}\partial(A \cup B) &= \overline{A \cup B} \cap \mathrm{cl}\big((A \cup B)^{\complement}\big) = \big(\bar{A} \cup \bar{B}\big) \cap \mathrm{cl}(A^{\complement} \cap B^{\complement}) \subseteq \big(\bar{A} \cup \bar{B}\big) \cap \big(\overline{A^{\complement}} \cap \overline{B^{\complement}}\big) \\ &= \big(\bar{A} \cap \overline{A^{\complement}} \cap \overline{B^{\complement}}\big) \cup \big(\bar{B} \cap \overline{A^{\complement}} \cap \overline{B^{\complement}}\big) \subseteq \big(\bar{A} \cap \overline{A^{\complement}}\big) \cup \big(\bar{B} \cap \overline{B^{\complement}}\big) = \partial A \cup \partial B\,.\end{aligned}$$

On the other hand, since $\partial A = \bar{A} \backslash \mathring{A}$ and $\mathring{A} \subseteq A$, we have

$$\bar{A} \subseteq A \cup \partial A \subseteq \mathring{A} \cup (\bar{A} \backslash \mathring{A}) = \bar{A}$$

which implies that $A \cup \partial A = \bar{A}$. Therefore,

$$\partial A \subseteq \bar{A} \subseteq \overline{A \cup B} = A \cup B \cup \partial(A \cup B)$$

and similarly $\partial B \subseteq A \cup B \cup \partial(A \cup B)$. Therefore,

$$\partial A \cup \partial B \subseteq \partial(A \cup B) \cup A \cup B\,.$$

Note that if $A = [-1, 0] \cup \big(\mathbb{Q} \cap [0, 1]\big)$ and $B = [-1, 0] \cup (\mathbb{Q}^{\complement} \cap [0, 1])$, then $A \cup B = [-1, 1]$, $\partial A = \partial B = \{-1\} \cup [0, 1]$ which implies that

$$\partial(A \cup B) = \{-1, 1\} \subsetneq \partial A \cup \partial B \subsetneq A \cup B = \partial(A \cup B) \cup A \cup B\,.$$

3. By 2, it suffices to shows that $\partial A \cup \partial B \subseteq \partial(A \cup B)$ if $\bar{A} \cap \bar{B} = \varnothing$. Let $x \in \partial A \cup \partial B$. W.L.O.G., assume that $x \in \partial A$. Then $x \in \bar{A}$; thus $x \notin \bar{B}$ which further implies that there exists $\varepsilon_0 > 0$ such that $B(x, \varepsilon_0) \cap B = \varnothing$ or equivalently, $B(x, \varepsilon_0) \subseteq B^{\complement}$. Therefore, for given $r > 0$, if $r < \varepsilon_0$, then

$$B(x, r) \cap (A \cup B) \supseteq B(x, r) \cap A \neq \varnothing$$

and

$$B(x, r) \cap \big((A \cup B)^{\complement}\big) = B(x, r) \cap (A^{\complement} \cap B^{\complement}) = B(x, r) \cap A^{\complement} \neq \varnothing$$

while if $r \geqslant \varepsilon_0$, then

$$B(x, r) \cap (A \cup B) \subseteq B(x, \varepsilon_0) \cap (A \cup B) \supseteq B(x, \varepsilon_0) \cap A \neq \varnothing$$

and

$$B(x, r) \cap \big((A \cup B)^{\complement}\big) \supseteq B(x, \varepsilon_0) \cap (A^{\complement} \cap B^{\complement}) = B(x, \varepsilon_0) \cap A^{\complement} \neq \varnothing\,.$$

As a consequence, for each $r > 0$,

$$B(x, r) \cap (A \cup B) \neq \varnothing \quad \text{and} \quad B(x, r) \cap (A \cup B)^{\complement}\,;$$

thus $x \in \overline{A \cup B}$ and $x \in \mathrm{cl}\big((A \cup B)^{\complement}\big)$ which implies that $x \in \partial(A \cup B)$.

4. Using 2 and 3 of Problem 8,

$$\begin{aligned}\partial(A \cap B) &= \overline{A \cap B} \cap \mathrm{cl}\big((A \cap B)^{\complement}\big) = \overline{A \cap B} \cap \mathrm{cl}(A^{\complement} \cup B^{\complement}) \subseteq \big(\bar{A} \cap \bar{B}\big) \cap \big(\overline{A^{\complement}} \cup \overline{B^{\complement}}\big) \\ &= \big[\big(\bar{A} \cap \bar{B}\big) \cap \overline{A^{\complement}}\big] \cup \big[\big(\bar{A} \cap \bar{B}\big) \cap \overline{B^{\complement}}\big] \subseteq \big(\bar{A} \cap \overline{A^{\complement}}\big) \cup \big(\bar{B} \cap \overline{B^{\complement}}\big) = \partial A \cup \partial B\,.\end{aligned}$$

Note that if $A = \mathbb{Q}$ and $B = \mathbb{Q}^{\complement}$, then $\partial A = \partial B = \mathbb{R}$ but

$$\partial(A \cap B) = \varnothing \subsetneq \mathbb{R} = \partial A \cap \partial B\,.$$

5. Since ∂A is closed, 1 implies that $\partial(\partial(\partial A)) = \partial(\partial A)$. □