分析導論

An Introduction to Mathematical Analysis

Ching-hsiao Arthur Cheng 鄭經斅

Copyright

Contents

0 Review of Contents from Basic Mathematics **i**

0.1 Sets i

0.2 Functions iii

0.3 Countability of Sets iv

1 The Real Number System and Completeness **1**

1.1 Ordered Fields 1

1.2 Sequences in Ordered Fields 9

1.3 Monotone Sequence Property 14

1.4 Least Upper Bound Property 16

1.5 Bolzano-Weierstrass Property 19

1.6 Cauchy Sequences 23

1.7 Completeness 24

1.8 Limit Inferior and Limit Superior 28

2 Normed Vector Spaces and Metric Spaces **34**

2.1 Euclidean Spaces and Vector Spaces 34

2.2 Normed Vector Spaces, Inner Product Spaces and Metric Spaces 36

2.3 Sequences in Metric Spaces 45

2.3.1 Sequences 45

2.3.2 Cauchy sequences, Banach spaces and Hilbert spaces 51

2.4 Series of Real Numbers and Vectors 52

3 Elementary Point-Set Topology **58**

3.1 Limit Points and Interior Points of Sets 58

3.2 Closed Sets and Open Sets 60

3.2.1 Closed sets . . . 60
3.2.2 Open sets . . . 64
3.3 Compactness（緊緻性） . . . 68
3.4 Connectedness（連通性） . . . 76
3.5 Subspace Topology . . . 78
3.6 Exercises . . . 81

4 Continuous Maps **92**
4.1 Continuity . . . 92
4.2 Operations on Continuous Maps . . . 98
4.3 Images under Continuous Maps . . . 100
4.3.1 Image of compact sets . . . 100
4.3.2 Image of connected sets . . . 105
4.4 Uniform Continuity（均勻連續） . . . 106

5 Differentiation of Maps **112**
5.1 Bounded Linear Maps . . . 112
5.2 Definition of Derivatives . . . 117
5.3 Continuity of Differentiable Maps . . . 125
5.4 Conditions for Differentiability . . . 126
5.5 The Product Rule and the Chain Rule . . . 130
5.6 Higher Derivatives of Functions . . . 132
5.7 Taylor's Theorem . . . 144

6 Integration of Functions **150**
6.1 Integrable Functions . . . 150
6.2 The Lebesgue Theorem . . . 159
6.2.1 Volume and sets of measure zero . . . 159
6.2.2 The Lebesgue theorem . . . 162
6.3 Properties of the Integrals . . . 169
6.4 The Fubini Theorem . . . 175
6.5 The Monotone and Bounded Convergence Theorems . . . 182
6.6 Improper Integrals . . . 189

6.6.1 The monotone convergence theorem and the dominated convergence theorem . . . 203
6.6.2 The Fubini theorem and the Tonelli theorem . . . 206

7 Uniform Convergence and the Space of Continuous Functions 210
7.1 Pointwise Convergence and Uniform Convergence（逐點收斂與均勻收斂） . 210
7.2 Series of Functions and The Weierstrass M-Test . . . 219
7.3 Integration and Differentiation of Series . . . 223
7.4 The Space of Continuous Functions . . . 227
7.5 The Arzelà-Ascoli Theorem . . . 230
7.5.1 Equi-continuous family of functions . . . 230
7.5.2 Compact sets in $\mathscr{C}(K;\mathcal{V})$ . . . 234
7.6 The Stone-Weierstrass Theorem . . . 237
7.7 Exercises . . . 244

8 Fourier Series 254
8.1 Basic Properties of the Fourier Series . . . 255
8.2 Uniform Convergence of the Fourier Series . . . 259
8.3 Cesàro Mean of Fourier Series . . . 266
8.4 Convergence of Fourier Series for Functions with Jump Discontinuity . . . 268
8.4.1 Uniform convergence on compact subsets . . . 269
8.4.2 Gibbs phenomenon . . . 272
8.5 The Inner-Product Point of View . . . 274
8.6 The Discrete Fourier "Transform" and the Fast Fourier "Transform" . . . 279
8.6.1 The inversion formula . . . 281
8.6.2 The fast Fourier transform . . . 282
8.7 Fourier Series for Functions of Two Variables . . . 284

9 Fourier Transforms 286
9.1 The Definition of the Fourier Transform . . . 287
9.2 Some Properties of the Fourier Transform . . . 288
9.3 The Fourier Inversion Formula . . . 292
9.4 The Fourier Transform of Generalized Functions . . . 303

Chapter 0

Review of Contents from Basic Mathematics

0.1 Sets

Definition 0.1. A ***set*** is a collection of objects called ***elements*** or ***members*** of the set. To denote a set, we make a complete list $\{x_1, x_2, \cdots, x_N\}$ or use the notation

$$\{x : P(x)\} \quad \text{or} \quad \{x \,|\, P(x)\},$$

where the sentence $P(x)$ describes the property that defines the set. A set A is said to be a ***subset*** of S if every member of A is also a member of S. We write $x \in A$ (or A contains x) if x is a member of A, and write $A \subseteq S$ (or S includes A) if A is a subset of S. The empty set, denoted $\varnothing$, is the set with no member.

Definition 0.2. Let S be a given set, and $A \subseteq S$, $B \subseteq S$. The set $A \cup B$, called the ***union*** of A and B, consists of members belonging to set A or set B, and the set $A \cap B$, called the ***intersection*** of A and B, consists of members belonging to both set A and set B.

Let $\mathscr{F}$ be a collection of subsets in S. The set $\bigcup\limits_{A \in \mathscr{F}} A$, called the ***union*** of sets in $\mathscr{F}$, is defined by

$$\bigcup_{A \in \mathscr{F}} A = \{x \in S \,|\, (\exists\, A \in \mathscr{F})(x \in A)\},$$

and $\bigcap\limits_{A \in \mathscr{F}} A = \{x \in S \,|\, (\forall\, A \in \mathscr{F})(x \in A)\}$ is the ***intersection*** of sets in $\mathscr{F}$. When $\mathscr{F} = \{A_\alpha \,|\, \alpha \in I\}$, the union and the intersection of sets in $\mathscr{F}$ can also be written as $\bigcup\limits_{\alpha \in I} A_\alpha$ and

$\bigcap\limits_{\alpha \in I} A_\alpha$, respectively. When $\mathscr{F} = \big\{A_1, A_2, \cdots, A_N\big\}$, the union and the intersection of sets in $\mathscr{F}$ can also be written as $\bigcup\limits_{i=1}^{N} A_i$ and $\bigcap\limits_{i=1}^{N} A_i$, respectively.

Example 0.3. Let $\mathscr{F}$ be the collection of open intervals with length 2 and mid-point in $[0,1]$. Then $\mathscr{F} = \big\{(\alpha - 1, \alpha + 1) \,\big|\, \alpha \in [0,1]\big\}$. Moreover,

$$\bigcup_{A \in \mathscr{F}} A = \bigcup_{\alpha \in [0,1]} (\alpha - 1, \alpha + 1) = (-1, 2) \quad \text{and} \quad \bigcap_{A \in \mathscr{F}} A = \bigcap_{\alpha \in [0,1]} (\alpha - 1, \alpha + 1) = (0, 1)\,.$$

Definition 0.4. Let S be a given set, and $A \subseteq S$, $B \subseteq S$. The ***complement*** of A relative to B, denoted $B \backslash A$, is the set consisting of members of B that are not members of A. When the universal set S under consideration is fixed, the complement of A relative to S or simply the complement of A, is denoted by $A^{\complement}$, or $S \backslash A$.

Theorem 0.5. (De Morgan's Law)

1. $B \backslash \bigcup\limits_{\alpha \in I} A_\alpha = \bigcap\limits_{\alpha \in I} (B \backslash A_\alpha)$ *or* $B \backslash \bigcup\limits_{A \in \mathscr{F}} A = \bigcap\limits_{A \in \mathscr{F}} (B \backslash A)$.
2. $B \backslash \bigcap\limits_{\alpha \in I} A_\alpha = \bigcup\limits_{\alpha \in I} (B \backslash A_\alpha)$ *or* $B \backslash \bigcap\limits_{A \in \mathscr{F}} A = \bigcup\limits_{A \in \mathscr{F}} (B \backslash A)$.

Proof. By definition,

$$\begin{aligned} x \in B \backslash \bigcup_{\alpha \in I} A_\alpha &\Leftrightarrow x \in B \text{ and } x \notin \bigcup_{\alpha \in I} A_\alpha \Leftrightarrow x \in B \text{ and } x \notin A_\alpha \text{ for all } \alpha \in I \\ &\Leftrightarrow x \in B \backslash A_\alpha \text{ for all } \alpha \in I \Leftrightarrow x \in \bigcap_{\alpha \in I} (B \backslash A_\alpha) \end{aligned}$$

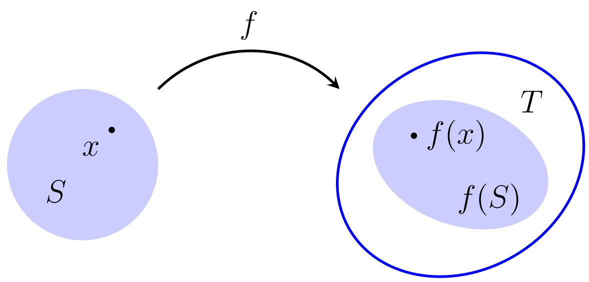
The proof of the second identity is similar, and is left as an exercise. □

Definition 0.6. An ***ordered pair*** (a, b) is an object formed from two objects a and b, where a is called the first coordinate and b the second coordinate. Two ordered pairs are equal whenever their corresponding coordinates are the same. An ***ordered*** n***-tuples*** $(a_1, a_2, \cdots, a_n)$ is an object formed from n objects $a_1, a_2, \cdots, a_n$, where for each j, a_j is called the j-th coordinate. Two n-tuples $(a_1, a_2, \cdots, a_n)$, $(c_1, c_2, \cdots, c_n)$ are equal if $a_j = c_j$ for all $j \in \{1, \cdots, n\}$.

Definition 0.7. Given sets A and B, the ***Cartesian product*** $A \times B$ of A and B is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$, $A \times B = \big\{(a, b) \,\big|\, a \in A \text{ and } b \in B\big\}$. The Cartesian of three or more sets are defined similarly.

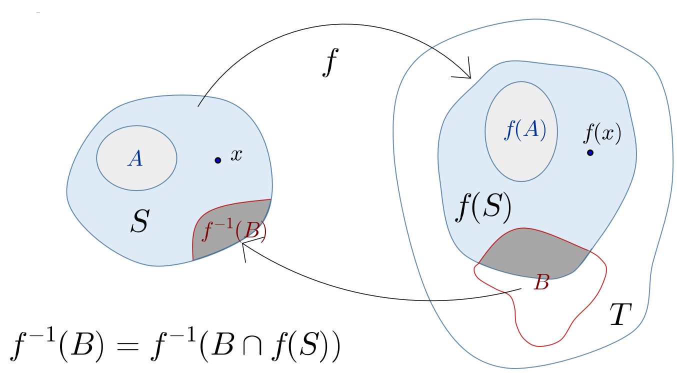
0.2 Functions

Definition 0.8. Let S and T be given sets. A ***function*** $f : S \to T$ consists of two sets S and T together with a "rule" that assigns to each $x \in S$ a special element of T denoted by $f(x)$. One writes $x \mapsto f(x)$ to denote that x is mapped to the element $f(x)$. S is called the ***domain*** (定義域) of f, and T is called the ***target*** or ***co-domain*** (對應域) of f. The ***range*** (值域) of f or the ***image*** of f, is the subset of T defined by $f(S) = \{f(x) \,|\, x \in S\}$.



Definition 0.9. A function $f : S \to T$ is called ***one-to-one*** (一對一), ***injective*** or an ***injection*** if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ (which is equivalent to that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$). A function $f : S \to T$ is called ***onto*** (映成), ***surjective*** or an ***surjection*** if $\forall\, y \in T,\ \exists\, x \in S,\ \ni f(x) = y$ (that is, $f(S) = T$). A function $f : S \to T$ is called an ***bijection*** if it is one-to-one and onto.

Definition 0.10. For $f : S \to T$, $A \subseteq S$, we call $f(A) = \{f(x) \,|\, x \in A\}$ the ***image*** of A under f. For $B \subseteq T$, we call $f^{-1}(B) = \{x \in S \,|\, f(x) \in B\}$ the ***pre-image*** of B under f.



Proposition 0.11. *Let $f : S \to T$ be a function, C_1 ,$C_2 \subseteq T$ and D_1, $D_2 \subseteq S$.*

(a) $f^{-1}(C_1 \cup C_2) = f^{-1}(C_1) \cup f^{-1}(C_2)$.

(b) $f(D_1 \cup D_2) = f(D_1) \cup f(D_2)$.

(c) $f^{-1}(C_1 \cap C_2) = f^{-1}(C_1) \cap f^{-1}(C_2)$.

(d) $f(D_1 \cap D_2) \subseteq f(D_1) \cap f(D_2)$.

(e) $f^{-1}(f(D_1)) \supseteq D_1$ (*"$=$" if f is one-to-one*).

(f) $f(f^{-1}(C_1)) \subseteq C_1$ (*"$=$" if $C_1 \subseteq f(S)$*).

Proof. We only prove (c) and (d), and the proof of the other statements are left as an exercise.

(c) We first show that $f^{-1}(C_1 \cap C_2) \subseteq f^{-1}(C_1) \cap f^{-1}(C_2)$. Suppose that $x \in f^{-1}(C_1 \cap C_2)$. Then $f(x) \in C_1 \cap C_2$. Therefore, $f(x) \in C_1$ and $f(x) \in C_2$, or equivalently, $x \in f^{-1}(C_1)$ and $x \in f^{-1}(C_2)$; thus $x \in f^{-1}(C_1) \cap f^{-1}(C_2)$.

Next, we show that $f^{-1}(C_1) \cap f^{-1}(C_2) \subseteq f^{-1}(C_1 \cap C_2)$. Suppose that $x \in f^{-1}(C_1) \cap f^{-1}(C_2)$. Then $x \in f^{-1}(C_1)$ and $x \in f^{-1}(C_2)$ which implies that $f(x) \in C_1$ and $f(x) \in C_2$; thus $f(x) \in C_1 \cap C_2$ or equivalently, $x \in f^{-1}(C_1 \cap C_2)$.

(d) Suppose that $y \in f(D_1 \cap D_2)$. Then there exists $x \in D_1 \cap D_2$ such that $y = f(x)$. As a consequence, $y \in f(D_1)$ and $y \in f(D_2)$ which implies that $y \in f(D_1) \cap f(D_2)$. □

0.3 Countability of Sets

Definition 0.12. A set S is called ***denumerable*** or ***countably infinite*** (無窮可數的) if S can be put into one-to-one correspondence with $\mathbb{N}$; that is, S is denumerable if and only if there exists $f : \mathbb{N} \to S$ which is one-to-one and onto. A set is called ***countable*** (可數的) if it is either finite or denumerable, and is called ***uncountable*** if it is not countable.

Remark 0.13. If $f : \mathbb{N} \xrightarrow[\textit{onto}]{1-1} S$, then $f^{-1} : S \xrightarrow[\textit{onto}]{1-1} \mathbb{N}$. Therefore,

$$S \text{ is denumerable} \Leftrightarrow \exists\, f : \mathbb{N} \xrightarrow[\textit{onto}]{1-1} S \Leftrightarrow \exists\, g = f^{-1} : S \xrightarrow[\textit{onto}]{1-1} \mathbb{N}.$$

f can be thought as a rule of counting/labeling elements in S since $S = \{f(1), f(2), \cdots\}$.

Example 0.14. $\mathbb{N}$ is countable since $f : \mathbb{N} \xrightarrow[\textit{onto}]{1-1} \mathbb{N}$ with $f(x) = x$, $\forall\, n \in \mathbb{N}$.

Example 0.15. $\mathbb{Z}$ is countable. $f:\mathbb{Z}\to\mathbb{N}$ with $f(x)=\begin{cases}1 & \text{if } x=0\\ 2x & \text{if } x>0\\ -2x+1 & \text{if } x<0\end{cases}.$

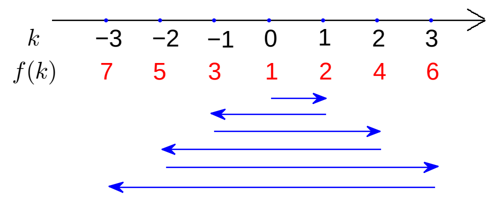


Figure 1: An illustration of how elements in $\mathbb{Z}$ are labeled

Example 0.16. The set $\mathbb{N}\times\mathbb{N}=\big\{(a,b)\,\big|\,a,b\in\mathbb{N}\big\}$ is countable. In fact, two ways of mapping are shown in the figures below.

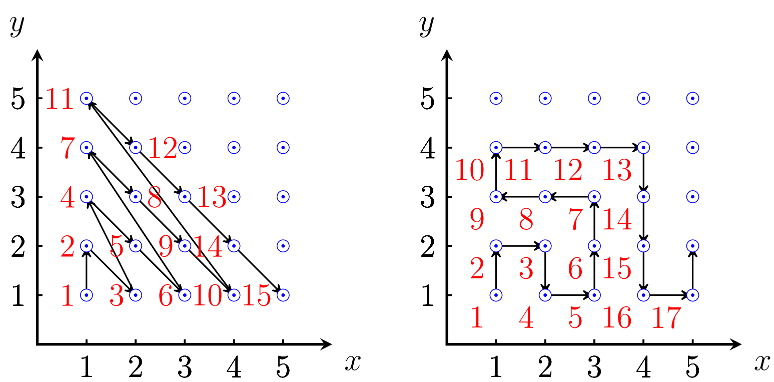


Figure 2: The illustration of two ways of labeling elements in $\mathbb{N}\times\mathbb{N}$

Proposition 0.17. *Let S be a non-empty set. The following three statements are equivalent:*

(a) *S is countable;*

(b) *there exists a surjection $f:\mathbb{N}\to S$;*

(c) *there exists an injection $f:S\to\mathbb{N}$.*

Proof. "(a) $\Rightarrow$ (b)" First suppose that $S=\{x_1,\cdots,x_n\}$ is finite. Define $f:\mathbb{N}\to S$ by

$$f(k)=\begin{cases}x_k & \text{if } k<n\,,\\ x_n & \text{if } k\geqslant n\,.\end{cases}$$

Then $f:\mathbb{N}\to S$ is a surjection. Now suppose that S is denumerable. Then by definition of countability, there exists $f:\mathbb{N}\xrightarrow[\textit{onto}]{1-1}S$.

"(a) $\Leftarrow$ (b)" W.L.O.G. (without loss of generality, 不失一般性) we assume that S is an infinite set. Let $k_1 = 1$. Since $\#(S) = \infty$, $S_1 \equiv S\backslash\{f(k_1)\} \neq \varnothing$; thus $N_1 \equiv f^{-1}(S_1)$ is a non-empty subset of $\mathbb{N}$. By the well-ordered property of $\mathbb{N}$ (that is, non-empty subset of $\mathbb{N}$ has least element), N_1 has a smallest element denoted by k_2. Since $\#(S) = \infty$, $S_2 = S\backslash\{f(k_1), f(k_2)\} \neq \varnothing$; thus $N_2 \equiv f^{-1}(S_2)$ is a non-empty subset of $\mathbb{N}$ and possesses a smallest element denoted by k_3. We continue this process and obtain a set $\{k_1, k_2, \cdots\} \subseteq \mathbb{N}$, where $k_1 < k_2 < \cdots$, and k_j is the smallest element of $N_{j-1} \equiv f^{-1}(S\backslash\{f(k_1), f(k_2), \cdots, f(k_{j-1})\})$.

Claim: $f : \{k_1, k_2, \cdots\} \to S$ is one-to-one and onto.

Proof of claim: The injectivity of f is due to that $f(k_j) \notin \big\{f(k_1), f(k_2), \cdots, f(k_{j-1})\big\}$ for all $j \geqslant 2$. For surjectivity, assume that there is $s \in S$ such that $s \notin f(\{k_1, k_2, \cdots\})$. Since $f : \mathbb{N} \to \mathbb{S}$ is onto, $f^{-1}(\{s\})$ is a non-empty subset of $\mathbb{N}$; thus possesses a smallest element k. Since $s \notin f(\{k_1, k_2, \cdots\})$, there exists $\ell \in \mathbb{N}$ such that $k_\ell < k < k_{\ell+1}$. As a consequence, there exists $k \in N_\ell$ such that $k < k_{\ell+1}$ which contradicts to the fact that $k_{\ell+1}$ is the smallest element of N_ℓ.

Define $g : \mathbb{N} \to \{k_1, k_2, \cdots\}$ by $g(j) = k_j$. Then $g : \mathbb{N} \to \{k_1, k_2, \cdots\}$ is one-to-one and onto; thus $h = g \circ f : \mathbb{N} \xrightarrow[onto]{1-1} S$.

"(a) $\Rightarrow$ (c)" If $S = \{x_1, \cdots, x_n\}$ is finite, we simply let $f : S \to \mathbb{N}$ be $f(x_n) = n$. Then f is clearly an injection. If S is denumerable, by definition there exists $g : \mathbb{N} \xrightarrow[onto]{1-1} S$ which shows that $f = g^{-1} : S \to \mathbb{N}$ is an injection.

"(a) $\Leftarrow$ (c)" Let $f : S \to \mathbb{N}$ be an injection. If f is also surjective, then $f : S \xrightarrow[onto]{1-1} \mathbb{N}$ which implies that S is denumerable. Now suppose that $f(S) \subsetneq \mathbb{N}$. Since S is non-empty, there exists $s \in S$. Let $g : \mathbb{N} \to S$ be defined by

$$g(n) = \begin{cases} f^{-1}(n) & \text{if } n \in f(S)\,, \\ s & \text{if } n \notin f(S)\,. \end{cases}$$

Then clearly $g : \mathbb{N} \to S$ is surjective; thus the equivalence between (a) and (b) implies that S is countable. □

Example 0.18. The set $\mathbb{N} \times \mathbb{N}$ is countable since the map $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f\big((m, n)\big) = 2^m 3^n$ is an injection.

Theorem 0.19. *Any non-empty subset of a countable set is countable.*

Proof. Let S be a countable set, and A be a non-empty subset of S. Since S is countable, by by Proposition 0.17 there exists a surjection $f : \mathbb{N} \to S$. On the other hand, since A is a non-empty subset of S, there exists $a \in A$. Define

$$g(x) = \begin{cases} x & \text{if } x \in A\,, \\ a & \text{if } x \notin A\,. \end{cases}$$

Then $h = g \circ f : \mathbb{N} \to A$ is a surjection, and Proposition 0.17 implies that A is countable. □

Theorem 0.20. *The union of countable countable sets is countable.* (可數個可數集的聯集是可數的)

Proof. Let A_i be countable, and define $A = \bigcup\limits_{i=1}^{\infty} A_i$. Write $A_i = \{x_{i1}, x_{i2}, x_{i3}, \cdots\}$. Then $A = \big\{x_{ij} \,\big|\, i = 1, 2, \cdots,\ j < \#(A_i) + 1\big\}$, where $\#(A_i) = \infty$ if A_i is countably infinite. Let $S = \big\{(i,j) \,\big|\, i = 1, 2, \cdots, j < \#(A_i) + 1\big\}$, and define $f : S \to A$ by $f\big((i,j)\big) = x_{ij}$. Then $f : S \to A$ is a surjection. On the other hand, since S is a subset of $\mathbb{N} \times \mathbb{N}$, Theorem 0.19 implies that S is countable; thus Proposition 0.17 guarantees the existence of a surjection $g : \mathbb{N} \to S$. Then $h = f \circ g : \mathbb{N} \to A$ is a surjection which, by Proposition 0.17 again, implies that A is countable. □

Example 0.21. $\mathbb{Z} \times \mathbb{Z}$ is countable.

Proof. For $i \in \mathbb{Z}$, let $A_i = \big\{(i,j) \,\big|\, j \in \mathbb{Z}\big\}$. By Example 0.15, A_i is countable for all $i \in \mathbb{Z}$. Since $\mathbb{Z} \times \mathbb{Z} = \bigcup_{i \in \mathbb{Z}} A_i$ which is countable union of countable sets, Theorem 0.20 implies that $\mathbb{Z} \times \mathbb{Z}$ is countable. □

Theorem 0.22. $\mathbb{Q}$ *is countable.*

Proof. Define

$$f(x) = \begin{cases} (p,q), & \text{if } x > 0, \quad x = \dfrac{q}{p}, \quad \gcd(p,q) = 1,\ p > 0. \\ (0,0), & \text{if } x = 0. \\ (p,-q), & \text{if } x < 0, \quad x = -\dfrac{q}{p}, \quad \gcd(p,q) = 1,\ p > 0. \end{cases}$$

Then $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ is one-to-one; thus $f : \mathbb{Q} \xrightarrow[onto]{1-1} f(\mathbb{Q})$. Since $\mathbb{Z} \times \mathbb{Z}$ is countable, its non-empty subset $f(\mathbb{Q})$ is also countable. As a consequence, there exists $g : f(\mathbb{Q}) \xrightarrow[onto]{1-1} \mathbb{N}$; thus $h = g \circ f : \mathbb{Q} \xrightarrow[onto]{1-1} \mathbb{N}$. □

Theorem 0.23. *The open interval* $(0,1)$ *is uncountable.*

Proof. Assume the contrary that there exists $f:\mathbb{N}\to(0,1)$ which is one-to-one and onto. Write $f(k)$ in decimal expansion (十進位展開); that is,

$$\begin{aligned} f(1) &= 0.d_{11}d_{21}d_{31}\cdots \\ f(2) &= 0.d_{12}d_{22}d_{32}\cdots \\ \vdots & \qquad \vdots \\ f(k) &= 0.d_{1k}d_{2k}d_{3k}\cdots \\ \vdots & \qquad \vdots \end{aligned}$$

Here we note that repeated 9's are chosen by preference over terminating decimals; that is, for example, we write $\dfrac{1}{4}=0.249999\cdots$ instead of $\dfrac{1}{4}=0.250000\cdots$.

Let $x\in(0,1)$ be such that $x=0.d_1d_2\cdots$, where

$$d_k=\begin{cases} 5 & \text{if} \quad d_{kk}\neq 5\,, \\ 3 & \text{if} \quad d_{kk}=5\,. \end{cases}$$

(建構一個 x 使其小數點下第 k 位數與 $f(k)$ 的小數點下第 k 位數不相等). Then $x\neq f(k)$ for all $k\in\mathbb{N}$, a contradiction; thus $(0,1)$ is uncountable. □

Corollary 0.24. *The collection of real numbers is uncountable.*

Chapter 1

The Real Number System and Completeness

1.1 Ordered Fields

Definition 1.1. A set $\mathbb{F}$ is said to be a ***field*** (體) if there are two operations $+$ and $\cdot$ such that

1. $x + y \in \mathbb{F}$, $x \cdot y \in \mathbb{F}$ if $x, y \in \mathbb{F}$. (封閉性)
2. $x + y = y + x$ for all $x, y \in \mathbb{F}$. (commutativity, 加法的交換性)
3. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}$. (associativity, 加法的結合性)
4. There exists $0 \in \mathbb{F}$, called the additive identity (加法單位元素), such that $x + 0 = x$ for all $x \in \mathbb{F}$. (the existence of zero)
5. For every $x \in \mathbb{F}$, there exists $y \in \mathbb{F}$ (usually y is denoted by $-x$ and is called the additive inverse (加法反元素) of x) such that $x + y = 0$. One writes $x - y \equiv x + (-y)$.
6. $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{F}$. (乘法的交換性)
7. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathbb{F}$. (乘法的結合性)
8. There exists $1 \in \mathbb{F}$, called the multiplicative identity (乘法單位元素), such that $x \cdot 1 = x$ for all $x \in \mathbb{F}$. (the existence of unity)
9. For every $x \in \mathbb{F}$, $x \neq 0$, there exists $y \in \mathbb{F}$ (usually y is denoted by x^{-1}, and is

called the multiplicative inverse (乘法反元素) of x) such that $x \cdot y = 1$. One writes $x \cdot y \equiv x \cdot x^{-1} = 1$.

10. $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{F}$. (distributive law, 分配律)
11. $0 \neq 1$.

Remark 1.2. Let x and y be both multiplicative inverse (乘法反元素) of a number a in $(\mathbb{F}, +, \cdot)$. Then

$$x \cdot a = 1 \quad \Rightarrow \quad (x \cdot a) \cdot y = 1 \cdot y = y \quad \Rightarrow \quad x \cdot 1 = x \cdot (a \cdot y) = y\,;$$

thus $x = y$. In other words, the multiplicative inverse of a number is unique. Similarly, the additive inverse of a number is also unique.

Remark 1.3. A set $\mathbb{F}$ satisfying properties 1 to 10 with $0 = 1$ consists of only one member: By distributive law, $x \cdot 0 = x \cdot (0+0) = x \cdot 0 + x \cdot 0$; thus $-(x \cdot 0) + (x \cdot 0) = -(x \cdot 0) + (x \cdot 0) + (x \cdot 0)$ which implies that $x \cdot 0 = 0$. Therefore, if $0 = 1$, then $x = x \cdot 1 = x \cdot 0 = 0$ for all $x \in \mathbb{F}$. Hence, the set $\mathbb{F}$ consists only one element 0.

Remark 1.4. If $x \in \mathbb{F}$, then $\big((1 + (-1)\big) \cdot x = 0$ which implies that $x + (-1) \cdot x = 0$. Therefore, $(-1) \cdot x = -x + x + (-1) \cdot x = -x + 0 = -x$.

Example 1.5. Let $\mathbb{F} = \{a, b, c\}$ with the operations $+$ and $\cdot$ defined by

$+$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\cdot$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

.

Then $\mathbb{F}$ is a field because of the following: Properties 1, 2, 3, 6, 7 are obvious.
Property 4: $\exists$ "0" $\ni x +$ "0" $= x$ for all $x \in \mathbb{F}$. In fact, "0" $= a$.
Property 5: $\forall\, x \in \mathbb{F}$, $\exists\, y \in \mathbb{F} \ni x + y = 0$, here $b = -c$, $c = -b$.
Property 8: $\exists$ "1" $\ni x \cdot$ "1" $= x$ for all $x \in \mathbb{F}$. In fact, "1" $= b$ (so Property 11 holds since $a \neq b$).
Property 9: $\forall\, x \neq 0$, $\in \mathbb{F}$, $\exists\, z \in \mathbb{F} \ni x \cdot z = 1$, here $z = x$.
The validity of Property 10 is left as an exercise.

Example 1.6. Let $(\mathbb{F}, +, \cdot)$ be a field. Consider the set $\mathcal{F} = \mathbb{F} \times \mathbb{F} = \big\{(a, b) \,\big|\, a, b \in \mathbb{F}\big\}$. Define

$$(a, b) \oplus (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b) \odot (c, d) = (a \cdot c - b \cdot d, a \cdot d + b \cdot c)\,.$$

Then $(\mathcal{F}, \oplus, \odot)$ is also a field. The ordered pair (a, b) in $\mathcal{F}$ is sometimes denoted by $a + bi$.

Example 1.7. Let $(\mathbb{F},+,\cdot)$ be a field. Then $(x-y)(x+y)=x^2-y^2$ for all $x,y\in\mathbb{F}$. In fact,

$$\begin{aligned}
(x-y)(x+y) &= (x-y)\cdot x+(x-y)\cdot y && \text{(by 分配律)}\\
&= x\cdot(x-y)+y\cdot(x-y) && \text{(by 乘法交換律)}\\
&= x\cdot x+x\cdot(-y)+y\cdot x+y\cdot(-y) && \text{(by 分配律)}\\
&= x^2-x\cdot y+x\cdot y-y^2 && \text{(by Remark 1.4 and 乘法交換律)}\\
&= x^2+0-y^2 && \text{(by Property 5)}\\
&= x^2-y^2 && \text{(by Property 4).}
\end{aligned}$$

Definition 1.8. An ***ordered field*** (有序體) is a field $(\mathbb{F},+,\cdot)$ equipped with a relation $\leqslant$ on $\mathbb{F}$ satisfying that

1. $x\leqslant x$ for all $x\in\mathbb{F}$ (reflexivity).
2. If $x,y\in\mathbb{F}$ satisfies that $x\leqslant y$ and $y\leqslant x$, then $x=y$ (anti-symmetry).
3. If $x,y\in\mathbb{F}$ satisfies that $x\leqslant y$ and $y\leqslant z$, then $x\leqslant z$ (transitivity).
4. For each $x,y\in\mathbb{F}$, either $x\leqslant y$ or $y\leqslant x$.
5. If $x\leqslant y$, then $x+z\leqslant y+z$ for all $z\in\mathbb{F}$ (compatibility of $\leqslant$ and $+$).
6. If $0\leqslant x$ and $0\leqslant y$, then $0\leqslant x\cdot y$ (compatibility of $\leqslant$ and $\cdot$).

Remark 1.9. A relation $\leqslant$ on a field $\mathbb{F}$ satisfying only 1-3 in the definition above is called a partial order. If in addition $\leqslant$ also satisfies 4, it is called a total order or linear order. Note that $\geqslant$ is a relation on $\mathbb{Q}$ satisfying 1-5 but not 6.

Remark 1.10. In an ordered field, the multiplicative inverse of $x\neq 0$ is sometimes denoted by $\dfrac{1}{x}$.

Definition 1.11. In an ordered field $(\mathbb{F},+,\cdot,\leqslant)$, the binary relations $<$, $\geqslant$ and $<$ are defined by:

1. $x<y$ if $x\leqslant y$ and $x\neq y$.
2. $x\geqslant y$ if $y\leqslant x$.
3. $x>y$ if $y<x$.

From now on, the total order $\preccurlyeq$ of an ordered field will be denoted by $\leqslant$, and the symbols $\prec$, $\succcurlyeq$ and $\succ$ will be denoted by $<$, $\geqslant$ and $>$, respectively.

Adopting the definition above, it is not immediately clear that $x \nleqslant y \Leftrightarrow x > y$. However, this is indeed the case, and to be more precise we have the following

Proposition 1.12. (Law of Trichotomy, 三一律) *If x and y are elements of an ordered field $(\mathbb{F}, +, \cdot, \leqslant)$, then exactly one of the relations $x < y$, $x = y$ or $y < x$ holds.*

Proof. Since $\mathbb{F}$ is a totally ordered field, x and y are comparable. Therefore, either $x \leqslant y$ or $y \leqslant x$. Assume that $x \leqslant y$.

1. If $x = y$, then $x \nless y$ and $x \ngtr y$.

2. If $x \neq y$, then $x < y$. If it also holds that $x > y$, then $x \geqslant y$; thus by the property of anit-symmetry of an order, we must have $x = y$, a contradiction. Therefore, it can only be that $x < y$.

The proof for the case $y \leqslant x$ is similar, and is left as an exercise. □

Proposition 1.13. *Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and $a, b, x, y, z \in \mathbb{F}$.*

1. *If $a + x = a$, then $x = 0$.*
   If $a \cdot x = a$ and $a \neq 0$, then $x = 1$.

2. *If $a + x = 0$, then $x = -a$.*
   If $a \cdot x = 1$ and $a \neq 0$, then $x = a^{-1}$.

3. *If $x \cdot y = 0$, then $x = 0$ or $y = 0$.*

4. *If $x \leqslant y < z$ or $x < y \leqslant z$, then $x < z$ (the transitivity of $<$).*

5. *If $a < b$, then $a + x < b + x$ (the compatibility of $<$ and $+$).*
   If $0 < a$ and $0 < b$, then $0 < a \cdot b$ (the compatibility of $<$ and $\cdot$).

6. *If $a + x = b + x$, then $a = b$.*
   If $a + x \leqslant (<)\, b + x$, then $a \leqslant (<)\, b$.
   If $a \cdot x = b \cdot x$ and $x \neq 0$, then $a = b$.
   If $a \cdot x \leqslant (<)\, b \cdot x$ and $x > 0$, then $a \leqslant (<)\, b$.

7. $0 \cdot x = 0$.

8. $-(-x) = x$.

9. $-x = (-1) \cdot x$.

10. *If* $x \neq 0$, *then* $x^{-1} \neq 0$ *and* $(x^{-1})^{-1} = x$.

11. *If* $x \neq 0$ *and* $y \neq 0$, *then* $x \cdot y \neq 0$ *and* $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$.

12. *If* $x \leqslant (<)\, y$ *and* $0 \leqslant (<)\, z$, *then* $x \cdot z \leqslant (<)\, y \cdot z$.
    If $x \leqslant (<)\, y$ *and* $0 \geqslant (>)\, z$, *then* $x \cdot z \geqslant (>)\, y \cdot z$.

13. *If* $x \leqslant (<)\, 0$ *and* $y \leqslant (<)\, 0$, *then* $x \cdot y \geqslant (>)\, 0$.
    If $x \leqslant (<)\, 0$ *and* $y \geqslant (>)\, 0$, *then* $x \cdot y \leqslant (<)\, 0$.

14. $0 < 1$ *and* $-1 < 0$.

15. $x \cdot x \equiv x^2 \geqslant 0$.

16. *If* $x > 0$, *then* $x^{-1} > 0$. *If* $x < 0$, *then* $x^{-1} < 0$.

Proof. 1. $(-a) + a + x = (-a) + a = 0 \Rightarrow x = 0$.
$(a^{-1}) \cdot a \cdot x = (a^{-1}) \cdot a = 1 \Rightarrow x = 1$.

2. $(-a) + a + x = (-a) + 0 = -a \Rightarrow x = -a$.
   $(a^{-1}) \cdot a \cdot x = (a^{-1}) \cdot 1 = a^{-1} \Rightarrow x = a^{-1}$.

3. Assume that $x \neq 0$, then $x^{-1} \cdot x \cdot y = x^{-1} \cdot 0 = 0 \Rightarrow y = 0$.
   Assume that $y \neq 0$, then $x \cdot y \cdot y^{-1} = 0 \cdot y^{-1} = 0 \Rightarrow x = 0$.

4 and 5 are Left as an exercise.

6. $a + 0 = a + x + (-x) = b + x + (-x) = b + 0 \Rightarrow a = b$.
   $a + 0 = a + x + (-x) \leqslant b + x + (-x) = b + 0 \Rightarrow a \leqslant b$ (compatibility of $\leqslant$ and $+$).
   $a \cdot x \cdot x^{-1} = b \cdot x \cdot x^{-1} \Rightarrow a = b$.
   Suppose the contrary that $b < a$. Then $0 = b + (-b) \leqslant a + (-b)$. Since $x > 0$, $x \geqslant 0$; thus
   $$0 \leqslant \big(a + (-b)\big) \cdot x = a \cdot x + (-b) \cdot x\,.$$
   As a consequence, $b \cdot x = 0 + b \cdot x \leqslant a \cdot x + (-b) \cdot x + b \cdot x = a \cdot x$. By assumption, we must have $a \cdot x = b \cdot x$ or $(a - b) \cdot x = 0$. Using 3, $x = 0$ (since $a \neq b$), a contradiction.

7. See Remark 1.3.

8. $(-x) + (-(-x)) = 0 = (-x) + x \Rightarrow x = -(-x)$.

9. See Remark 1.4.

10. Assume $x^{-1} = 0$, $1 = x \cdot x^{-1} = x \cdot 0 = 0$, a contradiction. Therefore, $x^{-1} \neq 0$; thus $(x^{-1})^{-1} \cdot x^{-1} = 1 = x \cdot x^{-1} \Rightarrow (x^{-1})^{-1} = x$ (by 4).

11. That $x \cdot y = 0$ cannot be true since it is against Property 3, so $x \cdot y \neq 0$. Moreover,
$$(x \cdot y)^{-1}(x \cdot y) = 1 = 1 \cdot 1 = (x \cdot x^{-1}) \cdot (y \cdot y^{-1}) = (x^{-1} \cdot y^{-1}) \cdot (x \cdot y)\,;$$
thus $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$ (by 4).

12. If $x \leqslant (<)\, y$, then $0 = x + (-x) \leqslant (<)\, y + (-x)$. Since $0 \leqslant (<)\, z$, by the compatibility of $\leqslant (<)$ and $\cdot$ we must have $0 \leqslant (<)\, (y + (-x)) \cdot z = y \cdot z + (-x) \cdot z$. Therefore, by the compatibility of $\leqslant (<)$ and $+$, $x \cdot z = 0 + x \cdot z \leqslant (<)\, y \cdot z + (-x) \cdot z + x \cdot z = y \cdot z$. The second statement can be proved in a similar fashion.

13. Left as an exercise.

14. If $1 \leqslant 0$, then compatibility of $\leqslant$ and $+$ implies that $0 \leqslant -1$. By the compatibility of $\leqslant$ and $\cdot$, using 8 and 9 we find that $0 \leqslant (-1) \cdot (-1) = -(-1) = 1$; thus we conclude that $1 = 0$, a contradiction. As a consequence, $0 < 1$; thus the compatibility of $<$ and $+$ implies that $-1 < 0$.

15. Left as an exercise.

16. If $x > 0$ but $x^{-1} \leqslant 0$, then $1 = x \cdot x^{-1} \leqslant x \cdot 0 = 0$, a contradiction. □

Proposition 1.14. *Let* $(\mathbb{F}, +, \cdot, \leqslant)$ *be an ordered field, and* $x, y \in \mathbb{F}$*.*

1. *If* $0 \leqslant x < y$*, then* $x^2 < y^2$*.*

2. *If* $0 \leqslant x, y$ *and* $x^2 < y^2$*, then* $x < y$*.*

Proof. 1. By definition of "$<$", $0 \leqslant x \leqslant y$ and $x \neq y$. Using 12 of Proposition 1.13,
$$x^2 \leqslant y \cdot x < y \cdot y = y^2\,.$$
By the transitivity of $<$, we conclude that $x^2 < y^2$.

2. Note that $x \neq y$, for if not, then $x^2 - y^2 = 0$ which contradicts to the assumption $x^2 < y^2$. Assume that $y < x$, then 1 implies that $y^2 < x^2$, a contradiction. □

Remark 1.15. Proposition 1.14 can be summarized as follows: if $x, y \geqslant 0$, then

$$x < y \Leftrightarrow x^2 < y^2 .$$

Moreover, Example 1.7, Proposition 1.13 and Proposition 1.14 together imply that if $x, y \geqslant 0$, then $x \leqslant y$ if and only if $x^2 \leqslant y^2$.

Definition 1.16. The ***magnitude*** or the ***absolute value*** of x, denoted $|x|$, is defined as

$$|x| = \begin{cases} x & \text{if } x \geqslant 0\,, \\ -x & \text{if } x < 0\,. \end{cases}$$

Proposition 1.17. *Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field. Then*

1. $|x| \geqslant 0$ *for all* $x \in \mathbb{F}$.

2. $|x| = 0$ *if and only if* $x = 0$.

3. $-|x| \leqslant x \leqslant |x|$ *for all* $x \in \mathbb{F}$.

4. $|x \cdot y| = |x| \cdot |y|$ *for all* $x, y \in \mathbb{F}$.

5. $|x + y| \leqslant |x| + |y|$ *for all* $x, y \in \mathbb{F}$ (***triangle inequality***, 三角不等式).

6. $\big||x| - |y|\big| \leqslant |x - y|$ *for all* $x, y \in \mathbb{F}$.

Proof. Left as an exercise. □

Definition 1.18. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field. The ***natural number system***, denoted by $\mathbb{N}$, is the collection of all the numbers 1, $1+1$, $1+1+1$, $1+1+\cdots+1$ and etc. in $\mathbb{F}$. We write $2 \equiv 1+1$, $3 \equiv 2+1$, and $n \equiv \underbrace{1+1+\cdots+1}_{\text{(n times)}}$. In other words, $\mathbb{N} = \{1, 2, 3, \cdots\}$.

The ***integer number system***, denoted by $\mathbb{Z}$, is the set $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$. The ***rational number system***, denoted by $\mathbb{Q}$, is the collection of all numbers of the form $\dfrac{q}{p} \equiv q \cdot p^{-1}$ with $p, q \in \mathbb{Z}$ and $p \neq 0$; that is,

$$\mathbb{Q} = \Big\{ x \in \mathbb{F} \,\Big|\, x = \frac{q}{p}, p, q \in \mathbb{Z}, p \neq 0 \Big\}.$$

Theorem 1.19. *Let* $(\mathbb{F}, +, \cdot, \leqslant)$ *be an ordered field. Then* $\mathbb{Q}$ *is an order field.*

- **Peano's Axiom for natural numbers**:

1. 1 is a natural number.

2. Every natural number has a unique successor which is a natural number (+1 is defined on natural numbers).

3. No two natural numbers have the same successor ($n+1 = m+1$ implies $n = m$).

4. 1 is not a successor for any natural number (1 is the "smallest" natural number).

5. If a property is possessed by 1 and is possessed by the successor of every natural number that possesses it, then the property is possessed by all natural numbers. (如果某個被自然數 1 所擁有的性質，也被其它擁有這個性質的自然數的下一個自然數所擁有，那麼所有的自然數都會擁有這個性質)

- **Principle of Mathematical Induction (PMI)**: If $S \subseteq \mathbb{N}$ has the property that

(1) $1 \in S$, and (2) $n+1 \in S$ whenever $n \in S$,

then $S = \mathbb{N}$.

- **Principle of Complete Induction (PCI)**: If $S \subseteq \mathbb{N}$ has the property that

$$\forall\, n \in \mathbb{N},\ n \in S \text{ whenever } \{1, 2, \cdots, n-1\} \subseteq S\,,$$

then $S = \mathbb{N}$.

- **Well-Ordering Principle (WOP)**: Every nonempty subset of $\mathbb{N}$ has a smallest element.

Theorem 1.20. PMI, **PCI** *and* **WOP** *are equivalent.*

Definition 1.21. An order field $(\mathbb{F}, +, \cdot, \leqslant)$ is said to satisfy ***Archimedean Property*** (**AP**) if for all $x \in \mathbb{F}$ there exists $n \in \mathbb{Z}$ such that $x < n$.

Example 1.22. The rational number system $\mathbb{Q}$ satisfies Archimedean Property. To see this, let $x \in \mathbb{Q}$ be given. If $x \leqslant 0$, we take $n = 1$. Otherwise if $0 < x = \dfrac{q}{p}$ with $p, q \in \mathbb{N}$, we take $n = q + 1$ and it is obvious that $\dfrac{q}{p} \leqslant q < q + 1 = n$.

1.2 Sequences in Ordered Fields

Definition 1.23. A ***sequence*** in a set S is a function $f : \mathbb{N} \to S$ (not necessary one-to-one or onto). The values of f are called the ***terms*** of the sequence, and $f(n)$ is called the n-th terms of the sequence.

Remark 1.24. A sequence in S is a countable list of elements in S arranged in a particular order, and is usually denoted by $\big\{f(n)\big\}_{n=1}^{\infty}$ or $\{x_n\}_{n=1}^{\infty}$ with $x_n = f(n)$.

Definition 1.25. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field. An ***"open" interval*** in $\mathbb{F}$ is a set of the form (a, b) which consists of all $x \in \mathbb{F}$ satisfying $a < x < b$. A ***"closed" interval*** in $\mathbb{F}$ is a set of the form $[a, b]$ which consists of all $x \in \mathbb{F}$ satisfying $a \leqslant x \leqslant b$.

Definition 1.26. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{F}$ is said to be ***convergent*** if there exists $x \in \mathbb{F}$ such that for every $\varepsilon > 0$ (and $\varepsilon \in \mathbb{F}$),

$$\#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} < \infty \,.$$

Such an x is called a ***limit*** of the sequence. In logic notation,

$$\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{F} \text{ is convergent} \quad \Leftrightarrow \quad (\exists\, x \in \mathbb{F})(\forall\, \varepsilon > 0)\big(\#\{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} < \infty\big).$$

If x is a limit of $\{x_n\}_{n=1}^{\infty}$, we say $\{x_n\}_{n=1}^{\infty}$ converges to x and write $x_n \to x$ as $n \to \infty$. If no such x exists we say that $\{x_n\}_{n=1}^{\infty}$ ***diverges*** (or the limit of $\{x_n\}_{n=1}^{\infty}$ does not exist).

Remark 1.27. The number N may depend on ε, and smaller ε usually requires larger N.

In the definition above, it could happen that there are two different limits of a convergent sequence. In fact, this is never the case because of the following

Proposition 1.28. *If $\{x_n\}_{n=1}^{\infty}$ is a sequence in an ordered field $\mathbb{F}$, and $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$. (The uniqueness of the limit).*

Proof. Assume the contrary that $x \neq y$. W.L.O.G. we may assume that $x < y$, and let $\varepsilon = \dfrac{y-x}{2} > 0$. Define

$$A_1 = \big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} \quad \text{and} \quad A_2 = \big\{n \in \mathbb{N} \,\big|\, x_n \notin (y-\varepsilon, y+\varepsilon)\big\} \,.$$

Then by the definition of the convergence of sequences, $\#A_1 < \infty$ and $\#A_2 < \infty$. Let $N_1 = \max A_1$, $N_2 = \max A_2$ and $N = \max\{N_1, N_2\}$. Since A_1, A_2 are finite, $N < \infty$. On the other hand, $N + 1 \notin A_1 \cup A_2$ which implies that $x_{N+1} \in (x-\varepsilon, x+\varepsilon) \cap (y-\varepsilon, y+\varepsilon) = \varnothing$, a contradiction. □

Notation: Since the limit of a convergent sequence is unique, if $\{x_n\}_{n=1}^\infty$ is a convergent sequence, we use $\lim\limits_{n\to\infty} x_n$, where n is a dummy index and can be change to other letters, to denote the limit of $\{x_n\}_{n=1}^\infty$.

Example 1.29. A ***permutation*** of a non-empty set A is a one-to-one function from A onto A. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$, and $\{x_n\}_{n=1}^\infty$ be a convergent sequence in an ordered field $\mathbb{F}$. Then $\{x_{\pi(n)}\}_{n=1}^\infty$ is also convergent since if x is the limit of $\{x_n\}_{n=1}^\infty$ and $\varepsilon > 0$,

$$\#\{n \in \mathbb{N} \,|\, x_{\pi(n)} \notin (x-\varepsilon, x+\varepsilon)\} = \#\{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} < \infty\,.$$

Proposition 1.30. *Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, $\{x_n\}_{n=1}^\infty \subseteq \mathbb{F}$ be a sequence, and $x \in \mathbb{F}$. Then $\lim\limits_{n\to\infty} x_n = x$ if and only if for every $\varepsilon > 0$, there exists $N > 0$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$. In logic notation,*

$$\lim_{n\to\infty} x_n = x \quad \Leftrightarrow \quad (\forall\, \varepsilon > 0)(\exists\, N > 0)(n \geqslant N \Rightarrow |x_n - x| < \varepsilon)\,.$$

Proof. "$\Rightarrow$" Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} x_n = x$, $\#\{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} < \infty$. If $\#\{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} > 0$, define $N = \max\{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} + 1$, otherwise define $N = 1$. Then if $n \geqslant N$, $x_n \in (x-\varepsilon, x+\varepsilon)$ or equivalently,

$$|x_n - x| < \varepsilon \quad \text{whenever} \quad n \geqslant N\,.$$

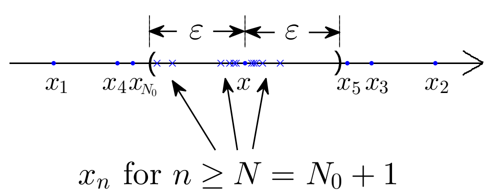


Figure 1.1: Let N_0 be the largest index of those x_n's outside $(x-\varepsilon, x+\varepsilon)$. Then $x_n \in (x-\varepsilon, x+\varepsilon)$ whenever $n \geqslant N = N_0 + 1$.

"$\Leftarrow$" Let $\varepsilon > 0$ be given. Then for some $N > 0$, if $n \geqslant N$, we have $|x_n - x| < \varepsilon$ or equivalently, if $n \geqslant N$, $x_n \in (x-\varepsilon, x+\varepsilon)$. This implies that

$$\#\{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} < N < \infty\,. \qquad \square$$

Remark 1.31. By the proposition above, $x_n \to x$ as $n \to \infty$ if and only if the sequence $\{|x_n - x|\}_{n=1}^\infty$ converges to 0; that is,

$$\lim_{n\to\infty} x_n = x \qquad \text{if and only if} \qquad \lim_{n\to\infty} |x_n - x| = 0\,.$$

Remark 1.32. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{F}$ diverges if (and only if)

$$\forall\, x \in \mathbb{F}, \exists\, \varepsilon > 0 \ni \#\{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} = \infty$$

which is equivalent to that

$$\forall\, x \in \mathbb{F}, \exists\, \varepsilon > 0 \ni \{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} = \{n_1 < n_2 < \cdots < n_j < \cdots\}\,.$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ diverges if (and only if)

$$\forall\, x \in \mathbb{F}, \exists\, \varepsilon > 0 \ni \forall\, N > 0, \exists\, n \geqslant N \text{ such that } |x_n - x| \geqslant \varepsilon\,.$$

Example 1.33. Now we use the ε-N argument as the definition of the convergence of sequences to re-establish the convergence of the sequence in Example 1.29.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence with limit x, and $\varepsilon > 0$ be given. Then there exists $N_1 > 0$ such that if $n \geqslant N_1$, we have $|x_n - x| < \varepsilon$. Let

$$N = \max\big\{\pi^{-1}(1), \pi^{-1}(2), \cdots, \pi^{-1}(N_1)\big\} + 1\,.$$

Then if $n \geqslant N$, $\pi(n) \geqslant N_1$ which implies that

$$\big|x_{\pi(n)} - x\big| < \varepsilon \quad \text{whenever} \quad n \geqslant N\,.$$

Therefore, $\lim\limits_{n\to\infty} x_{\pi(n)} = x$.

From the example above, we notice that proving the convergence using the ε-N argument seems more complicated; however, it is a necessary evil so we encourage the readers to major it.

Lemma 1.34 (Sandwich)**.** *If $\lim\limits_{n\to\infty} x_n = L$, $\lim\limits_{n\to\infty} y_n = L$, $\{z_n\}_{n=1}^{\infty}$ is a sequence such that $x_n \leqslant z_n \leqslant y_n$, then $\lim\limits_{n\to\infty} z_n = L$.*

Proof. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} x_n = L$ and $\lim\limits_{n\to\infty} y_n = L$, by definition

$$\exists\, N_1 > 0 \ni L - \varepsilon < x_n < L + \varepsilon \;\text{ whenever }\; n \geqslant N_1$$

and

$$\exists\, N_2 > 0 \ni L - \varepsilon < y_n < L + \varepsilon \;\text{ whenever }\; n \geqslant N_2\,.$$

Let $N = \max\{N_1, N_2\}$. Then for $n \geqslant N$, $L - \varepsilon < x_n \leqslant z_n \leqslant y_n < L + \varepsilon$; thus $\lim\limits_{n\to\infty} z_n = L$. □

Proposition 1.35. *If $x_n \leqslant y_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$, then $x \leqslant y$.*

Proof. Assume the contrary that $x > y$. Let $\varepsilon = \dfrac{x-y}{2}$. By the fact that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, there exists $N_1, N_2 > 0$ such that

$$|x_n - x| < \frac{x-y}{2} \quad \text{whenever} \quad n \geqslant N_1 \quad \text{and} \quad |y_n - y| < \frac{x-y}{2} \quad \text{whenever} \quad n \geqslant N_2 \,.$$

Let $N = \max\{N_1, N_2\}$. Then for $n \geqslant N$,

$$y_n < y + \frac{x-y}{2} = \frac{x+y}{2} = x - \frac{x-y}{2} < x_n \,,$$

a contradiction. □

Corollary 1.36. 1. *If $a \leqslant x_n$ (or $x_n \leqslant b$) and $\lim\limits_{n\to\infty} x_n = x$, then $a \leqslant x$ (or $x \leqslant b$).*

2. *If $a < x_n$ (or $x_n < b$) and $\lim\limits_{n\to\infty} x_n = x$, then $a \leqslant x$ (or $x \leqslant b$).*

Definition 1.37. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{F}$.

1. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded from above*** (有上界) if there exists $B \in \mathbb{F}$, called an ***upper bound*** of the sequence, such that $x_n \leqslant B$ for all $n \in \mathbb{N}$.

2. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded from below*** (有下界) if there exists $A \in \mathbb{F}$, called a ***lower bound*** of the sequence, such that $A \leqslant x_n$ for all $n \in \mathbb{N}$.

3. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded*** (有界的) if it is bounded from above and from below.

Remark 1.38. An equivalent definition of bounded sequences is stated as follows: $\{x_n\}_{n=1}^{\infty}$ is said to be bounded if there exists $M > 0$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$.

Proposition 1.39. *A convergent sequence is bounded* (數列收斂必有界).

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence with limit x. Then there exists $N > 0$ such that

$$x_n \in (x-1, x+1) \qquad \forall\, n \geqslant N \,.$$

Let $M = \max\big\{|x_1|, |x_2|, \cdots, |x_{N-1}|, |x|+1\big\}$. Then $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. □

Theorem 1.40. *Suppose that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then*

1. *$x_n \pm y_n \to x \pm y$ as $n \to \infty$.*

2. *$x_n \cdot y_n \to x \cdot y$ as $n \to \infty$.*

3. *If $y_n, y \neq 0$, then $\dfrac{x_n}{y_n} \to \dfrac{x}{y}$ as $n \to \infty$.*

Proof. 1. Let $\varepsilon > 0$ be given. Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$, there exist $N_1, N_2 \in \mathbb{N}$ such that $|x_n - x| < \dfrac{\varepsilon}{2}$ for all $n \geqslant N_1$ and $|y_n - x| < \dfrac{\varepsilon}{2}$ whenever $n \geqslant N_2$. Define $N = \max\{N_1, N_2\}$. Then $N \in \mathbb{N}$ and if $n \geqslant N$,

$$|(x_n \pm y_n) - (x \pm y)| \leqslant |x_n - x\,| + |y_n - y\,| < \varepsilon\,;$$

thus $x_n \pm y_n \to x \pm y$ as $n \to \infty$.

2. Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$, by Proposition 1.39 there exists $M > 0$ such that $|x_n| \leqslant M$ and $|y_n| \leqslant M$. Let $\varepsilon > 0$ be given. Then

$$(\exists\, N_1 \in \mathbb{N})\big(n \geqslant N_1 \Rightarrow |x_n - x\,| < \frac{\varepsilon}{2M}\big)\,,$$

and

$$(\exists\, N_2 \in \mathbb{N})\big(n \geqslant N_2 \Rightarrow |y_n - y\,| < \frac{\varepsilon}{2M}\big)\,.$$

Define $N = \max\{N_1, N_2\}$. Then $N \in \mathbb{N}$, and if $n \geqslant N$,

$$\begin{aligned}|x_n \cdot y_n - x \cdot y| &= |x_n \cdot y_n - x_n \cdot y + x_n \cdot y - x \cdot y| \leqslant |x_n \cdot (y_n - y)| + |y \cdot (x_n - x)| \\ &\leqslant M \cdot |y_n - y| + M \cdot |x_n - x| < M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.\end{aligned}$$

3. It suffices to show that $\lim\limits_{n\to\infty} \dfrac{1}{y_n} = \dfrac{1}{y}$ if $y_n, y \neq 0$ (because of 2). Since $\lim\limits_{n\to\infty} y_n = y$, there exists $N_1 \in \mathbb{N}$ such that $|y_n - y| < \dfrac{|y|}{2}$ whenever $n \geqslant N_1$. Therefore, $|y| - |y_n| < \dfrac{|y|}{2}$ for all $n \geqslant N_1$ which further implies that $|y_n| > \dfrac{|y|}{2}$ for all $n \geqslant N_1$.

Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} y_n = y$, there exists $N_2 \in \mathbb{N}$ such that $|y_n - y| < \dfrac{|y|^2}{2}\varepsilon$ whenever $n \geqslant N_2$. Define $N = \max\{N_1, N_2\}$. Then $N \in \mathbb{N}$ and if $n \geqslant N$,

$$\Big|\frac{1}{y_n} - \frac{1}{y}\Big| = \frac{|y_n - y|}{|y_n||y|} < \frac{|y|^2}{2}\varepsilon \cdot \frac{1}{|y|}\frac{2}{|y|} = \varepsilon\,. \qquad \square$$

Theorem 1.41. *An ordered field* $(\mathbb{F}, +, \cdot, \leqslant)$ *has Archimedean Property if and only if the sequence* $\Big\{\dfrac{1}{n}\Big\}_{n=1}^{\infty}$ *converges to* 0.

Proof. "$\Rightarrow$" Let $\varepsilon > 0$ be given. Define $x = \dfrac{1}{\varepsilon}$. Then $x > 0$ by Proposition 1.13. Moreover, Archimedean Property of $\mathbb{F}$ implies that there exists N such that $x < N$. Then $N > 0$ and if $n \geqslant N$,

$$\Big|\frac{1}{n} - 0\Big| = \frac{1}{n} \leqslant \frac{1}{N} < \frac{1}{x} = \varepsilon\,.$$

"$\Leftarrow$" Let $x \in \mathbb{F}$ be given. If $x \leqslant 0$, we choose $n = 1$ so that $x < 1$. If $x > 0$, let $\varepsilon = \frac{1}{x}$. Then $\varepsilon > 0$ by Proposition 1.13. Since $\frac{1}{n} \to 0$ as $n \to \infty$, there exists $N > 0$ such that

$$\frac{1}{n} = \left|\frac{1}{n} - 0\right| < \varepsilon = \frac{1}{x} \qquad \text{whenever} \quad n \geqslant N\,.$$

In particular, $\frac{1}{N} < \frac{1}{x}$ which implies that $x < N$. □

Remark 1.42. There are ordered fields that do not have Archimedean Property, and these fields are called non-Archimedean ordered fields (while an ordered field satisfying Archimedean Property is called Archimedean ordered fields - **AP 有序體**). In a non-Archimedean ordered field, the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ does not converge to 0.

1.3 Monotone Sequence Property

Definition 1.43. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{F}$.

1. $\{x_n\}_{n=1}^{\infty}$ is said to be ***increasing/non-decreasing*** if $x_n \leqslant x_{n+1}$ for all $n \in \mathbb{N}$.
2. $\{x_n\}_{n=1}^{\infty}$ is said to be ***decreasing/non-increasing*** if $x_n \geqslant x_{n+1}$ for all $n \in \mathbb{N}$.
3. $\{x_n\}_{n=1}^{\infty}$ is said to be ***strictly increasing*** if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.
4. $\{x_n\}_{n=1}^{\infty}$ is said to be ***strictly decreasing*** if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.

A sequence is called (strictly) ***monotone*** if it is either (strictly) increasing or (strictly) decreasing.

Definition 1.44. An ordered field $\mathbb{F}$ is said to satisfy the ***monotone sequence property*** (**MSP**) if every bounded monotone sequence converges to a limit in $\mathbb{F}$.

Remark 1.45. An equivalent definition of the monotone sequence property is that every monotone *increasing* sequence *bounded from above* converges; that is, if each sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{F}$ satisfying

(i) $x_n \leqslant x_{n+1}$ for all $n \in \mathbb{N}$,

(ii) there exists $M \in \mathbb{F}$ such that $x_n \leqslant M$ for all $n \in \mathbb{N}$,

is convergent, then we say $\mathbb{F}$ satisfies the monotone sequence property.

Example 1.46. Consider the sequence $\{y_n\}_{n=1}^\infty$ in $\mathbb{Q}$ defined by

$$y_1 = \frac{1}{2}, \qquad y_{n+1} = \frac{1}{2 + \dfrac{1}{2 + y_n}}.$$

1. $\{y_n\}_{n=1}^\infty$ is bounded from below by zero.

2. $\{y_n\}_{n=1}^\infty$ is a decreasing sequence in $\mathbb{Q}$ (which can be proved by induction).

If $\lim\limits_{n\to\infty} y_n = y$, then Theorem 1.40 implies that $y = \dfrac{1}{2 + \dfrac{1}{2+y}}$ from which we conclude that $y = -1 + \sqrt{2}$. Since $y \notin \mathbb{Q}$, $\{y_n\}_{n=1}^\infty$ does not converge (to a limit) in $\mathbb{Q}$. In other words, $\mathbb{Q}$ does not satisfy the monotone sequence property.

Proposition 1.47. *An ordered field satisfying the monotone sequence property satisfies Archimedean Property; that is, if $\mathbb{F}$ is an ordered field satisfying the monotone sequence property, then for all $x \in \mathbb{F}$, there exists $n \in \mathbb{N}$ such that $x < n$.*

Proof. Assume the contrary that there exists $x \in \mathbb{F}$ such that $n \leqslant x$ for all $n \in \mathbb{N}$. Let $x_n = n$. Then $\{x_n\}_{n=1}^\infty$ is increasing and bounded from above. By the monotone sequence property of $\mathbb{F}$, there exists $\widehat{x} \in \mathbb{F}$ such that $x_n \to \widehat{x}$ as $n \to \infty$; thus there exists $N > 0$ such that

$$|x_n - \widehat{x}| < \frac{1}{4} \qquad \text{whenever} \quad n \geqslant N.$$

In particular, $|N - \widehat{x}| < \frac{1}{4}$, $|N + 1 - \widehat{x}| < \frac{1}{4}$; thus

$$1 = |N + 1 - N| \leqslant |N + 1 - \widehat{x}| + |\widehat{x} - N| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

a contradiction. □

Example 1.48. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the monotone sequence property, and $y \in \mathbb{F}$ be a given positive number (that is, $y > 0$). Define $x_n = \dfrac{N_n}{2^n}$, where N_n is the largest integer such that $x_n^2 \leqslant y$; that is, $\left(\dfrac{N_n}{2^n}\right)^2 \leqslant y$ but $\left(\dfrac{N_n+1}{2^n}\right)^2 > y$ (for example, if $y = 2$, then $x_1 = \dfrac{2}{2^1}$, $x_2 = \dfrac{5}{2^2}$, $x_3 = \dfrac{11}{2^3}$, $\cdots$). Then

1. x_n is bounded from above: since $x_n^2 \leqslant y \leqslant 2y + y^2 + 1 = (y+1)^2$, by the non-negativity of x_n and y and Remark 1.15 we must have $0 \leqslant x_n \leqslant y + 1$.

2. x_n is increasing: by the definition of N_n,

$$N_n^2 \leqslant 2^{2n} \cdot y \Rightarrow 4 \cdot N_n^2 \leqslant 2^{2n+2} \cdot y = 2^{2(n+1)} \cdot y \Rightarrow \Big(\frac{2N_n}{2^{n+1}}\Big)^2 \leqslant y \Rightarrow 2N_n \leqslant N_{n+1}\,.$$

   Therefore, $x_n = \dfrac{N_n}{2^n} = \dfrac{2N_n}{2^{n+1}} \leqslant \dfrac{N_{n+1}}{2^{n+1}} = x_{n+1}$. Since $\mathbb{F}$ satisfies the monotone sequence property, there exists $x \in \mathbb{F}$ such that $x_n \to x$ as $n \to \infty$. By Theorem 1.40, $x_n^2 \to x^2$, and by Proposition 1.35, $x^2 \leqslant y$.

Now we show $x^2 = y$. To this end observe that

$$\Big(x_n + \frac{1}{2^n}\Big)^2 = \Big(\frac{N_n}{2^n} + \frac{1}{2^n}\Big) = \Big(\frac{N_n + 1}{2^n}\Big)^2 > y;$$

thus $x_n^2 \leqslant y \leqslant \big(x_n + \dfrac{1}{2^n}\big)^2$. By Archimedean propery of $\mathbb{F}$ (Proposition 1.47), $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$; thus Theorem 1.40 implies that $x^2 = \lim\limits_{n\to\infty} x_n^2 = \lim\limits_{n\to\infty} \big(x_n + \dfrac{1}{2^n}\big)^2 = y$. Note that Proposition 1.14 implies that such an x is unique if $x > 0$.

In general, one can define the n-th root of non-negative number y in an ordered field satisfying the monotone sequence property. The construction of the n-th root of $y \in \mathbb{F}$ is left as an exercise.

Definition 1.49. For $n \in \mathbb{N}$, the ***n-th root*** of a non-negative number y in an ordered field satisfying the monotone sequence property is the unique non-negative number x satisfying $x^n = y$. One writes $y^{1/n}$ or $\sqrt[n]{y}$ to denote n-th root of y.

1.4 Least Upper Bound Property

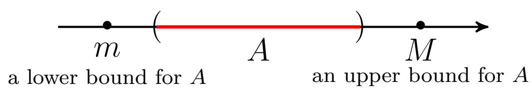
Definition 1.50. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and $\varnothing \neq A \subseteq \mathbb{F}$. A number $M \in \mathbb{F}$ is called an ***upper bound*** (上界) for A if $x \leqslant M$ for all $x \in A$, and a number $m \in \mathbb{F}$ is called a ***lower bound*** (下界) for A if $x \geqslant m$ for all $x \in A$. If there is an upper bound for A, then A is said to be ***bounded from above***, while if there is a lower bound for A, then A is said to be ***bounded from below***. A number $b \in \mathbb{F}$ is called a ***least upper bound*** (最小上界) of A if

1. b is an upper bound for A, and

2. if M is an upper bound for A, then $M \geqslant b$.

A number a is called a ***greatest lower bound*** (最大下界) of A if

1. a is a lower bound for A, and
2. if m is a lower bound for A, then $m \leqslant a$.



If A is not bounded from above, the least upper bound of A is set to be ∞, while if A is not bounded from below, the greatest lower bound of A is set to be $-\infty$. The least upper bound of A is also called the ***supremum*** of A and is usually denoted by lub$A$ or $\sup A$, and "the" greatest lower bound of A is also called the ***infimum*** of A, and is usually denoted by glb$A$ or $\inf A$. If $A = \varnothing$, then $\sup A = -\infty$, $\inf A = \infty$.

We emphasize that "$\sup A = \infty$" is purely a notation denoting that A is not bounded from above; however, $\infty \notin \mathbb{F}$ and $\sup A$ does not exist.

Remark 1.51. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field.

1. If $b_1, b_2 \in \mathbb{F}$ are least upper bounds for a set $A \subseteq \mathbb{F}$, then $b_1 = b_2$ (since $b_1 \leqslant b_2$ and $b_2 \leqslant b_1$ for b_1, b_2 are also upper bounds for A). Therefore, $\sup A$ is a well-defined concept. Similarly, $\inf A$ is a well-defined concept.
2. Since the sentence "$x \in \varnothing \Rightarrow x \leqslant M$" is true for all $M \in \mathbb{F}$, we conclude that $\sup \varnothing = -\infty$. Similarly, $\inf \varnothing = \infty$.

Example 1.52. In the ordered field $\mathbb{R}$ (pretended that you know what $\mathbb{R}$ is),

1. $\sup(0, 3) = 3$ and $\inf(0, 3) = 0$.
2. $\sup \mathbb{N}$ does not exist, but $\inf \mathbb{N} = 1$.
3. Let $A = \{2^{-k} \,|\, k \in \mathbb{N}\}$. Then $\inf A = 0$ and $\sup A = \dfrac{1}{2}$.
4. Let $B = \{x \in \mathbb{Q} \,|\, x^2 < 2\}$. Then $\inf B = -\sqrt{2}$ and $\sup B = \sqrt{2}$.

How about considering the supremum and infimum for the sets above in the ordered field $\mathbb{Q}$?

Proposition 1.53. *Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and A be a non-empty subset of $\mathbb{F}$.*

1. $b = \sup A \in \mathbb{F}$ *if and only if*

   (i) *b is an upper bound for A.* (ii) $(\forall\, \varepsilon > 0)(\exists\, x \in A)(x > b - \varepsilon)$.

2. $a = \inf A \in \mathbb{F}$ *if and only if*

   (i) *a is a lower bound for A.* (ii) $(\forall\, \varepsilon > 0)(\exists\, x \in A)(x < a + \varepsilon)$.

Proof. It suffices to prove 1.

"$\Rightarrow$" (i) is part of the definition of being a least upper bound.

(ii) If M is an upper bound for A, then we must have $M \geqslant b$; thus $b - \varepsilon$ is not an upper bound for A. Therefore, there exists $x \in A$ such that $x > b - \varepsilon$.

"$\Leftarrow$" We show that if M is an upper bound of A, then $M \geqslant b$. Assume the contrary that there exists an upper bound M for A satisfying $M < b$. Let $\varepsilon = b - M$. Then $\varepsilon > 0$ and there is no $x \in A$ satisfying $x > b - \varepsilon$, a contradiction. □

Corollary 1.54. *Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and A be a non-empty subset of $\mathbb{F}$.*

1. $b = \sup A \in \mathbb{F}$ *if and only if*

   (i) $(\forall\, \varepsilon > 0)(\forall\, x \in A)(x < b + \varepsilon)$. (ii) $(\forall\, \varepsilon > 0)(\exists\, x \in A)(x > b - \varepsilon)$.

2. $a = \inf A \in \mathbb{F}$ *if and only if*

   (i) $(\forall\, \varepsilon > 0)(\forall\, x \in A)(x > a - \varepsilon)$. (ii) $(\forall\, \varepsilon > 0)(\exists\, x \in A)(x < a + \varepsilon)$.

Proof. By Proposition 1.53, it suffices to show that

$$\text{condition 1(i)} \quad \Leftrightarrow \quad b \text{ is an upper bound for } A.$$

Since the direction "$\Leftarrow$" is trivial, we only need to prove the direction "$\Rightarrow$". Suppose the contrary that b is not an upper bound for A. Then there exists $x \in A$ such that $b < x$. Let $\varepsilon = x - s$. Then $\varepsilon > 0$ and we do not have 1(i) since $x \in A$ but $x \nless s + \varepsilon$. □

Proposition 1.55. *Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an order field, and $\varnothing \neq A \subseteq B \subseteq \mathbb{F}$. Then* $\inf B \leqslant \inf A \leqslant \sup A \leqslant \sup B$ *whenever those numbers exist in $\mathbb{F}$ or are $\pm\infty$.*

Proof. We proceed as follows.

1. $\sup A \leqslant \sup B$: Let $b = \sup B$, then for all $x \in B$, $x \leqslant b$. Since $A \subseteq B$, then for all $x \in A$, $x \leqslant b$; hence b is also an upper bound for A. Since $\sup A$ is the least upper bound of A and b is an upper bound for A, then $\sup A \leqslant b = \sup B$.

2. It is similar to prove $\inf B \leqslant \inf A$.

3. It is trivially true that $\inf A \leqslant \sup A$. □

Definition 1.56 (Least Upper Bound Property)**.** Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field. $\mathbb{F}$ is said to satisfy the ***least upper bound property*** (**LUBP**) if every non-empty subset of $\mathbb{F}$ that has an upper bound in $\mathbb{F}$ has a supremum that is an element of $\mathbb{F}$（非空有上界的集合必有最小上界）. The ***greatest lower bound property*** (**GLBP**) for ordered fields is defined similarly.

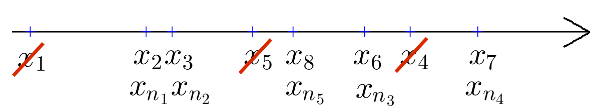
Proposition 1.57. *Every ordered field satisfying the least upper bound property satisfies Archimedean Property; that is, if $\mathbb{F}$ is an ordered field satisfying the least upper bound property, then for all $x \in \mathbb{F}$, there exists $n \in \mathbb{N}$ such that $x < n$.*

Proof. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field with the least upper bound property, and $x \in \mathbb{F}$ be given. If $x < 1$, then the choice $n = 1$ validates $n > x$. Suppose $x \geqslant 1$. Define $A = \{n \in \mathbb{N} \,|\, n \leqslant x\}$. Then $1 \in A$ and x is an upper bound for A. By the least upper bound property of $\mathbb{F}$, $s \equiv \sup A \in \mathbb{F}$ exists. Since s is the least upper bound of A, $s - 1$ is not an upper bound for A; thus there exists $m \in A$ such that $m > s - 1$ or $s < m + 1$. Then $m + 1 \notin A$ which implies that $m + 1 \nleqslant x$. The choice $n = m + 1$ then satisfies $n > x$. □

1.5 Bolzano-Weierstrass Property

Definition 1.58. A sequence $\{y_j\}_{j=1}^{\infty}$ is called a ***subsequence*** (子數列) of a sequence $\{x_n\}_{n=1}^{\infty}$ if there exists a strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that $y_j = x_{\phi(j)}$. In this case, we often write $\phi(j) = n_j$ and $y_j = x_{n_j}$.

In other words, a subsequence is a sequence that can be derived from another sequence by deleting some elements without changing the order of remaining elements. Let $f : \mathbb{N} \to \mathbb{F}$ be a sequence and $x_n = f(n)$. A subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ is the image of an infinite subset $\{n_1, n_2, \cdots\}$ of $\mathbb{N}$ under the map f (or simply the sequence $f \circ \phi : \mathbb{N} \to \mathbb{F}$).



Example 1.59. Consider the sequence $\{x_n\}_{n=1}^\infty$ defined recursively by

$$x_1 = \frac{1}{2}, \qquad x_2 = \frac{1}{2+\frac{1}{2}}, \qquad \cdots, \qquad x_{n+1} = \frac{1}{2+x_n}.$$

Then the sequence $\{y_n\}_{n=1}^\infty$ given in Example 1.46 is a subsequence of $\{x_n\}_{n=1}^\infty$. In fact,

$$y_n = x_{2n-1} \quad \forall\, n \in \mathbb{N} \qquad (\text{with the choice of } \phi(n) = 2n-1).$$

The following proposition concerns equivalent conditions for the convergence of sequences.

Proposition 1.60. *Let* $(\mathbb{F}, +, \cdot, \leqslant)$ *be an ordered field,* $\{x_n\}_{n=1}^\infty$ *be a sequence in* $\mathbb{F}$*, and* $x \in \mathbb{F}$*. Then*

1. $x_n \to x$ *as* $n \to \infty$ *if and only if every proper subsequence of* $\{x_n\}_{n=1}^\infty$ *converges to* x*.*

2. $x_n \to x$ *as* $n \to \infty$ *if and only if every proper subsequence of* $\{x_n\}_{n=1}^\infty$ *has a further subsequence that converges to* x*.*

Proof. 1. "$\Rightarrow$" Let $\{x_{n_j}\}_{j=1}^\infty$ be a subsequence of a convergent sequence $\{x_n\}_{n=1}^\infty$ with limit x, and $\varepsilon > 0$ be given. Then there exists $N > 0$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$. Since $n_j \geqslant j$ for all $j \in \mathbb{N}$, we find that $|x_{n_j} - x| < \varepsilon$ whenever $j \geqslant N$.

"$\Leftarrow$" Let $\varepsilon > 0$ be given. Since every proper subsequence of $\{x_n\}_{n=1}^\infty$ converges to x, the subsequence $\{x_{n+1}\}_{n=1}^\infty$ converges to x. Therefore, there exists $N_1 > 0$ such that

$$|x_{n+1} - x| < \varepsilon \qquad \text{whenever} \quad n \geqslant N_1 .$$

Let $N = N_1 + 1$. Then $|x_n - x| < \varepsilon$ whenever $n \geqslant N$.

2. The direction "$\Rightarrow$" follows from 1. For the direction "$\Leftarrow$", assume the contrary that $x_n \nrightarrow x$ as $n \to \infty$. Then

$$(\exists\, \varepsilon > 0)\big(\#\big\{n \in \mathbb{N} \,\big|\, |x_n - x| \geqslant \varepsilon\big\} = \infty\big).$$

Let $\big\{n \in \mathbb{N} \,\big|\, |x_n - x| \geqslant \varepsilon\big\} = \big\{n_j \in \mathbb{N} \,\big|\, n_j < n_{j+1} \text{ for all } j \in \mathbb{N}\big\}$. The subsequence $\{x_{n_j}\}_{j=1}^\infty$ clearly does not have any subsequence $\big\{x_{n_{j_k}}\big\}_{k=1}^\infty$ which converge to x, a contradiction. □

Remark 1.61. 1 of Proposition 1.60 indeed can be rephrased as "$\{x_n\}_{n=1}^{\infty}$ converges if and only if every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ converges". This fact is left as an exercise.

Recall that sequence $\{x_n\}_{n=1}^{\infty}$ converges to x if and only if

$$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x - \varepsilon, x + \varepsilon)\big\} < \infty\,.$$

The statement above implies that

$$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \in (x - \varepsilon, x + \varepsilon)\big\} = \infty\,; \tag{1.5.1}$$

however, if x satisfies (1.5.1), x might not be the limit of the sequence. Nevertheless, a candidate for the limit of a sequence must satisfy (1.5.1), and we call such a point a cluster point of $\{x_n\}_{n=1}^{\infty}$. To be more precise, we have the following

Definition 1.62. A point x is called a ***cluster point*** of a sequence $\{x_n\}_{n=1}^{\infty}$ if (1.5.1) holds.

We note that (1.5.1) is equivalent to that

$$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, |x_n - x| < \varepsilon\big\} = \infty\,.$$

Example 1.63. Let $x_n = (-1)^n$. Then 1 and -1 are the only two cluster points of $\{x_n\}_{n=1}^{\infty}$.

Example 1.64. Let $x_n = (-1)^n + \dfrac{1}{n}$.
Claim: 1 and -1 are cluster points of $\{x_n\}_{n=1}^{\infty}$.
Let $\varepsilon > 0$ be given. We observe that

$$\big\{n \in \mathbb{N} \,\big|\, x_n \in (1 - \varepsilon, 1 + \varepsilon)\big\} \supseteq \Big\{n \in \mathbb{N} \,\Big|\, n \text{ is even}, \frac{1}{n} < \varepsilon\Big\}\,;$$

thus $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in (1 - \varepsilon, 1 + \varepsilon)\big\} = \infty$. Similarly, -1 is a cluster point. Moreover, if $a \neq \pm 1$, a is not a cluster point of $\{x_n\}_{n=1}^{\infty}$.

Example 1.65. Let $S = \mathbb{Q} \cap [0, 1]$. Then S is countable since it is a subset of a countable set $\mathbb{Q}$. Therefore, there exists $f : \mathbb{N} \xrightarrow[\textit{onto}]{1-1} S$ or equivalently $S = \{q_1, q_2, \cdots, q_n, \cdots\}$. The collection of all cluster points of $\{q_n\}_{n=1}^{\infty}$ is $[0, 1]$ since every open interval (with mid-point in $[0, 1]$) contains infinitely many rational numbers in S.

Definition 1.66 (Bolzano-Weierstrass Property)**.** Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field. $\mathbb{F}$ is said to satisfy the ***Bolzano-Weierstrass property*** (**BWP**) if every bounded sequence in $\mathbb{F}$ has a convergent subsequence; that is, every bounded sequence in $\mathbb{F}$ has a subsequence that converges to a limit in $\mathbb{F}$.

Remark 1.67. $\mathbb{Q}$ does not satisfy the Bolzano-Weierstrass property. Example 1.46 provides an counterexample of a bounded divergent sequence in $\mathbb{Q}$.

Proposition 1.68. *Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the Bolzano-Weierstrass Property, $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{F}$, and $x \in \mathbb{F}$. Then $x_n \to x$ as $n \to \infty$ if and only if $\{x_n\}_{n=1}^{\infty}$ is bounded and x is the only cluster point of $\{x_n\}_{n=1}^{\infty}$.*

Proof. "$\Rightarrow$" The boundedness is concluded by Proposition 1.39. For uniqueness, suppose the contrary that there is another cluster point $y \neq x$. Let $\varepsilon = \dfrac{|x-y|}{2}$. Then

$$\#\{n \in \mathbb{N} \,\big|\, |x_n - y| < \varepsilon\} = \infty\,.$$

Since $\{n \in \mathbb{N} \,\big|\, |x_n - x| > \varepsilon\} \supseteq \{n \in \mathbb{N} \,\big|\, |x_n - y| < \varepsilon\}$ (this inclusion is left as an exercise), we find that

$$\#\{n \in \mathbb{N} \,\big|\, |x_n - x| > \varepsilon\} = \infty\,,$$

a contradiction to that $x_n \to x$ as $n \to \infty$.

"$\Leftarrow$" Suppose that $\{x_n\}_{n=1}$ is a bounded sequence in $\mathbb{F}$ and has x as the only cluster point but $\{x_n\}_{n=1}^{\infty}$ does not converge to x. Then

$$\exists\, \varepsilon > 0 \ni \#\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} = \infty\,.$$

Write $\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} = \{n_1, n_2, \cdots, n_k, \cdots\}$. Then we find a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ lying outside $(x-\varepsilon, x+\varepsilon)$. Since $\{x_{n_k}\}_{k=1}^{\infty}$ is bounded, the Bolzano-Weierstrass Property implies that there exists a convergent subsequence $\{x_{n_{k_j}}\}_{j=1}^{\infty}$ with limit y. Since $x_{n_{k_j}} \notin (x-\varepsilon, x+\varepsilon)$, $y \notin (x-\varepsilon, x+\varepsilon)$ by Proposition 1.35; thus $y \neq x$. On the other hand, the limit $\lim\limits_{j\to\infty} x_{n_{k_j}} = y$ implies that for every $\varepsilon > 0$,

$$\{j \in \mathbb{N} \,\big|\, |x_{n_{k_j}} - y| < \varepsilon\} \supseteq \{j \in \mathbb{N} \,\big|\, j \geqslant J\}$$

for some $J > 0$; thus $\#\{j \in \mathbb{N} \,\big|\, |x_{n_{k_j}} - y| < \varepsilon\} = \infty$ which shows that y is also a cluster point of $\{x_n\}_{n=1}^{\infty}$, a contradiction to the assumption that x is the only cluster point of $\{x_n\}_{n=1}^{\infty}$. □

Example 1.69. Consider the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_n = \begin{cases} n & \text{if } n \text{ is odd}\,, \\ 1 & \text{if } n \text{ is even}\,. \end{cases}$$

Then 1 is the only cluster point of $\{x_n\}_{n=1}^{\infty}$, but $\{x_n\}_{n=1}^{\infty}$ does not converge to 1 (since x_n is not bounded).

1.6 Cauchy Sequences

If a sequence $\{x_n\}_{n=1}^{\infty}$ in an ordered field $\mathbb{F}$ converges, then

$$\exists!\, x \in \mathbb{F} \ni \forall\, \varepsilon > 0, \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} < \infty\,.$$

We note that the statement above implies that if $\{x_n\}_{n=1}^{\infty}$ converges, then

$$(\forall\, \varepsilon > 0)(\exists \text{ an interval } I \text{ of length } 2\varepsilon)\big(\#\big\{n \in \mathbb{N} \,\big|\, x_n \notin I\big\} < \infty\big)\,. \tag{$\star$}$$

The statement above motivates the following

Definition 1.70. A sequence $\{x_n\}_{n=1}^{\infty}$ in an ordered field is said to be ***Cauchy*** if

$$(\forall\, \varepsilon > 0)(\exists\, N > 0)\big(n, m \geqslant N \Rightarrow |x_n - x_m| < \varepsilon\big)\,.$$

Remark 1.71. ($\star$) 這個敘述的中心思想是：給定任一正數 ε, 我們都能找到一個長度是 2ε 的區間使得落在此區間外的 x_n 只有有限個。因為當對每個長度我們都能找到這樣的區間時，才有機會找到 $\{x_n\}_{n=1}^{\infty}$ 的極限（而這個極限一定落在所有這樣的區間之內）。

Example 1.72. In $\mathbb{Q}$, $x_1 = 3, x_2 = 3.1, x_3 = 3.14, x_4 = 3.141, \cdots$. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, but is not convergent. Therefore, a Cauchy sequence in an ordered field may not converge.

Example 1.73. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an Archimedean ordered field, and $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{F}$ be a sequence satisfying $|x_n - x_{n+1}| < \dfrac{1}{2^{n+1}}$ for all $n \in \mathbb{N}$.
Claim: $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Given $\varepsilon > 0$, choose $N > 0$ such that $\dfrac{1}{2^N} < \varepsilon$ (such an N exists because of Theorem 1.41 and the fact that $2^N > N$ for all $N \in \mathbb{N}$). Then if $N \leqslant n < m$,

$$\begin{aligned}
|x_n - x_m| &\leqslant |x_n - x_{n+1}| + |x_{n+1} - x_m| \\
&\leqslant \cdots \\
&\leqslant |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\
&\leqslant \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots + \frac{1}{2^m} \\
&\leqslant \frac{1}{2^n} \leqslant \frac{1}{2^N} < \varepsilon\,;
\end{aligned}$$

thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{F}$.

Proposition 1.74. *Every convergent sequence is Cauchy.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a convergent sequence with limit x, and $\varepsilon > 0$ be given. By the definition of the convergence of sequences, there exists $N > 0$ such that $|x_n - x| < \dfrac{\varepsilon}{2}$ whenever $n \geqslant N$. Then by triangle inequality, if $n, m \geqslant N$,

$$|x_n - x_m| \leqslant |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,;$$

thus $\{x_n\}_{n=1}^\infty$ is Cauchy. □

Lemma 1.75. *Every Cauchy sequence is bounded.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be Cauchy. There exists $N > 0$ such that $|x_n - x_m| < 1$ for all $n, m \geqslant N$. In particular, $|x_n - x_{_N}| < 1$ if $n \geqslant N$ or equivalently,

$$x_{_N} - 1 < x_n < x_{_N} + 1 \qquad \forall\, n \geqslant N\,.$$

Let $M = \max\big\{|x_1|, |x_2|, \cdots, |x_{_{N-1}}|, |x_{_N}| + 1\big\}$. Then $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. □

Lemma 1.76. *If a subsequence of a Cauchy sequence is convergent, then this Cauchy sequence also converges.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence with a convergent subsequence $\{x_{n_j}\}_{j=1}^\infty$ whose limit is x, and $\varepsilon > 0$ be given. Then there exist $K, N > 0$ such that

$$|x_{n_j} - x| < \frac{\varepsilon}{2} \ \text{ whenever } \ j \geqslant K, \quad \text{ and } \quad |x_n - x_m| < \frac{\varepsilon}{2} \ \text{ whenever } \ n, m \geqslant N\,.$$

Choose $j \geqslant \max\{K, N\}$. Then $n_j \geqslant N$; thus if $n \geqslant N$,

$$|x_n - x| \leqslant |x_n - x_{n_j}| + |x_{n_j} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

□

Remark 1.77. Combining Proposition 1.60 and Lemma 1.76, we conclude that a Cauchy sequence converges if and only if a subsequence converges.

1.7 Completeness

In this section, we establish the equivalency between those properties introduced in the previous sections. These equivalent properties lead to an important concept, the completeness of ordered fields. There is exactly one ordered field satisfying all these properties, and this ordered field will be the real number system $\mathbb{R}$.

Theorem 1.78. *An ordered field satisfies the monotone sequence property if and only if it satisfies the least upper bound property*（有序體中 **MSP** 與 **LUBP** 為等價性質）.

Proof. Let $(\mathbb{F},+,\cdot,\leqslant)$ be an ordered field.

"$\Leftarrow$" Suppose that $\mathbb{F}$ satisfies the least upper bound property, and let $\{x_n\}_{n=1}^{\infty}\subseteq\mathbb{F}$ be an increasing sequence bounded from above. By the least upper bound property of $\mathbb{F}$, the set $\{x_1,x_2,\cdots,x_n,\cdots\}$ has a least upper bound $x\in\mathbb{F}$. We next show that x is the limit of $\{x_n\}_{n=1}^{\infty}$.

Let $\varepsilon>0$ be given. By Corollary 1.54, there exists x_N such that $x_N > x-\varepsilon$. Therefore, the fact that $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded from above by x imply that

$$x-\varepsilon<x_n\leqslant x<x+\varepsilon \qquad \forall\, n\geqslant N\,.$$

This shows that $|x_n-x|<\varepsilon$ whenever $n\geqslant N$; thus $x_n\to x$ as $n\to\infty$.

"$\Rightarrow$ Suppose that $\mathbb{F}$ satisfies the monotone sequence property, and let A be a non-empty subset of $\mathbb{F}$ bounded from above. Let N_n is the largest integer satisfying that $\dfrac{N_n}{2^n}$ is not an upper bound for A but $\dfrac{N_n+1}{2^n}$ is an upper bound for A, and define $x_n=\dfrac{N_n}{2^n}$. If M is an upper bound for A, $x_n\leqslant M$ for all $n\in\mathbb{N}$; thus $\{x_n\}_{n=1}^{\infty}$ is bounded from above. Moreover, by the fact that N_{n+1} is the largest integer satisfying $\dfrac{N_{n+1}}{2^{n+1}}$ is not an upper bound for A, we must have

$$x_n=\frac{N_n}{2^n}=\frac{2N_n}{2^{n+1}}\leqslant\frac{N_{n+1}}{2^{n+1}}=x_{n+1}\,;$$

thus $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence. Therefore, the monotone sequence property implies that $\{x_n\}_{n=1}^{\infty}$ converges to a limit $x\in\mathbb{F}$. Next we show that x is the least upper bound of A.

Let $\varepsilon>0$ be given. Then $x+\varepsilon$ must be an upper bound for A for otherwise if $\varepsilon>\dfrac{1}{2^k}$ for some $k\in\mathbb{N}$, then N_k is not the largest integer satisfying the required property. On the other hand, since $x_n\to x$ as $n\to\infty$, there exists $N>0$ such that $|x_n-x|<\varepsilon$ whenever $n\geqslant N$. Therefore, $x_N>x-\varepsilon$ which shows that $x-\varepsilon$ cannot be an upper bound for A; thus there exists $y\in A$ such that $y>x-\varepsilon$. By Corollary 1.54, we conclude that $x=\sup A$. □

Theorem 1.79. *An ordered field satisfying the monotone sequence property satisfies the Bolzano-Weierstrass property*（具 **MSP** 的有序體亦有 **BWP**）.

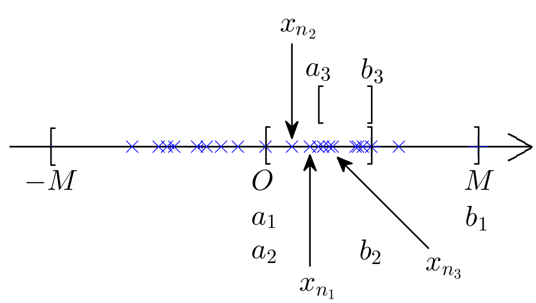
Proof. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the monotone sequence property, and $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence satisfying $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. Divide $[-M, M]$ into two intervals $[-M, 0], [0, M]$, and denote one of the two intervals containing infinitely many x_n as $[a_1, b_1]$; that is, $\#\{n \in \mathbb{N} \,|\, x_n \in [a_1, b_1]\} = \infty$. Divide $[a_1, b_1]$ into two intervals $\big[a_1, \frac{a_1+b_1}{2}\big], \big[\frac{a_1+b_1}{2}, b_1\big]$, and denote one of the two intervals containing infinitely many x_n as $[a_2, b_2]$. We continue this process, and obtain a sequence of intervals $[a_k, b_k]$ such that $[a_{k+1}, b_{k+1}] \subseteq [a_k, b_k]$, $|b_k - a_k| = \frac{M}{2^{k-1}}$ and $\#\{n \in \mathbb{N} \,|\, x_n \in [a_k, b_k]\} = \infty$ for all $k \in \mathbb{N}$.

Since $[a_k, b_k] \supseteq [a_{k+1}, b_{k+1}]$ for all $k \in \mathbb{N}$, we find that $\{a_k\}_{k=1}^{\infty}$ is increasing and $\{b_k\}_{k=1}^{\infty}$ is decreasing. Moreover, $a_k \leqslant M$, $b_k \geqslant -M$. Therefore, the monotone sequence property implies that a_k converges to $a \in \mathbb{F}$ and b_k converges to $b \in \mathbb{F}$. On the other hand,

$$b - a = \lim_{k\to\infty} (b_k - a_k) = \lim_{k\to\infty} \frac{M}{2^{k-1}} = 0\,,$$

where we have used Proposition 1.47 along with Theorem 1.41 (and Theorem 1.40) to conclude the limit. Therefore, $a = b$.

Finally, we construct a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$. Let x_{n_1} be an element belonging to $[a_1, b_1]$. Since $\#\{n \in \mathbb{N} \,|\, x_n \in [a_1, b_1]\} = \infty$, we can choose $n_2 > n_1$ such that $x_{n_2} \in [a_2, b_2]$, and for the same reason we can choose $n_3 > n_2$ such that $x_{n_3} \in [a_3, b_3]$. We continue this process and obtain a subsequence $x_{n_k} \in [a_k, b_k]$ with $n_k > n_{k-1}$.



Since $a_k \leqslant x_{n_k} \leqslant b_k$ for all $k \in \mathbb{N}$, by Sandwich Lemma $\lim\limits_{k\to\infty} x_{n_k} = a = b$. □

Theorem 1.80. *Every Cauchy sequence in an ordered field satisfying the Bolzano-Weierstrass property converges*（具 **BWP** 的有序體中的柯西數列必收斂）.

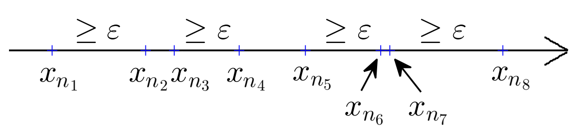
Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in an ordered field satisfying the Bolzano-Weierstrass property. By Lemma 1.75, $\{x_n\}_{n=1}^{\infty}$ is bounded; thus the Bolzano-Weierstrass property provides a convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$. The convergence of $\{x_n\}_{n=1}^{\infty}$ is then guaranteed by Lemma 1.76. □

Theorem 1.81. *An Archimedean ordered field satisfying the property that every Cauchy sequence converges satisfies the monotone sequence property*（**滿足柯西數列必收斂的 AP 有序體也有 MSP**）.

Proof. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an Archimedean ordered field. Suppose the contrary that there is a bounded increasing sequence $\{x_n\}_{n=1}^{\infty}$ that does not converge to a limit in $\mathbb{F}$. By the assumption that every Cauchy sequence converges, $\{x_n\}_{n=1}^{\infty}$ cannot be Cauchy; thus

$$(\exists\, \varepsilon > 0)(\forall\, N > 0)(\exists\, n, m \geqslant N)(|x_n - x_m| \geqslant \varepsilon)\,.$$

Let $N = 1$, there exists $n_2 > n_1 \geqslant 1$ such that $|x_{n_1} - x_{n_2}| \geqslant \varepsilon$. Let $N = n_2 + 1$, there exists $n_4 > n_3 \geqslant n_2 + 1$ such that $|x_{n_3} - x_{n_4}| \geqslant \varepsilon$. We continue this process and obtain a sequence $\{x_{n_j}\}_{j=1}^{\infty}$ satisfying $\big|x_{n_{2k-1}} - x_{n_{2k}}\big| \geqslant \varepsilon$ for all $k \in \mathbb{N}$.



Suppose that $\{x_n\}_{n=1}^{\infty}$ is bounded from above by M; that is, $x_n \leqslant M$ for all $n \in \mathbb{N}$. Then for each $k \in \mathbb{N}$,

$$\begin{aligned} M \geqslant x_{n_{2k}} &= x_{n_{2k}} - x_{n_{2k-1}} + x_{n_{2k-1}} \geqslant \varepsilon + x_{n_{2k-2}} = \varepsilon + x_{n_{2k-2}} - x_{n_{2k-3}} + x_{n_{2k-3}} \\ &\geqslant \varepsilon + \varepsilon + x_{n_{2k-4}} = 2\varepsilon + x_{n_{2k-4}} - x_{n_{2k-5}} + x_{n_{2k-5}} \geqslant \cdots \geqslant (k-1)\varepsilon + x_{n_1}\,; \end{aligned}$$

thus

$$k \leqslant 1 + \frac{M - x_{n_1}}{\varepsilon} \qquad \forall\, k \in \mathbb{N}\,,$$

a contradiction to Archimedean Property. □

Summary: In an Archimedean ordered field, the following four properties are equivalent:

1. the monotone sequence property (單調有界數列必收斂),
2. the least upper bound property (非空集合有上界必有最小上界),
3. the Bolzano-Weierstrass property (有界數列必有收斂子數列),
4. the property that every Cauchy sequence converges (柯西數列必收斂).

Such property is called the completeness（完備性）, and we have the following

Definition 1.82. An ordered field $\mathbb{F}$ is said to be ***complete*** (完備) (or have the completeness property, 具備完備性) if it satisfies the monotone sequence property.

Theorem 1.83. *There is a complete ordered field. Moreover, if* $(\mathbb{F}, +, \cdot, \leqslant)$ *and* $(\mathcal{F}, \oplus, \odot, \preccurlyeq)$ *are complete ordered fields, there exists a bijection* $\phi : \mathbb{F} \to \mathcal{F}$ *such that*

1. $\phi(x + y) = \phi(x) \oplus \phi(y)$ *and* $\phi(x \cdot y) = \phi(x) \odot \phi(y)$ *for all* $x, y \in \mathbb{F}$.

2. *The order* $\leqslant$ *and* $\preccurlyeq$ *is consistent under the map* ϕ*; that is, if* $x \leqslant y$*, then* $\phi(x) \preccurlyeq \phi(y)$.

In other words, two complete order fields are isomorphic (so that there is one and only one complete ordered field).

Axiom of the completeness of real number system $\mathbb{R}$: The real number system $\mathbb{R}$ is complete.

Theorem 1.84. *Every Cauchy sequence in* $\mathbb{R}$ *is convergent.*

Remark 1.85. Let $f : A \to \mathbb{R}$ be a real-valued function. Then the supremum of the image of A under f is denoted by $\sup\limits_{A} f$ or $\sup\limits_{x \in A} f(x)$. In other words,

$$\sup_{A} f = \sup_{x \in A} f(x) = \sup \big\{ f(x) \, \big| \, x \in A \big\} .$$

Similarly, the infimum of the image of A under f is denoted by $\inf\limits_{A} f$ or $\inf\limits_{x \in A} f(x)$.

1.8 Limit Inferior and Limit Superior

Definition 1.86. A sequence $\{x_n\}_{n=1}^{\infty}$ is said to ***diverge to infinity*** if for all $M > 0$, there exists $N > 0$ such that $x_n > M$ whenever $n \geqslant N$. It is said to ***diverge to negative infinity*** if $\{-x_n\}_{n=1}^{\infty}$ diverge to infinity. We use $\lim\limits_{n \to \infty} x_n = \infty$ or $-\infty$ to denote that $\{x_n\}_{n=1}^{\infty}$ diverges to infinity or negative infinity.

Remark 1.87. By Definition 1.26, the limit of a sequence $\{x_n\}_{n=1}^{\infty}$ does not exist if $\lim\limits_{n \to \infty} x_n = \infty$ or $-\infty$; however, we sometimes also call ∞ or $-\infty$ the limit of $\{x_n\}_{n=1}^{\infty}$.

Definition 1.88. The ***extended real number system***, denoted by $\mathbb{R}^*$, is the number system $\mathbb{R} \cup \{\infty, -\infty\}$, where ∞ and $-\infty$ are two symbols satisfying $-\infty < x < \infty$ for all $x \in \mathbb{R}$.

Remark 1.89. 1. $\mathbb{R}^*$ is not a field since ∞ and $-\infty$ do not have multiplicative inverse.

2. The definition of the least upper bound of a set can be simplified as follows: Let $S \subseteq \mathbb{R}^*$ be a set (not necessary non-empty set). A number $b \in \mathbb{R}^*$ is said to be the least upper bound of S if

   (a) b is an upper bound for S (that is, $s \leqslant b$ for all $s \in S$);

   (b) If $M \in \mathbb{R}^*$ is an upper bound for S, then $b \leqslant M$.

   No further discussion (such as $S = \varnothing$ or S is not bounded from above) has to be made. The greatest lower bound can be defined in a similar fashion.

3. Any sets in $\mathbb{R}^*$ has a least upper bound and a greatest lower bound in $\mathbb{R}^*$, even the empty set and unbounded set.

4. Proposition 1.53 for the case $\mathbb{F} = \mathbb{R}$ can be rephrased as follows: Let $S \subseteq \mathbb{R}^*$. Then $b = \sup S \in \mathbb{R}$ if and only if

   (a) b is an upper bound for S;

   (b) for all $\varepsilon > 0$, there exists $s \in S$ such that $s > b - \varepsilon$.

   Note that $b \in \mathbb{R}$ is crucial since there is no $s \in \mathbb{R}^*$ such that $s > \infty - \varepsilon = \infty$. The greatest lower bound counterpart can be made in a similar fashion.

5. In light of Definition 1.86, we can redefine cluster points of a real sequence as follows: A number $x \in \mathbb{R}^*$ is said to be a cluster point of a sequence $\{x_n\}_{n=1}^\infty \subseteq \mathbb{R}$ if there exists a subsequence $\{x_{n_j}\}_{j=1}^\infty$ such that $\lim\limits_{j\to\infty} x_{n_j} = x$. Note that now we can talk about if ∞ or $-\infty$ is a cluster points of a real sequence.

In the rest of the section, one is allowed to find the least upper bound and the greatest lower bound of a subset in $\mathbb{R}^*$.

Definition 1.90. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}$.

1. The ***limit superior*** of $\{x_n\}_{n=1}^\infty$, denoted by $\limsup\limits_{n\to\infty} x_n$ or $\overline{\lim\limits_{n\to\infty}}\, x_n$, is the infimum of the set $\Big\{ \sup\{x_n \,|\, n \geqslant k\} \,\Big|\, k \in \mathbb{N} \Big\}$.

2. The ***limit inferior*** of $\{x_n\}_{n=1}^\infty$, denoted by $\liminf\limits_{n\to\infty} x_n$ or $\underline{\lim\limits_{n\to\infty}}\, x_n$, is the supremum of the set $\Big\{ \inf\{x_n \,|\, n \geqslant k\} \,\Big|\, k \in \mathbb{N} \Big\}$.

Remark 1.91. Let $\sup\limits_{n \geqslant k} x_n$ denote the number $\sup\big\{x_n \,\big|\, n \geqslant k\big\}$ and $\inf\limits_{n \geqslant k} x_n$ denote the number $\inf\big\{x_n \,\big|\, n \geqslant k\big\}$. Then the limit superior and the limit inferior can be written as

$$\limsup_{n\to\infty} x_n = \inf_{k\geqslant 1} \sup_{n\geqslant k} x_n \quad \text{and} \quad \liminf_{n\to\infty} x_n = \sup_{k\geqslant 1} \inf_{n\geqslant k} x_n\,.$$

Remark 1.92. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, and $y_k = \sup\limits_{n\geqslant k} x_n$ and $z_k = \inf\limits_{n\geqslant k} x_n$. Then $\{y_k\}_{k=1}^{\infty}$ is a decreasing sequence, and $\{z_k\}_{k=1}^{\infty}$ is an increasing sequence. Therefore, the limit of $\{y_k\}_{k=1}^{\infty}$ and the limit of $\{z_k\}_{k=1}^{\infty}$ both "exist" in the sense of Definition 1.26 and Remark 1.87. In fact, the limit of $\{y_k\}_{k=1}^{\infty}$ is the infimum of $\{y_k\}_{k=1}^{\infty}$, and the limit of $\{z_k\}_{k=1}^{\infty}$ is the supremum of $\{z_k\}_{k=1}^{\infty}$. In other words,

$$\lim_{k\to\infty} \sup_{n\geqslant k} x_n = \inf_{k\geqslant 1} \sup_{n\geqslant k} x_n \quad \text{and} \quad \lim_{k\to\infty} \inf_{n\geqslant k} x_n = \sup_{k\geqslant 1} \inf_{n\geqslant k} x_n\,;$$

thus

$$\limsup_{n\to\infty} x_n = \lim_{k\to\infty} \sup_{n\geqslant k} x_n \quad \text{and} \quad \liminf_{n\to\infty} x_n = \lim_{k\to\infty} \inf_{n\geqslant k} x_n\,.$$

Example 1.93. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence given by $x_n = (-1)^n$. Then

$$y_k = \sup_{n\geqslant k} x_n = 1 \quad \text{and} \quad z_k = \inf_{n\geqslant k} x_n = -1\,.$$

Therefore, $\limsup\limits_{n\to\infty} x_n = \lim_{k\to\infty} y_k = 1$ and $\liminf\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} z_k = -1$.

Example 1.94. Let $\{x_n\}_{n=1}^{\infty}$ be a real sequence given by $x_n = \dfrac{1}{n}$. Then

$$y_k = \sup_{n\geqslant k} x_n = \frac{1}{k} \quad \text{and} \quad z_k = \inf_{n\geqslant k} x_n = 0\,.$$

Therefore, $\limsup\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} y_k = 0$ and $\liminf\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} z_k = 0$.

Example 1.95. Let $x_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$; that is, $\{x_n\}_{n=1}^{\infty} = \{1, 0, 3, 0, 5, \cdots\}$. Then

$$y_k = \sup_{n\geqslant k} x_n \quad \text{and} \quad z_k = \inf_{n\geqslant k} x_n = 0\,.$$

Therefore, $\limsup\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} y_k = \infty$ and $\liminf\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} z_k = 0$.

Example 1.96. Let $\{x_n\}_{n=1}^{\infty}$ be a real sequence defined by $x_n = (-1)^n + \dfrac{1}{n}$ or

$$\{x_n\}_{n=1}^{\infty} = \Big\{-1 + \frac{1}{1}, 1 + \frac{1}{2}, -1 + \frac{1}{3}, 1 + \frac{1}{4}, -1 + \frac{1}{5}, 1 + \frac{1}{6}, \cdots\Big\}\,.$$

Then for each $k \in \mathbb{N}$,

$$\sup_{n \geqslant k} x_n = 1 + \frac{1}{2[(k+1)/2]} \quad \text{and} \quad \inf_{n \geqslant k} x_n = -1 .$$

Therefore, $\limsup_{n\to\infty} x_n = 1$ and $\liminf_{n\to\infty} x_n = -1$.

Proposition 1.97. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then*

$$\limsup_{n\to\infty} -x_n = -\liminf_{n\to\infty} x_n \quad \textit{and} \quad \liminf_{n\to\infty} -x_n = -\limsup_{n\to\infty} x_n .$$

Proof. By the fact that $\sup_{n \geqslant k} -x_n = -\inf_{n \geqslant k} x_n$,

$$\limsup_{n\to\infty} -x_n = \lim_{k\to\infty} \sup_{n \geqslant k}(-x_n) = \lim_{k\to\infty} \big(-\inf_{n \geqslant k} x_n\big) = -\lim_{k\to\infty} \inf_{n \geqslant k} x_n = -\liminf_{n\to\infty} x_n .$$

The second identity holds simply by replacing x_n by $-x_n$ in the first identity. □

Proposition 1.98. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then*

1. $a = \liminf_{n\to\infty} x_n \in \mathbb{R}$ *if and only if the following two statements hold*
   (a) *for all $\varepsilon > 0$, there exists $N > 0$ such that $a - \varepsilon < x_n$ whenever $n \geqslant N$; that is,*
   $$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \leqslant a - \varepsilon\big\} < \infty ;$$
   (b) *for all $\varepsilon > 0$ and $N > 0$, there exists $n \geqslant N$ such that $x_n < a + \varepsilon$; that is,*
   $$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n < a + \varepsilon\big\} = \infty .$$
2. $b = \limsup_{n\to\infty} x_n \in \mathbb{R}$ *if and only if the following two statements hold*
   (a) *for all $\varepsilon > 0$, there exists $N > 0$ such that $b + \varepsilon > x_n$ whenever $n \geqslant N$; that is,*
   $$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \geqslant b + \varepsilon\big\} < \infty ;$$
   (b) *for all $\varepsilon > 0$ and $N > 0$, there exists $n \geqslant N$ such that $x_n > b - \varepsilon$; that is,*
   $$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n > b - \varepsilon\big\} = \infty .$$

Proof. We only prove 1 since the proof of 2 is similar. Let $z_k = \inf_{n \geqslant k} x_n$, and

$$\sup_{k \geqslant 1} z_k = \lim_{k\to\infty} z_k = a \in \mathbb{R}^* .$$

We show that $a \in \mathbb{R}$ if and only if 1-(a) and 1-(b) both hold. Nevertheless, by Proposition 1.53 (or Remark 1.89), $a \in \mathbb{R}$ if and only if

(i) a is an upper bound for $\{z_k\}_{k=1}^{\infty}$.

(ii) $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N} \ni z_N > a - \varepsilon$.

We justify the equivalency between 1-(a) and (ii), as well as the equivalency between 1-(b) and (i) as follows:

(i) a is an upper bound for $\{z_k\}_{k=1}^{\infty}$ $\Leftrightarrow$ $a \geqslant z_k$ for all $k \in \mathbb{N}$ $\Leftrightarrow$ $\forall\, \varepsilon > 0, a + \varepsilon > z_k$ for all $k \in \mathbb{N}$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$ and $k \in \mathbb{N}$, $a + \varepsilon > \inf\limits_{n \geqslant k} x_n$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$ and $k \in \mathbb{N}$, $a + \varepsilon$ is not a lower bound for $\{x_n\}_{n \geqslant k}^{\infty}$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$ and $k \in \mathbb{N}$, $\exists\, n \geqslant k \ni a + \varepsilon > x_n$ $\Leftrightarrow$ 1-(b).

(ii) $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N} \ni z_N > a - \varepsilon$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, N > 0 \ni \inf\limits_{n \geqslant N} x_n > a - \varepsilon$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, N > 0$ such that $a - \varepsilon$ is a lower bound for $\{x_N, x_{N+1}, \cdots\}$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, N > 0$ such that $a - \varepsilon \leqslant x_n$ for all $n \geqslant N$ $\Leftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, N > 0$ such that $a - \varepsilon < x_n$ for all $n \geqslant N$ $\Leftrightarrow$ 1-(a). □

Remark 1.99. By Proposition 1.98, if $a = \liminf\limits_{n \to \infty} x_n \in \mathbb{R}$, then

$$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \in (a - \varepsilon, a + \varepsilon)\big\} = \infty$$

which implies that a is a cluster point of $\{x_n\}_{n=1}^{\infty}$. Moreover, 1-(a) of Proposition 1.98 implies that no other cluster points can be smaller than a. In other words, if $a = \liminf\limits_{n \to \infty} x_n \in \mathbb{R}$, then a is the smallest cluster point of $\{x_n\}_{n=1}^{\infty}$. Similarly, b is the largest cluster point of $\{x_n\}_{n=1}^{\infty}$ if $b = \limsup\limits_{n \to \infty} x_n \in \mathbb{R}$.

Theorem 1.100. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then*

1. $\liminf\limits_{n \to \infty} x_n \leqslant \limsup\limits_{n \to \infty} x_n$.
2. *If $\{x_n\}_{n=1}^{\infty}$ is bounded from above by M, then $\limsup\limits_{n \to \infty} x_n \leqslant M$.*
3. *If $\{x_n\}_{n=1}^{\infty}$ is bounded from below by m, then $\liminf\limits_{n \to \infty} x_n \geqslant m$.*
4. $\limsup\limits_{n \to \infty} x_n = \infty$ *if and only if $\{x_n\}_{n=1}^{\infty}$ is not bounded from above.*
5. $\liminf\limits_{n \to \infty} x_n = -\infty$ *if and only if $\{x_n\}_{n=1}^{\infty}$ is not bounded from below.*
6. *If x is a cluster point of $\{x_n\}_{n=1}^{\infty}$, then $\liminf\limits_{n \to \infty} x_n \leqslant x \leqslant \limsup\limits_{n \to \infty} x_n$.*
7. *If $a = \liminf\limits_{n \to \infty} x_n$ is finite, then a is a cluster point.*

8. *If* $b = \limsup_{n\to\infty} x_n$ *is finite, then* b *is a cluster point.*

9. *If* $\{x_n\}_{n=1}^{\infty}$ *converges to* x *in* $\mathbb{R}$ *if and only if* $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x \in \mathbb{R}$.

Proof. Left as an exercise. □

Remark 1.101. Using the definition of cluster points of a sequence in Remark 1.89, Remark 1.99 and Theorem 1.100 together imply that the limit superior/inferior of a sequence is the largest/smallest cluster point of that sequence.

Example 1.102. Let $\mathbb{Q} \cap [0,1] = \{q_1, q_2, \cdots, q_n, \cdots\}$. Then $\{q_n\}_{n=1}^{\infty}$ does not converge since $\limsup_{n\to\infty} q_n = 1$ while $\liminf_{n\to\infty} q_n = 0$ by Example 1.65.

Chapter 2

Normed Vector Spaces and Metric Spaces

2.1 Euclidean Spaces and Vector Spaces

Definition 2.1. ***Euclidean*** ***n-space***, denoted by $\mathbb{R}^n$, consists of all ordered n-tuples of real numbers. Symbolically,

$$\mathbb{R}^n = \left\{ \boldsymbol{x} \,\middle|\, \boldsymbol{x} = (x_1, x_2, \cdots, x_n), x_i \in \mathbb{R} \right\}.$$

Elements of $\mathbb{R}^n$ are generally denoted by single letters that stand for n-tuples such as $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, and speak of $\boldsymbol{x}$ as a "point" in $\mathbb{R}^n$.

Remark 2.2. Let $\mathbb{C}$ denote the collection of ordered pairs $\mathbb{C} = \left\{ (a, b) \,\middle|\, a, b \in \mathbb{R} \right\}$ on which $+$ and $\cdot$ are given by Example 1.6. Then $\mathbb{C}$ is a field, and is called the complex number system. The ordered pair (a, b) in $\mathbb{C}$ is usually denoted by $a + bi$, where $i^2 = -1$ according to the definition of the multiplication. The space $\mathbb{C}^n$ can be defined similarly by

$$\mathbb{C}^n = \left\{ \boldsymbol{z} \,\middle|\, \boldsymbol{z} = (z_1, z_2, \cdots, z_n), z_i \in \mathbb{C} \right\}.$$

Definition 2.3. A ***vector space*** $\mathcal{V}$ over a scalar field $\mathbb{F}$ is a set of elements called vectors, with given operations of vector addition $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and scalar multiplication $\cdot : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$ such that

1. $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$ for all $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$.
2. $(\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{u} = \boldsymbol{v} + (\boldsymbol{u} + \boldsymbol{w})$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$.
3. there exists $\mathbf{0}$, the zero vector, such that $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathcal{V}$.

4. for each $\boldsymbol{v} \in \mathcal{V}$ there exists $\boldsymbol{w} \in \mathcal{V}$ such that $\boldsymbol{v} + \boldsymbol{w} = \mathbf{0}$.
5. $\lambda \cdot (\boldsymbol{v} + \boldsymbol{w}) = \lambda \cdot \boldsymbol{v} + \lambda \cdot \boldsymbol{w}$ for all $\lambda \in \mathbb{F}$ and $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$.
6. $(\lambda + \mu) \cdot \boldsymbol{v} = \lambda \cdot \boldsymbol{v} + \mu \cdot \boldsymbol{v}$ for all $\lambda, \mu \in \mathbb{F}$ and $\boldsymbol{v} \in \mathcal{V}$.
7. $(\lambda \cdot \mu) \cdot \boldsymbol{v} = \lambda \cdot (\mu \cdot \boldsymbol{v})$ for all $\lambda, \mu \in \mathbb{F}$ and $\boldsymbol{v} \in \mathcal{V}$.
8. $1 \cdot \boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathcal{V}$.

Remark 2.4. In general the scalar field $\mathbb{F}$ can be the rational number system $\mathbb{Q}$, the real number system $\mathbb{R}$, or even the complex number system $\mathbb{C}$. In this lecture note, $\mathbb{F}$ is taken as either the real number system $\mathbb{R}$ or the complex number system $\mathbb{C}$ (and mostly $\mathbb{R}$ if not specified).

Example 2.5. Let the vector addition and scalar multiplication on $\mathbb{F}^n$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, be defined by

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \cdots, x_n + y_n) \quad \text{if} \quad \boldsymbol{x} = (x_1, \cdots, x_n), \boldsymbol{y} = (y_1, \cdots, y_n)$$

and

$$\lambda \cdot \boldsymbol{x} = (\lambda x_1, \cdots, \lambda x_n) \quad \text{if} \quad \lambda \in \mathbb{F}, \boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{F}^n .$$

Then $\mathbb{F}^n$ is a vector space over $\mathbb{F}$ if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Moreover, $\mathbb{C}^n$ is a vector space over $\mathbb{R}$; however, $\mathbb{R}^n$ is not a vector space over $\mathbb{C}$.

Example 2.6. Let $\mathcal{M}_{n\times m}$ be the collection of all $n \times m$ real matrices; that is, $\mathcal{M}_{n\times m} \equiv \{n \times m \text{ matrix with entries in } \mathbb{R}\}$. Define

$$A + B \equiv [a_{ij} + b_{ij}], \quad \lambda \cdot A \equiv [\lambda \cdot a_{ij}] \quad \text{if} \quad \lambda \in \mathbb{R}, A = [a_{ij}], B = [b_{ij}] \in \mathcal{M} .$$

Then $\mathcal{M}_{n\times m}$ is a vector space over $\mathbb{R}$.

Definition 2.7. $\mathcal{W}$ is called a ***subspace*** of a vector space $\mathcal{V}$ over a scalar field $\mathbb{F}$ if

1. $\mathcal{W}$ is a subset of $\mathcal{V}$.
2. $(\mathcal{W}, +, \cdot)$, with vector addition and scalar multiplication in $\mathcal{V}$, is a vector space over $\mathbb{F}$.

Example 2.8. $\mathcal{V} = \mathbb{R}^3$, $W = \mathbb{R}^2 \times \{0\} \equiv \{(x, y, 0) \,|\, x, y \in \mathbb{R}\}$. $\mathcal{W}$ is a subspace of $\mathcal{V}$.

Lemma 2.9. *If $\mathcal{W}$ is a subset of a vector space $\mathcal{V}$ over a scalar field $\mathbb{F}$, then $\mathcal{W}$ is a subspace if and only if $\lambda \cdot \boldsymbol{v} + \mu \cdot \boldsymbol{w} \in \mathcal{W}$ for all $\lambda, \mu \in \mathbb{F}$, $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{W}$.*

Remark 2.10. "n" is called the ***dimension*** of $\mathbb{R}^n$.
There are n linearly independent vectors $\mathbf{e}_1 = (1, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, 0, \cdots, 0), \cdots, \mathbf{e}_n = (0, 0, \cdots, 0, 1)$, but if $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{n+1}$ are $(n+1)$ vectors in $\mathbb{R}^n$, there exists $\lambda_1, \cdots, \lambda_{n+1} \in \mathbb{R}$ such that $(\lambda_1, \cdots, \lambda_{n+1}) \neq (0, \cdots, 0)$ and $\lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_{n+1} \boldsymbol{v}_{n+1} = \mathbf{0}$.

On the other hand, the dimension of $\mathbb{C}^n$ depends on the scalar field $\mathbb{F}$.

1. If $\mathbb{F} = \mathbb{R}$, then the dimension of $\mathbb{C}^n$ is $2n$ since $\mathbf{e}_1, \cdots, \mathbf{e}_n, i\mathbf{e}_1, \cdots, i\mathbf{e}_n$ are $2n$ linearly independent vectors in $\mathbb{C}^n$, and any $(2n + 1)$ non-zeros vectors in $\mathbb{C}^n$ are not linearly independent; thus the dimension of $\mathbb{C}^n$ over $\mathbb{R}$ is $2n$. When $\mathbb{F} = \mathbb{R}$, we usually identify $\mathbb{C}^n$ as $\mathbb{R}^{2n}$.

2. If $\mathbb{F} = \mathbb{C}$, $\mathbf{e}_1, \cdots, \mathbf{e}_n$ are linearly independent in $\mathbb{C}^n$ and any $(n + 1)$ non-zeros vectors in $\mathbb{C}^n$ are not linearly independent; thus the dimension of $\mathbb{C}^n$ over $\mathbb{C}$ is n.

Definition 2.11. Let $\mathcal{V}$ be a vector space (over a scalar field $\mathbb{F}$), and A, B be subsets of $\mathcal{V}$. The sum of A and B, denoted by $A + B$, is the set $\{\boldsymbol{a} + \boldsymbol{b} \,|\, \boldsymbol{a} \in A, \boldsymbol{b} \in B\}$. If A consists of only one single vector $\boldsymbol{a}$, $A + B$ is usually denoted by $\boldsymbol{a} + B$ instead of $\{\boldsymbol{a}\} + B$.

The following theorem should be clear to the readers, and is left as an exercise.

Theorem 2.12. *Let $\mathcal{V}$ be a vector space (over a field $\mathbb{F}$), and A, B be subsets of $\mathcal{V}$. Then*

$$A + B = \bigcup_{\boldsymbol{a} \in A} (\boldsymbol{a} + B) = \bigcup_{\boldsymbol{b} \in B} (\boldsymbol{b} + A).$$

Definition 2.13. A subset $H \subseteq \mathbb{R}^n$ is called a ***hyperplane*** or ***hyperspace*** if H is $(n-1)$-dimensional subspace of $\mathbb{R}^n$. An ***affine hyperplane*** is a set $\boldsymbol{x} + H$ for some $\boldsymbol{x} \in \mathbb{R}^n$ and hyperplane H.

Example 2.14. A straight line on the plane is a hyperplane, and a plane on the (3-dimensional) space is also a hyperplane. However, a straight line on the (3-dimensional) space is not a hyperplane.

2.2 Normed Vector Spaces, Inner Product Spaces and Metric Spaces

Definition 2.15. A ***normed vector space*** (or simply ***normed space***) $(\mathcal{V}, \|\cdot\|)$ is a vector space $\mathcal{V}$ over a scalar field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, associated with a function $\|\cdot\| : \mathcal{V} \to \mathbb{R}$ such that

(a) $\|\boldsymbol{x}\| \geqslant 0$ for all $\boldsymbol{x} \in \mathcal{V}$.

(b) $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$.

(c) $\|\lambda \cdot \boldsymbol{x}\| = |\lambda| \cdot \|\boldsymbol{x}\|$ for all $\lambda \in \mathbb{F}$ and $\boldsymbol{x} \in \mathcal{V}$.

(d) $\|\boldsymbol{x} + \boldsymbol{y}\| \leqslant \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$.

A function $\|\cdot\|$ satisfying (a)-(d) is called a ***norm*** on $\mathcal{V}$.

Remark 2.16. Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space. Treating a vector in $\mathcal{V}$ as a point in $\mathcal{V}$, the number $\|\boldsymbol{x} - \boldsymbol{y}\|$ can be viewed as the distance (induced by the norm) between $\boldsymbol{x}$ and $\boldsymbol{y}$, and (d) implies that

$$\|\boldsymbol{x} - \boldsymbol{y}\| \leqslant \|\boldsymbol{x} - \boldsymbol{z}\| + \|\boldsymbol{z} - \boldsymbol{y}\| \qquad \forall\, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}\,.$$

The inequality above states that the distance between $\boldsymbol{x}$ and $\boldsymbol{y}$ is not greater than the sum of the distance between $\boldsymbol{x}$ and $\boldsymbol{z}$ and the distance between $\boldsymbol{y}$ and $\boldsymbol{z}$; thus the inequality in (d) is called the ***triangle inequality***.

Example 2.17. Let $\mathcal{V} = \mathbb{R}^n$, and define

$$\|\boldsymbol{x}\|_p \equiv \begin{cases} \Big(\sum\limits_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty, \\ \max\big\{|x_1|, \cdots, |x_n|\big\} & \text{if } p = \infty, \end{cases} \qquad \text{for all } \boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n.$$

Then $\|\cdot\|_p$ is a norm, called p-norm, on $\mathbb{R}^n$. Property (d) in Definition 2.15; that is, $\|\boldsymbol{x} + \boldsymbol{y}\|_p \leqslant \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$ (so-called the ***Minkowski inequality***), is left as an exercise.

Example 2.18. Let $\mathcal{V} = \mathbb{C}$ and the norm $\|\cdot\|$ is the usual absolute value of complex numbers; that is, $\|a + ib\| \equiv |a + ib| = \sqrt{a^2 + b^2}$. Then $(\mathbb{C}, \|\cdot\|)$ is a normed vector space.

Example 2.19. Let $\mathcal{M}_{n\times m} \equiv \big\{n \times m$ matrix with entries in $\mathbb{R}\big\}$, and $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ be standard basis of $\mathbb{R}^m$. For $A = \big[a_{ij}\big] \in \mathcal{M}_{n\times m}$ and $\boldsymbol{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_m\mathbf{e}_m$, we use $A\boldsymbol{x}$ to denote the n-vector

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}.$$

Define

$$\|A\|_p = \sup_{\|\boldsymbol{x}\|_p=1} \|A\boldsymbol{x}\|_p = \sup_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{\|A\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p} \qquad \forall\, A \in \mathcal{M}_{n\times m}\,;$$

that is, $\|A\|_p$ is the least upper bound of the set $\left\{\frac{\|A\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p} \,\Big|\, \boldsymbol{x}\neq\boldsymbol{0}, \boldsymbol{x}\in\mathbb{R}^m\right\}$. Then $\|\cdot\|_p$ satisfies the triangle inequality for the following reason: Suppose that $A, B \in \mathcal{M}_{n\times m}$. If $\|\boldsymbol{x}\|_p = 1$,

$$\begin{aligned}\|(A+B)\boldsymbol{x}\|_p &= \|A\boldsymbol{x}+B\boldsymbol{x}\|_p \leqslant \|A\boldsymbol{x}\|_p + \|B\boldsymbol{x}\|_p \\ &\leqslant \sup_{\|\boldsymbol{x}\|_p=1}\|A\boldsymbol{x}\|_p + \sup_{\|\boldsymbol{x}\|_p=1}\|B\boldsymbol{x}\|_p = \|A\|_p + \|B\|_p\,;\end{aligned}$$

thus

$$\|A+B\|_p = \sup_{\|\boldsymbol{x}\|_p=1}\|(A+B)\boldsymbol{x}\|_p \leqslant \|A\|_p + \|B\|_p\,.$$

Since property (a), (b), (c) in Definition 2.15 are obvious, we conclude that $\|\cdot\|_p$ is a norm on $\mathcal{M}_{n\times m}$. Moreover, by the definition of the p-norm we have $\frac{\|A\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p} \leqslant \|A\|_p$ for all $\boldsymbol{x}\neq\boldsymbol{0}$; thus

$$\|A\boldsymbol{x}\|_p \leqslant \|A\|_p\|\boldsymbol{x}\|_p \qquad \forall\, \boldsymbol{x}\in\mathbb{R}^m\,.$$

Consider the case $p=1, p=2$ and $p=\infty$ respectively.

1. $p=2$: Let $(\cdot,\cdot)_{\mathbb{R}^k}$ denote the inner product in Euclidean space $\mathbb{R}^k$. Then

$$\|A\boldsymbol{x}\|_2^2 = (A\boldsymbol{x}, A\boldsymbol{x})_{\mathbb{R}^n} = (\boldsymbol{x}, A^{\mathrm{T}}A\boldsymbol{x})_{\mathbb{R}^m} = (\boldsymbol{x}, P\Lambda P^{\mathrm{T}}\boldsymbol{x})_{\mathbb{R}^m} = (P^{\mathrm{T}}\boldsymbol{x}, \Lambda P^{\mathrm{T}}\boldsymbol{x})_{\mathbb{R}^n}\,,$$

in which we use the fact that $A^{\mathrm{T}}A$ is symmetric; thus diagonalizable by an orthonormal matrix P (that is, $A^{\mathrm{T}}A = P\Lambda P^{\mathrm{T}}$, $P^{\mathrm{T}}P = I$, Λ is a diagonal matrix with non-negative entries since $A^{\mathrm{T}}A$ is positive semi-definite). Let $\boldsymbol{y} = P^{\mathrm{T}}\boldsymbol{x}$. Since P is orthonormal, $\|\boldsymbol{x}\|_2 = 1$ if and only if $\|\boldsymbol{y}\|_2 = 1$; thus

$$\begin{aligned}\sup_{\|\boldsymbol{x}\|_2=1}\|A\boldsymbol{x}\|_2^2 &= \sup_{\|\boldsymbol{x}\|_2=1}(P^{\mathrm{T}}\boldsymbol{x}, \Lambda P^{\mathrm{T}}\boldsymbol{x}) = \sup_{\|\boldsymbol{y}\|_2=1}(\boldsymbol{y}, \Lambda\boldsymbol{y}) \\ &= \sup_{\|\boldsymbol{y}\|_2=1}(\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_m y_m^2) \\ &= \max\left\{\lambda_1, \cdots, \lambda_m\right\} = \text{maximum eigenvalue of } A^{\mathrm{T}}A\end{aligned}$$

which implies that $\|A\|_2 = \sqrt{\text{maximum eigenvalue of } A^{\mathrm{T}}A}$.

2. $p = \infty$: In this case we will show that

$$\|A\|_\infty = \sup_{\|\boldsymbol{x}\|_\infty=1} \|A\boldsymbol{x}\|_\infty = \max\left\{\sum_{j=1}^m |a_{1j}|, \sum_{j=1}^m |a_{2j}|, \cdots \sum_{j=1}^m |a_{nj}|\right\} = \max_{1\leqslant i\leqslant n} \sum_{j=1}^m |a_{ij}| .$$

Reason: Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ and $A = \big[a_{ij}\big]_{n\times m}$ (W.L.O.G. we can assume that A is not zero matrix). Then

$$A\boldsymbol{x} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1m}x_m \\ a_{21}x_1 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m \end{bmatrix} .$$

If $\|\boldsymbol{x}\|_\infty = 1$, then for each $1 \leqslant i \leqslant n$,

$$|a_{i1}x_1 + a_{i2}x_2 + \cdots a_{im}x_m| \leqslant \sum_{j=1}^m |a_{ij}| \leqslant \max_{1\leqslant i\leqslant n} \sum_{j=1}^m |a_{ij}| ;$$

thus the absolute value of each component of $A\boldsymbol{x}$, under the constraint $\|\boldsymbol{x}\|_\infty = 1$, has an upper bound $\max\limits_{1\leqslant i\leqslant n} \sum\limits_{j=1}^m |a_{ij}|$. Therefore,

$$\|A\|_\infty = \sup_{\|\boldsymbol{x}\|_\infty=1} \|A\boldsymbol{x}\|_\infty = \sup_{\|\boldsymbol{x}\|_\infty=1} \max_{1\leqslant i\leqslant n} |a_{i1}x_1+a_{i2}x_2+\cdots a_{im}x_m| \leqslant \max_{1\leqslant i\leqslant n} \sum_{j=1}^m |a_{ij}| . \tag{2.2.1}$$

On the other hand, assume $\max\limits_{1\leqslant i\leqslant n} \sum\limits_{j=1}^m |a_{ij}| = \sum\limits_{j=1}^m |a_{kj}|$ for some $1 \leqslant k \leqslant n$. Let

$$\boldsymbol{x} = \big(\mathrm{sgn}(a_{k1}), \mathrm{sgn}(a_{k2}), \cdots, \mathrm{sgn}(a_{kn})\big) .$$

Then $\|\boldsymbol{x}\|_\infty = 1$ (since A is not zero matrix), and $\|A\boldsymbol{x}\|_\infty = \sum\limits_{j=1}^m |a_{kj}|$; thus

$$\|A\|_\infty = \sup_{\|\boldsymbol{x}\|_\infty=1} \|A\boldsymbol{x}\|_\infty \geqslant \sum_{j=1}^m |a_{kj}| = \max_{1\leqslant i\leqslant n} \sum_{j=1}^m |a_{ij}| . \tag{2.2.2}$$

The combination of (2.2.1) and (2.2.2) implies that

$$\|A\|_\infty = \max\left\{\sum_{j=1}^m |a_{1j}|, \sum_{j=1}^m |a_{2j}|, \cdots \sum_{j=1}^m |a_{nj}|\right\} . \tag{2.2.3}$$

In other words, $\|A\|_\infty$ is the largest sum of the absolute value of row entries.

3. $p = 1$: $\|A\|_1 = \max\left\{\sum_{i=1}^{n}|a_{i1}|, \sum_{i=1}^{n}|a_{i2}|, \cdots, \sum_{i=1}^{n}|a_{im}|\right\}$. This result is left as an exercise.

In general, we can also define

$$\|A\|_{p,q} = \sup_{\|\boldsymbol{x}\|_p=1} \|A\boldsymbol{x}\|_q = \sup_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{\|A\boldsymbol{x}\|_q}{\|\boldsymbol{x}\|_p}\,.$$

Then $\|A\|_p = \|A\|_{p,p}$.

Example 2.20. For $1 \leqslant p < \infty$, let ℓ^p denote the collection of all sequences $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ satisfying $\sum_{n=1}^{\infty} |x_n|^p < \infty$; that is

$$\ell^p \equiv \left\{\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R} \,\middle|\, \text{the series } \sum_{n=1}^{\infty} |x_n|^p \text{ converges}\right\}.$$

Then ℓ^p is a vector space over $\mathbb{R}$. The function $\|\cdot\| : \ell^p \to \mathbb{R}$ defined by $\left\|\{x_n\}_{n=1}^{\infty}\right\| = \left(\sum_{n=1}^{\infty}|x_n|^p\right)^{\frac{1}{p}}$ is a norm on ℓ^p.

Example 2.21. Let $\mathscr{C}([a,b];\mathbb{R})$ be the collection of all continuous real-valued functions on the interval $[a,b]$; that is,

$$\mathscr{C}([a,b];\mathbb{R}) = \left\{f : [a,b] \to \mathbb{R} \,\middle|\, f \text{ is continuous on } [a,b]\right\}.$$

For each $f \in \mathscr{C}([a,b];\mathbb{R})$, we define

$$\|f\|_p = \begin{cases} \left[\int_a^b |f(x)|^p dx\right]^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty\,, \\ \max\limits_{x\in[a,b]} |f(x)| & \text{if } p = \infty\,. \end{cases}$$

The function $\|\cdot\|_p : \mathscr{C}([a,b];\mathbb{R}) \to \mathbb{R}$ is a norm on $\mathscr{C}([a,b];\mathbb{R})$ (Minkowski's inequality).

Definition 2.22. An ***inner product space*** $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is a vector space $\mathcal{V}$ over a scalar field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, associated with a function $\langle\cdot,\cdot\rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ such that

(1) $\langle\boldsymbol{x},\boldsymbol{x}\rangle \geqslant 0,\ \forall\, \boldsymbol{x} \in \mathcal{V}$.

(2) $\langle\boldsymbol{x},\boldsymbol{x}\rangle = 0$ if and only if $\boldsymbol{x} = 0$.

(3) $\langle\boldsymbol{x},\boldsymbol{y}+\boldsymbol{z}\rangle = \langle\boldsymbol{x},\boldsymbol{y}\rangle + \langle\boldsymbol{x},\boldsymbol{z}\rangle$ for all $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathcal{V}$.

(4) $\langle \lambda \boldsymbol{x}, \boldsymbol{y} \rangle = \lambda \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ for all $\lambda \in \mathbb{F}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$.

(5) $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, where $\bar{c}$ denotes the complex conjugate of c.

A function $\langle \cdot, \cdot \rangle$ satisfying (1)-(5) is called an ***inner product*** on $\mathcal{V}$.

Example 2.23. Let $(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} x_i y_i \quad \forall\, \boldsymbol{x} = (x_1, \cdots, x_n), \boldsymbol{y} = (y_1, \cdots, y_n)\,.$$

Then $(\cdot, \cdot)$ is an inner product on $\mathbb{R}^n$. Moreover, $\langle \cdot, \cdot \rangle : \mathbb{C}^n \to \mathbb{C}$ defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i} \qquad \forall\, \boldsymbol{x} = (x_1, \cdots, x_n), \boldsymbol{y} = (y_1, \cdots, y_n)$$

is an inner product on $\mathbb{C}^n$.

Example 2.24. Let $\mathscr{C}([a, b]; \mathbb{R})$ be defined as in Example 2.21. Define

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx\,.$$

Then $\langle \cdot, \cdot \rangle : \mathscr{C}([a, b]; \mathbb{R}) \times \mathscr{C}([a, b]; \mathbb{R}) \to \mathbb{R}$ satisfies all the properties that an inner product has. Note that $\langle f, f \rangle = \|f\|_2^2$.

Similar to the inner product given above, one can also consider an inner product on $\mathscr{C}([a, b]; \mathbb{C})$, where $\mathscr{C}([a, b]; \mathbb{C})$ denotes the collection of continuous complex-valued functions defined on $[a, b]$. Note that $\mathscr{C}([a, b]; \mathbb{C})$ is a vector space over $\mathbb{R}$ and over $\mathbb{C}$, and we always viewed $\mathscr{C}([a, b], \mathbb{C})$ as a vector space over $\mathbb{C}$. On $\mathscr{C}([a, b]; \mathbb{C})$, define

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)}\, dx\,.$$

Then $\langle \cdot, \cdot \rangle : \mathscr{C}([a, b]; \mathbb{C}) \times \mathscr{C}([a, b]; \mathbb{C}) \to \mathbb{C}$ satisfies all the properties that an inner product has.

Proposition 2.25. *Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space $\mathcal{V}$ over a scalar field $\mathbb{F}$.*

1. $\langle \lambda \boldsymbol{v} + \mu \boldsymbol{w}, \boldsymbol{u} \rangle = \lambda \langle \boldsymbol{v}, \boldsymbol{u} \rangle + \mu \langle \boldsymbol{w}, \boldsymbol{u} \rangle$ *for all* $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ *and* $\lambda, \mu \in \mathbb{F}$.

2. $\langle \boldsymbol{u}, \lambda \boldsymbol{v} + \mu \boldsymbol{w} \rangle = \bar{\lambda} \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \bar{\mu} \langle \boldsymbol{u}, \boldsymbol{w} \rangle$ *for all* $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ *and* $\lambda, \mu \in \mathbb{F}$.

3. $\langle \boldsymbol{0}, \boldsymbol{w} \rangle = \langle \boldsymbol{w}, \boldsymbol{0} \rangle = 0$ *for all* $\boldsymbol{w} \in \mathcal{V}$.

Theorem 2.26. *The inner product* $\langle\cdot,\cdot\rangle$ *on a vector space* $\mathcal{V}$ *induces a norm* $\|\cdot\|$ *given by* $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}$ *and satisfies the **Cauchy-Schwarz inequality***

$$\big|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\big| \leqslant \|\boldsymbol{x}\|\cdot\|\boldsymbol{y}\| \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}\,. \tag{2.2.4}$$

Moreover, for non-zero vectors $\boldsymbol{x}, \boldsymbol{y}$*, the equality holds if and only if there exists* $\gamma \in \mathbb{F}$ *such that* $\boldsymbol{x} = \gamma \boldsymbol{y}$*.*

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$. Define $\alpha = \langle \boldsymbol{x}, \boldsymbol{y}\rangle$. W.L.O.G. we can assume that $\alpha \neq 0$ (for otherwise (2.2.4) holds trivially). Then there exists $\beta \in \mathbb{F}$ such that $\alpha \cdot \beta = |\alpha|$ (so $|\beta| = 1$). For any $\lambda \in \mathbb{R}$,

$$\begin{aligned} 0 &\leqslant \langle \lambda\beta\boldsymbol{x} + \boldsymbol{y}, \lambda\beta\boldsymbol{x} + \boldsymbol{y}\rangle = \lambda^2|\beta|^2\|\boldsymbol{x}\|^2 + \langle\lambda\beta\boldsymbol{x}, \boldsymbol{y}\rangle + \langle \boldsymbol{y}, \lambda\beta\boldsymbol{x}\rangle + \|\boldsymbol{y}\|^2 \\ &= \lambda^2\|\boldsymbol{x}\|^2 + \lambda\beta\langle \boldsymbol{x}, \boldsymbol{y}\rangle + \lambda\overline{\langle\beta\boldsymbol{x}, \boldsymbol{y}\rangle} + \|\boldsymbol{y}\|^2 \\ &= \lambda^2\|\boldsymbol{x}\|^2 + 2\lambda\big|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\big| + \|\boldsymbol{y}\|^2\,. \end{aligned} \tag{2.2.5}$$

Since the right-hand side in the inequality above is always non-negative for all real λ, we must have

$$\big|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\big|^2 - \|\boldsymbol{x}\|^2\cdot\|\boldsymbol{y}\|^2 \leqslant 0$$

which implies (2.2.4).

It should be clear that (a)-(c) in Definition 2.15 are satisfied. To show that $\|\cdot\|$ satisfies the triangle inequality, by (2.2.4) we find that

$$\begin{aligned} \big(\|\boldsymbol{x}\| + \|\boldsymbol{y}\|\big)^2 - \|\boldsymbol{x}+\boldsymbol{y}\|^2 &= \|\boldsymbol{x}\|^2 + 2\|\boldsymbol{x}\|\|\boldsymbol{y}\| + \|\boldsymbol{y}\|^2 - \langle \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y}\rangle \\ &= 2\big(\|\boldsymbol{x}\|\|\boldsymbol{y}\| - \mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle\big) \geqslant 2\big(\|\boldsymbol{x}\|\|\boldsymbol{y}\| - \big|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\big|\big) \geqslant 0\,; \end{aligned}$$

thus the triangle inequality is also valid.

Finally, suppose that $\boldsymbol{x}, \boldsymbol{y} \neq \boldsymbol{0}$ and $\big|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\big| = \|\boldsymbol{x}\|\|\boldsymbol{y}\|$. Then with $\lambda \in \mathbb{R}$ given by $\lambda = -\dfrac{\|\boldsymbol{y}\|}{\|\boldsymbol{x}\|}$, (2.2.5) shows that

$$0 \leqslant \|\lambda\beta\boldsymbol{x} + \boldsymbol{y}\|^2 = \lambda^2\|\boldsymbol{x}\|^2 + 2\lambda\|\boldsymbol{x}\|\|\boldsymbol{y}\| + \|\boldsymbol{y}\|^2 = (\lambda\|\boldsymbol{x}\| + \|\boldsymbol{y}\|)^2 = 0\,;$$

thus $\lambda\beta\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{0}$. □

Corollary 2.27. *Let* $f, g : [a,b] \to \mathbb{R}$ *be continuous. Then*

$$\left|\int_a^b f(x)g(x)dx\right| \leqslant \left(\int_a^b |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_a^b |g(x)|^2 dx\right)^{\frac{1}{2}}.$$

Example 2.28. Let $\mathcal{V}$ be a finite dimensional vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be a basis of $\mathcal{V}$; that is, every $\boldsymbol{x} \in \mathcal{V}$ can be uniquely expressed as

$$\boldsymbol{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j = x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n$$

for some n-tuple $(x_1, \cdots, x_n) \in \mathbb{F}^n$. Define $\|\boldsymbol{x}\|_2 = \Big(\sum\limits_{j=1}^{n} |x_j|^2 \Big)^{\frac{1}{2}}$. Then $\|\cdot\|_2$ is a norm on $\mathcal{V}$. In fact, $\|\cdot\|_2$ is induced by the inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{j=1}^{n} x_i \overline{y_i} \qquad \text{if } \boldsymbol{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j \text{ and } \boldsymbol{y} = \sum_{j=1}^{n} y_j \mathbf{e}_j .$$

It is also possible to talk about the notion of distance between points in a general set. A set with a distance function is called a metric space.

Definition 2.29. A ***metric space*** (M, d) is a set M associated with a function $d : M \times M \to \mathbb{R}$ such that

(i) $d(x, y) \geqslant 0$ for all $x, y \in M$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$ for all $x, y \in M$.

(iv) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in M$.

A function d satisfying (i)-(iv) is called a ***metric*** on M.

Example 2.30 (Discrete metric)**.** Let M be a non-empty set. Define a function d_0 by

$$d_0(x, y) = \begin{cases} 0 & \text{if } x = y\,, \\ 1 & \text{if } x \neq y\,. \end{cases}$$

Then $d_0 : M \times M \to \mathbb{R}$ is a metric on M, and we call d_0 the discrete metric.

Example 2.31 (Bounded metric)**.** Let (M, d) be a metric space. Define a function ρ by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}\,.$$

Then $\rho : M \times M \to \mathbb{R}$ is also a metric on M.

Proposition 2.32. *Let (M, d) be a metric space, and A be a non-empty subset of M. Then (A, d) is a metric space.*

Proposition 2.33. *If $(\mathcal{V}, \|\cdot\|)$ is a normed vector space, then the function $d : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ is a metric on $\mathcal{V}$. In other words, $(\mathcal{V}, d)$ is a metric space, and we usually write $(\mathcal{V}, \|\cdot\|)$ as the metric space.*

Definition 2.34. Let (M, d) be a metric space. For each $x \in M$ and $r > 0$, the set

$$B(x, r) = \{y \in M \mid d(x, y) < r\}$$

is called the r***-ball*** about x or the ball centered at x with radius r.

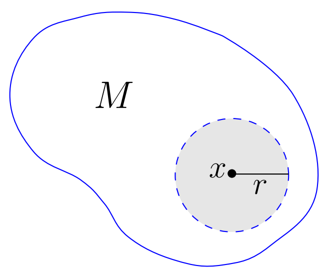


Figure 2.1: The r-ball about x in a metric space

Example 2.35. In $\mathbb{R}$, $B(x, r) = (x - r, x + r)$.

Example 2.36. Consider the 1-ball about the origin in $(\mathbb{R}^2, \|\cdot\|_p)$ for $p = 1, 2, \infty$, respectively.

1. $p = 1$: $\|\boldsymbol{x}\|_1 = |x_1| + |x_2|$, $\|\boldsymbol{x} - \boldsymbol{y}\|_1 = |x_1 - y_1| + |x_2 - y_2|$.
2. $p = 2$: $\|\boldsymbol{x}\|_2 = \sqrt{x_1^2 + x_2^2}$, $\|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.
3. $p = \infty$: $\|\boldsymbol{x}\|_\infty = \max\{|x_1|, |x_2|\}$, $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

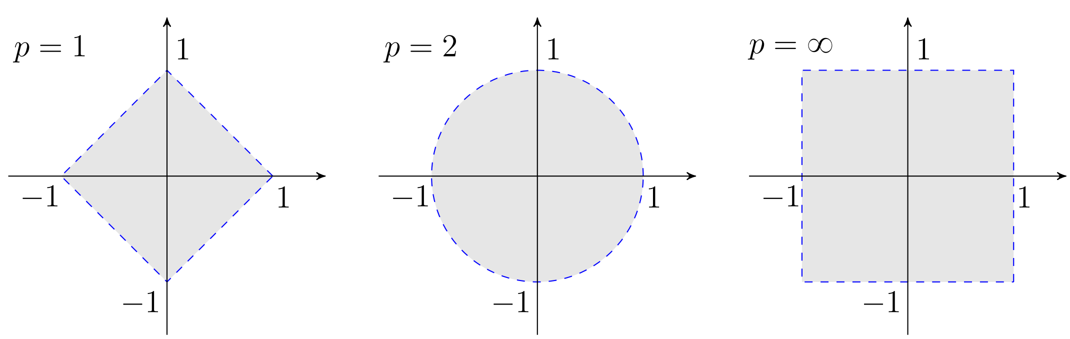


Figure 2.2: The 1-ball about $\mathbf{0}$ in $\mathbb{R}^2$ with different p

Definition 2.37. Let $\mathcal{V}$ be a vector space. Two norms $\|\cdot\|_1$, $\|\cdot\|_2$ on $\mathcal{V}$ are said to be ***equivalent*** if there exist two positive constant C, c such that

$$c\|\boldsymbol{x}\|_1 \leqslant \|\boldsymbol{x}\|_2 \leqslant C\|\boldsymbol{x}\|_1 \qquad \forall\, x \in \mathcal{V}\,.$$

Note that the constant c and C must be independent of $\boldsymbol{x}$.

Example 2.38. For $1 \leqslant p, q \leqslant \infty$, the p-norm $\|\cdot\|_p$ and q-norm $\|\cdot\|_q$ on $\mathbb{R}^n$ are equivalent; however, the p-norm $\|\cdot\|_p$ and the q-norm $\|\cdot\|_p$ on $\mathscr{C}([a,b];\mathbb{R})$ are **NOT** equivalent. The result is left as an exercise.

Theorem 2.39. *Let $\mathcal{V}$ be a vector space (over field $\mathbb{F}$), and $\|\cdot\|_1$, $\|\cdot\|_2$ are equivalent norms on $\mathcal{V}$. Then every ball in $(\mathcal{V}, \|\cdot\|_1)$ contains some balls in $(\mathcal{V}, \|\cdot\|_2)$ and is contained in some balls in $(\mathcal{V}, \|\cdot\|_2)$.*

Proof. Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, there exist positive constants c and C such that

$$c\|\boldsymbol{x}\|_1 \leqslant \|\boldsymbol{x}\|_2 \leqslant C\|\boldsymbol{x}\|_1 \qquad \forall\, x \in \mathcal{V}\,.$$

Let $B_1(\boldsymbol{x}, r) = \big\{\boldsymbol{y} \in \mathcal{V} \,\big|\, \|\boldsymbol{y}-\boldsymbol{x}\|_1 < r\big\}$ be a ball in $(\mathcal{V}, \|\cdot\|_1)$. Let $\delta = cr$ and $R = Cr$. Then with $B_2(\boldsymbol{x}, r)$ denoting the set $\big\{\boldsymbol{y} \in \mathcal{V} \,\big|\, \|\boldsymbol{y}-\boldsymbol{x}\|_2 < r\big\}$, we have

$$\|\boldsymbol{y}-\boldsymbol{x}\|_1 \leqslant \frac{1}{c}\|\boldsymbol{y}-\boldsymbol{x}\|_2 < r \quad \forall\, \boldsymbol{y} \in B_2(\boldsymbol{x}, \delta) \quad \text{and} \quad \|\boldsymbol{y}-\boldsymbol{x}\|_2 \leqslant C\|\boldsymbol{y}-\boldsymbol{x}\|_1 < R$$

In other words, $B_2(\boldsymbol{x}, \delta) \subseteq B_1(\boldsymbol{x}, r)$ and $B_1(\boldsymbol{x}, r) \subseteq B_2(\boldsymbol{x}, R)$. □

2.3 Sequences in Metric Spaces

2.3.1 Sequences

Recall that a ***sequence*** in a set S is a function $f : \mathbb{N} \to S$, and $f(n)$ is called the n-th terms of the sequence. A sequence in S is usually denoted by $\big\{f(n)\big\}_{n=1}^{\infty}$ or $\{x_n\}_{n=1}^{\infty}$ with $x_n = f(n)$.

Definition 2.40. Let (M, d) be a metric space. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$ is said to be ***convergent*** if there exists $x \in M$ such that for every $\varepsilon > 0$, there exists $N > 0$ such that

$$d(x_n, x) < \varepsilon \quad \text{whenever} \quad n \geqslant N\,.$$

Such an x is called a ***limit*** of the sequence. In notation,

$$\{x_n\}_{n=1}^{\infty} \subseteq M \text{ is convergent} \quad \Leftrightarrow \quad (\exists\, x \in M)(\forall\, \varepsilon > 0)(\exists\, N > 0)\big(n \geqslant N \Rightarrow d(x_n, x) < \varepsilon\big)\,.$$

If x is a limit of $\{x_n\}_{n=1}^{\infty}$, we say $\{x_n\}_{n=1}^{\infty}$ converges to x and write $x_n \to x$ as $n \to \infty$. If no such x exists we say that $\{x_n\}_{n=1}^{\infty}$ ***diverges*** or $\lim\limits_{n\to\infty} x_n$ does not exist.

Remark 2.41. Similar to Definition 1.26, the convergence of a sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space can be stated as follows: a sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$ is said to be ***convergent*** if there exists $x \in M$ such that for every $\varepsilon > 0$, there exists $N > 0$ such that

$$\#\big\{n \in \mathbb{N}\,\big|\, x_n \notin B(x, \varepsilon)\big\} < \infty\,.$$

Similar to Proposition 1.28, we have the following

Proposition 2.42. *Let (M, d) be a metric space. If $\{x_n\}_{n=1}^{\infty}$ is a sequence in M, and $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$. (The uniqueness of the limit).*

Proof. Assume the contrary that $x \neq y$. Then $\varepsilon = \dfrac{d(x,y)}{2} > 0$. Then there exist $N_1, N_2 > 0$ such that $d(x_n, x) < \varepsilon$ for all $n \geqslant N_1$ and $d(x_n, y) < \varepsilon$ for all $n \geqslant N_2$. Let $N = \max\{N_1, N_2\}$. Then if $n \geqslant N$,

$$d(x, y) \leqslant d(x, x_n) + d(x_n, y) < 2\varepsilon = d(x, y)\,,$$

a contradiction. □

Notation: Similar to the notation used to denote the unique limit of a convergent sequence in an ordered field, we also use $\lim\limits_{n\to\infty} x_n$ to denote the limit of a convergent sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$.

Remark 2.43. Similar to Remark 1.31, the proposition above implies that $x_n \to x$ as $n \to \infty$ if and only if $\lim\limits_{n\to\infty} d(x_n, x) = 0$.

Remark 2.44. Let (M, d) be a metric space. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$ diverges if (and only if)

$$\forall\, x \in M, \exists\, \varepsilon > 0 \ni \forall\, N > 0, \exists\, n \geqslant N \text{ such that } d(x_n, x) \geqslant \varepsilon\,.$$

Definition 2.45. Let (M, d) be a metric space. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$ is said to be ***bounded***（有界的）if there exist $y \in M$ and $r > 0$ such that $d(x_n, y) < r$ for all $n \in \mathbb{N}$. In other words, sequence $\{x_n\}_{n=1}^{\infty}$ is bounded if it is contained in some r-ball.

Remark 2.46. In a normed vector space $(\mathcal{V}, \|\cdot\|)$, the boundedness of a sequence $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ is equivalent to that there exists $r > 0$ such that $\|\boldsymbol{x}_n\| < r$ for all $n \in \mathbb{N}$. In other words, the point $\boldsymbol{y}$ in the definition above is the zero vector.

Proposition 2.47. *A convergent sequence is bounded*（收斂數列必有界）.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence with limit x. Then there exists $N > 0$ such that $x_n \in B(x, 1)$ whenever $n \geqslant N$, or equivalently,

$$d(x_n, x) < 1 \quad \text{whenever} \quad n \geqslant N.$$

Let $r = \max\left\{d(x_1, x), d(x_2, x), \cdots, d(x_{N-1}, x)\right\} + 1$. Then $d(x_n, x) < r$ for all $n \in \mathbb{N}$. □

Theorem 2.48. *Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space over a scalar field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$, $\{\boldsymbol{y}_n\}_{n=1}^{\infty}$ be sequences in $\mathcal{V}$, and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{F}$. Suppose that $\boldsymbol{x}_n \to \boldsymbol{x}$, $\boldsymbol{y}_n \to \boldsymbol{y}$ and $\lambda_n \to \lambda$ as $n \to \infty$. Then*

1. $\boldsymbol{x}_n \pm \boldsymbol{y}_n \to \boldsymbol{x} \pm \boldsymbol{y}$ *as* $n \to \infty$.
2. $\lambda_n \boldsymbol{x}_n \to \lambda \boldsymbol{x}$ *as* $n \to \infty$.
3. *If* $\lambda_n, \lambda \neq 0$, *then* $\dfrac{x_n}{\lambda_n} \to \dfrac{x}{\lambda}$ *as* $n \to \infty$.

If in addition that $\mathcal{V}$ is an inner product space equipped with inner product $\langle\cdot,\cdot\rangle$ which induces the norm $\|\cdot\|$, then

4. $\langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle \to \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ *as* $n \to \infty$.

Proof. We only prove 3 and 4. The proof of 1 and 2 are left as an exercise.

3. It suffices to show that $\lim\limits_{n\to\infty} \dfrac{1}{\lambda_n} = \dfrac{1}{\lambda}$ if $\lambda_n, \lambda \neq 0$ (because of 2). Since $\lim\limits_{n\to\infty} \lambda_n = \lambda$, there exists $N_1 > 0$ such that $|\lambda_n - \lambda| < \dfrac{|\lambda|}{2}$ whenever $n \geqslant N_1$. Therefore, $|\lambda| - |\lambda_n| < \dfrac{|\lambda|}{2}$ for all $n \geqslant N_1$ which further implies that $|\lambda_n| > \dfrac{|\lambda|}{2}$ for all $n \geqslant N_1$.

   Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} \lambda_n = \lambda$, there exists $N_2 > 0$ such that $|\lambda_n - \lambda| < \dfrac{|\lambda|^2}{2}\varepsilon$ whenever $n \geqslant N_2$. Define $N = \max\{N_1, N_2\}$. Then if $n \geqslant N$,

$$\Big|\frac{1}{\lambda_n} - \frac{1}{\lambda}\Big| = \frac{|\lambda_n - \lambda|}{|\lambda_n||\lambda|} < \frac{|\lambda|^2}{2}\varepsilon \cdot \frac{1}{|\lambda|}\frac{2}{|\lambda|} = \varepsilon\,.$$

4. Let $\varepsilon > 0$ be given. Since $\boldsymbol{x}_n \to \boldsymbol{x}$ and $\boldsymbol{y}_n \to \boldsymbol{y}$ as $n \to \infty$, by Proposition 2.47 and Remark 2.46 there exists $M > 0$ such that $\|\boldsymbol{x}_n\| \leqslant M$ and $\|\boldsymbol{y}_n\| \leqslant M$. Moreover,
$$\exists\, N_1 > 0 \ni \|\boldsymbol{x}_n - \boldsymbol{x}\| < \frac{\varepsilon}{2M} \quad \text{whenever} \quad n \geqslant N_1$$
and
$$\exists\, N_2 > 0 \ni \|\boldsymbol{y}_n - \boldsymbol{y}\| < \frac{\varepsilon}{2M} \quad \text{whenever} \quad n \geqslant N_2\,.$$
Define $N = \max\{N_1, N_2\}$. Then if $n \geqslant N$,
$$\begin{aligned} \big|\langle \boldsymbol{x}_n, \boldsymbol{y}_n\rangle - \langle \boldsymbol{x}, \boldsymbol{y}\rangle\big| &= \big|\langle \boldsymbol{x}_n, \boldsymbol{y}_n\rangle - \langle \boldsymbol{x}_n, \boldsymbol{y}\rangle + \langle \boldsymbol{x}_n, \boldsymbol{y}\rangle - \langle \boldsymbol{x}, \boldsymbol{y}\rangle\big| \\ &\leqslant \big|\langle \boldsymbol{x}_n, \boldsymbol{y}_n - \boldsymbol{y}\rangle\big| + \big|\langle \boldsymbol{y}, \boldsymbol{x}_n - \boldsymbol{x}\rangle\big| \\ &\leqslant M\|\boldsymbol{y}_n - \boldsymbol{y}\| + M\|\boldsymbol{x}_n - \boldsymbol{x}\| < M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon\,. \end{aligned}$$ □

Proposition 2.49. *In $\mathbb{R}^n$, a sequence of vectors converges if and only if every component of the vectors converges. In other words, in $\mathbb{R}^n$*

Componentwise convergence $\Leftrightarrow$ *Convergence.*

Proof. Let $\{\boldsymbol{v}_k\}_{k=1}^{\infty}$, $\boldsymbol{v}_k = (v_k^{(1)}, v_k^{(2)}, \cdots, v_k^{(n)})$, be a sequence of vectors in $\mathbb{R}^n$.

"$\Leftarrow$" Suppose that $\boldsymbol{v}_k \to \boldsymbol{v} = (v^{(1)}, \cdots, v^{(n)})$ as $k \to \infty$. Let $\varepsilon > 0$ be given. There exists $N > 0$ such that
$$\|\boldsymbol{v}_k - \boldsymbol{v}\|_2 < \varepsilon \quad \text{whenever} \quad k \geqslant N\,;$$
thus if $k \geqslant N$,
$$|v_k^{(i)} - v^{(i)}| \leqslant \sqrt{(v_k^{(1)} - v^{(1)})^2 + \cdots + (v_k^{(n)} - v^{(n)})^2} = \|\boldsymbol{v}_k - \boldsymbol{v}\|_2 < \varepsilon.$$

"$\Rightarrow$" Suppose that $v_k^{(i)} \to v^{(i)}$ as $k \to \infty$ for each $1 \leqslant i \leqslant n$. Let $\varepsilon > 0$ be given. For each $1 \leqslant i \leqslant n$, there exist $N_i > 0$ such that
$$\big|v_k^{(i)} - v^{(i)}\big| < \frac{\varepsilon}{\sqrt{n}} \quad \text{whenever} \quad k \geqslant N_i\,.$$
Let $\boldsymbol{v} = \big(v^{(1)}, v^{(2)}, \cdots, v^{(n)}\big)$ and $N = \max\{N_1, N_2, \cdots, N_n\}$. Then if $k \geqslant N$,
$$\|\boldsymbol{v}_k - \boldsymbol{v}\|_2 = \sqrt{(v_k^{(1)} - v^{(1)})^2 + \cdots + (v_k^{(n)} - v^{(n)})^2} < \sqrt{\frac{\varepsilon^2}{n} + \cdots + \frac{\varepsilon^2}{n}} = \varepsilon.$$ □

Example 2.50. Let $\boldsymbol{v}_k = \big(\frac{1}{k}, \frac{1}{k^2}\big) \in \mathbb{R}^2$. Then $\boldsymbol{v}_k \to (0,0)$ as $k \to \infty$ since

$$\sqrt{(\frac{1}{k} - 0)^2 + (\frac{1}{k^2} - 0)^2} = \frac{1}{k^2}\sqrt{k^2+1} \to 0 \text{ as } k \to \infty\,.$$

On the other hand, since $\frac{1}{k} \to 0$ and $\frac{1}{k^2} \to 0$ as $k \to \infty$, Proposition 2.49 implies that $\boldsymbol{v}_k \to (0,0)$ as $k \to \infty$.

Theorem 2.51 (Bolzano-Weierstrass)**.** *Every bounded sequence in* $(\mathbb{R}^n, \|\cdot\|_2)$ *has a convergent subsequence.*

Proof. We prove by induction. Let

$$S = \Big\{ n \in \mathbb{N} \,\Big|\, \text{every bounded sequence in } (\mathbb{R}^n, \|\cdot\|_2) \text{ has a convergent subsequence} \Big\}\,.$$

Then $1 \in S$ because of the Bolzano-Weierstrass Property of $\mathbb{R}$.

Suppose that $n \in S$. Let $\{\boldsymbol{x}_k\}_{k=1}^\infty$ be a bounded sequence in $\mathbb{R}^{n+1}$. Write $\boldsymbol{x}_k = \big(x_k^{(1)}, x_k^{(2)}, \cdots, x_k^{(n)}, x_k^{(n+1)}\big)$, and let $\boldsymbol{y}_k = \big(x_k^{(1)}, x_k^{(2)}, \cdots, x_k^{(n)}\big)$. Since $\{\boldsymbol{x}_k\}_{k=1}^\infty$ is bounded, there exists $M > 0$ such that $\|\boldsymbol{x}_k\|_2 \leqslant M$ for all $k \in \mathbb{N}$; thus

$$\|\boldsymbol{y}_k\|_2 \leqslant M \quad \forall\, k \in \mathbb{N} \qquad \text{and} \qquad \big|x_k^{(n+1)}\big| \leqslant M \qquad \forall\, k \in \mathbb{N}\,.$$

that is, $\{\boldsymbol{y}_k\}_{k=1}^\infty$ is bounded in $\mathbb{R}^n$ and $\big\{x_k^{(n+1)}\big\}_{k=1}^n$ is bounded in $\mathbb{R}$. By the assumption that $n \in S$, $\{y_k\}_{k=1}^\infty$ has a convergent subsequence $\{\boldsymbol{y}_{k_j}\}_{j=1}^\infty$ of $\{\boldsymbol{y}_k\}_{k=1}^\infty$ which converges to $\boldsymbol{y} = \big(y^{(1)}, y^{(2)}, \cdots, y^{(n)}\big)$. Applying the Bolzano-Weierstrass Property to the sequence $\big\{x_{k_j}^{(n+1)}\big\}_{j=1}^\infty$, we obtain a subsequence $\big\{x_{k_{j_\ell}}^{(n+1)}\big\}_{\ell=1}^\infty$ of $\big\{x_{k_j}^{(n+1)}\big\}_{k=1}^\infty$ so that $x_{k_{j_\ell}}^{(n+1)} \to y^{(n+1)}$ as $\ell \to \infty$. Let $\boldsymbol{z}_\ell = \boldsymbol{x}_{k_{j_\ell}}$. Then $\{\boldsymbol{z}_\ell\}_{\ell=1}^\infty$ is a subsequence of $\{\boldsymbol{x}_k\}_{k=1}^\infty$ and $\{\boldsymbol{z}_\ell\}_{\ell=1}^\infty$ converges to $\boldsymbol{z} = \big(y^{(1)}, y^{(2)}, \cdots, y^{(n+1)}\big)$ by Proposition 2.49. Therefore, $n+1 \in S$.

By induction, $S = \mathbb{N}$; thus the theorem is proved. □

Remark 2.52. By identifying $\mathbb{C}^n$ as $\mathbb{R}^{2n}$, Theorem 2.51 also implies that every bounded sequence in $\mathbb{C}^n$ has a convergent subsequence.

Definition 2.53. Let (M, d) be a metric space, and $\{x_n\}_{n=1}^\infty$ be a sequence in M. A point $x \in M$ is called a ***cluster point*** of $\{x_n\}_{n=1}^\infty$ if

$$\forall\, \varepsilon > 0,\ \#\big\{ n \in \mathbb{N} \,\big|\, x_n \in B(x, \varepsilon) \big\} = \infty\,.$$

Example 2.54. Let $x_n = (-1)^n$. Then 1 and -1 are the only two cluster points of $\{x_n\}_{n=1}^\infty$.

Example 2.55. Let $x_n = (-1)^n + \dfrac{1}{n}$. Then 1 and -1 are cluster points of $\{x_n\}_{n=1}^{\infty}$: Let $\varepsilon > 0$ be given. We observe that

$$\big\{n \in \mathbb{N} \,\big|\, x_n \in (1-\varepsilon, 1+\varepsilon)\big\} \supseteq \Big\{n \in \mathbb{N} \,\Big|\, n \text{ is even}, \frac{1}{n} < \varepsilon\Big\};$$

thus $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in (1-\varepsilon, 1+\varepsilon)\big\} = \infty$. Similarly, -1 is a cluster point.

On the other hand, each $a \neq \pm 1$ is not a cluster point of $\{x_n\}_{n=1}^{\infty}$.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Recall that a subsequence of $\{x_n\}_{n=1}^{\infty}$ is a sequence $\{y_j\}_{j=1}^{\infty}$ satisfying that $y_j = x_{f(j)}$ for some strictly increasing function $f : \mathbb{N} \to \mathbb{N}$. In other words, each strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ corresponds to a subsequence of $\{x_n\}_{n=1}^{\infty}$ and vice versa.

Similar to Proposition 1.68, we have the following

Proposition 2.56. *Let (M, d) be a metric space, $\{x_n\}_{n=1}^{\infty}$ be a sequence in M, and $x \in M$.*

1. *x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if for each $\varepsilon > 0$ and $N > 0$, there exists $n \geqslant N$ such that $d(x_n, x) < \varepsilon$.*

2. *x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges to x.*

3. *$x_n \to x$ as $n \to \infty$ if and only if every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to x.*

4. *$x_n \to x$ as $n \to \infty$ if and only if every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ has a further subsequence that converges to x.*

Proof. We only prove 1 and 2 since the proof of 3 and 4 are similar to the one given in Proposition 1.60.

1. "$\Rightarrow$" Let $\varepsilon > 0$ and $N > 0$ be given. Since $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(x, \varepsilon)\big\} = \infty$, there exist natural numbers $n_1 < n_2 < n_3 < \cdots$ such that

$$\big\{n \in \mathbb{N} \,\big|\, x_n \in B(x, \varepsilon)\big\} = \big\{n_1, n_2, n_3, \cdots\big\}.$$

Note that $n_j \geqslant j$; thus $n_N \geqslant N$.

"$\Leftarrow$" Let $\varepsilon > 0$ be given. Pick $n_1 \geqslant 1$ such that $d(x_{n_1}, x) < \varepsilon$, then pick $n_2 \geqslant n_1 + 1$ such that $d(x_{n_2}, x) < \varepsilon$. We continue this process and obtain a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ satisfying $d(x_{n_j}, x) < \varepsilon$ for all $j \in \mathbb{N}$. Then $\big\{n \in \mathbb{N} \,\big|\, x_n \in B(x, \varepsilon)\big\} \supseteq \{n_1, n_2, \cdots\}$ which implies that $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(x, \varepsilon)\big\} = \infty$.

2. "⇒" By 1, we can pick $n_1 \geqslant 1$ such that $d(x_{n_1}, x) < 1$ and then pick $n_2 \geqslant n_1 + 1$ such that $d(x_{n_2}, x) < \frac{1}{2}$. In general, we can pick $n_{k+1} \geqslant n_k + 1$ so that

$$d(x_{n_k}, x) < \frac{1}{k} \qquad \forall\, k \in \mathbb{N}\,.$$

Therefore, $\lim\limits_{k\to\infty} x_{n_k} = x$.

"⇐" Let $\varepsilon > 0$ be given. By assumption there exists $J > 0$ such that $d(x_{n_j}, x) < \varepsilon$ whenever $j \geqslant J$. Then $\big\{n \in \mathbb{N} \,\big|\, x_n \in B(x,\varepsilon)\big\} \supseteq \{n_J, n_{J+1}, \cdots\}$ which implies that $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(x,\varepsilon)\big\} = \infty$. □

2.3.2 Cauchy sequences, Banach spaces and Hilbert spaces

Definition 2.57. A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (M,d) is said to be ***Cauchy*** if

$$(\forall\, \varepsilon > 0)(\exists\, N > 0)\big(n, m \geqslant N \Rightarrow d(x_n, x_m) < \varepsilon\big)\,.$$

Similar to Proposition 1.74, Lemma 1.75 and 1.76, we have the following

Proposition 2.58. 1. *Every convergent sequence (in a metric space (M,d)) is Cauchy.*

2. *Every Cauchy sequence (in a metric space (M,d)) is bounded.*

3. *If a subsequence of Cauchy sequence (in a metric space (M,d)) converges, then this Cauchy sequence also converges.*

Proof. See the proof of Proposition 1.74 and Lemma 1.76 by changing $|x-y|$ to $d(x,y)$ with appropriate x and y. □

Remark 2.59. By 2 of Proposition 2.56 and 3 of Proposition 2.58, we find that if $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence but $\{x_k\}_{k=1}^{\infty}$ does not converge, then

$$(\forall\, y)(\exists\, r > 0)\big(\#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(y,r)\big\} < \infty\big)\,.$$

Theorem 2.60. *A sequence in $\mathbb{R}^n$ converges if and only if the sequence is Cauchy* (*because of the inequality* $\max\limits_{1\leqslant i\leqslant n} \big|v_k^{(i)} - v_\ell^{(i)}\big| \leqslant \|\boldsymbol{v}_k - \boldsymbol{v}_\ell\|_2 \leqslant \sqrt{n} \max\limits_{1\leqslant i\leqslant n} |v_k^{(i)} - v_\ell^{(i)}|$).

Now we would like to define the completeness of a normed vector space. Recall that in an Archimedean ordered field, the following four properties are equivalent:

1. the Bolzano-Weierstrass property (有界數列必有收斂子數列),

2. the monotone sequence property (單調有界數列必收斂),
3. the least upper bound property (非空集合有上界必有最小上界),
4. the property that every Cauchy sequence converges (柯西數列必收斂).

Since in general a normed vector space cannot be an ordered field, we cannot define the completeness through the monotone sequence property or the least upper bound property. For the completeness of normed vector spaces, we use the ***Cauchy completeness***.

Definition 2.61. A metric space (M, d) is said to be complete if every Cauchy sequence in M converges. A ***Banach space*** is a complete normed vector space, and a ***Hilbert space*** is a complete inner product space (that is, a Banach space whose norm is induced by the inner product).

Example 2.62. The Euclidean n-space $\mathbb{R}^n$, equipped with p-norm, is a Banach space for all $1 \leqslant p \leqslant \infty$. To see this, let $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^n$. Then the sequence $\{x_k^{(i)}\}_{k=1}^{\infty}$, consisting of the i-th components of $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$, is Cauchy for all $1 \leqslant i \leqslant n$ since

$$\left|x_k^{(i)} - x_\ell^{(i)}\right| \leqslant \|\boldsymbol{x}_k - \boldsymbol{x}_\ell\|_p \qquad \forall\, 1 \leqslant i \leqslant n \text{ and } 1 \leqslant p \leqslant \infty\,.$$

By the completeness of $\mathbb{R}$, the real sequence $\{x_k^{(i)}\}_{k=1}^{\infty}$ converges for all $1 \leqslant i \leqslant n$; thus each component of $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ converges. Proposition 2.49 implies that $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ converges.

2.4 Series of Real Numbers and Vectors

Definition 2.63. Let $(\mathcal{V}, \|\cdot\|)$ be a normed space. A series $\sum\limits_{k=1}^{\infty} \boldsymbol{x}_k$, where $\{\boldsymbol{x}_k\}_{k=1}^{\infty} \subseteq \mathcal{V}$, is said to ***converge*** to $\boldsymbol{S} \in \mathcal{V}$ if the partial sum $\boldsymbol{S}_n = \sum\limits_{k=1}^{n} \boldsymbol{x}_k$ converges to $\boldsymbol{S}$, and one writes $\boldsymbol{S} = \sum\limits_{k=1}^{\infty} \boldsymbol{x}_k$ if this is the case. A series in $\mathcal{V}$ is said to converge or be convergent if it converges to some element in $\mathcal{V}$.

The following proposition is a direct consequence of the monotone sequence property of the real number system $\mathbb{R}$.

Proposition 2.64. *Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of real numbers, and $x_k \geqslant 0$ for all $k \in \mathbb{N}$. Then $\sum\limits_{k=1}^{\infty} x_k$ converges if and only if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$, where $S_n = \sum\limits_{k=1}^{n} x_k$, is bounded (from above).*

Theorem 2.65 (Cauchy's criterion)**.** *Let $(\mathcal{V}, \|\cdot\|)$ be a Banach space. A series $\sum_{k=1}^{\infty} \boldsymbol{x}_k$ converges if and only if*

$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni \|\boldsymbol{x}_k + \boldsymbol{x}_{k+1} + \cdots + \boldsymbol{x}_{k+p}\| < \varepsilon \quad \textit{whenever} \quad k \geqslant N, p \geqslant 0\,.$$

Proof. Let $\boldsymbol{S}_n = \sum_{k=1}^{n} \boldsymbol{x}_k$ be partial sum of $\sum_{k=1}^{\infty} \boldsymbol{x}_k$. Then

$$\begin{aligned}
\{\boldsymbol{S}_n\}_{n=1}^{\infty} &\text{ converges in } \mathcal{V} \Leftrightarrow \{\boldsymbol{S}_n\}_{n=1}^{\infty} \text{ is Cauchy} \\
&\Leftrightarrow \forall\, \varepsilon > 0, \exists\, N > 0 \ni \|\boldsymbol{S}_n - \boldsymbol{S}_m\| < \varepsilon \text{ whenever } n, m \geqslant N \\
&\Leftrightarrow \forall\, \varepsilon > 0, \exists\, N > 0 \ni \|\boldsymbol{x}_{n+1} + \boldsymbol{x}_{n+2} + \cdots + \boldsymbol{x}_m\| < \varepsilon \text{ whenever } m > n \geqslant N \\
&\Leftrightarrow \forall\, \varepsilon > 0, \exists\, N > 0 \ni \|\boldsymbol{x}_k + \boldsymbol{x}_{k+1} + \cdots + \boldsymbol{x}_{k+p}\| < \varepsilon \text{ whenever } k \geqslant N+1, p \geqslant 0.
\end{aligned}$$ □

Corollary 2.66 (n-th term test)**.** *If $\sum_{k=1}^{\infty} \boldsymbol{x}_k$ converges, then $\|\boldsymbol{x}_k\| \to 0$ as $k \to \infty$, and if $\|\boldsymbol{x}_k\| \nrightarrow 0$ as $k \to \infty$, then $\sum_{k=1}^{\infty} x_k$ diverges.*

Proof. Take $p = 0$ in Theorem 2.65. □

Definition 2.67. A series $\sum_{k=1}^{\infty} \boldsymbol{x}_k$ is said to ***converge absolutely*** if $\sum_{k=1}^{\infty} \|\boldsymbol{x}_k\|$ converges in $\mathbb{R}$. A series that is convergent but not absolutely convergent is said to be ***conditionally convergent***.

Example 2.68. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ is conditionally convergent. See Theorem 2.70 for the reason.

Theorem 2.69. *Let $(\mathcal{V}, \|\cdot\|)$ be a Banach space, and $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{V}$. If $\sum_{k=1}^{\infty} \boldsymbol{x}_k$ converges absolutely, then $\sum_{k=1}^{\infty} \boldsymbol{x}_k$ converges.*

Proof. If $\sum_{k=1}^{\infty} \boldsymbol{x}_k$ converges absolutely, then $S_n = \sum_{k=1}^{n} \|\boldsymbol{x}_k\|$ converges in $\mathbb{R}$. Then

$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni \big|\|\boldsymbol{x}_k\| + \|\boldsymbol{x}_{k+1}\| + \cdots + \|\boldsymbol{x}_{k+p}\|\big| < \varepsilon \quad \text{whenever} \quad k \geqslant N, p \geqslant 0\,.$$

Therefore, if $k \geqslant N, p \geqslant 0$,

$$\|\boldsymbol{x}_k + \boldsymbol{x}_{k+1} + \cdots + \boldsymbol{x}_{k+p}\| \leqslant \|\boldsymbol{x}_k\| + \cdots + \|\boldsymbol{x}_{k+p}\| < \varepsilon\,,$$

and the convergence of $\sum_{k=1}^{\infty} \boldsymbol{x}_k$ is guaranteed by the Cauchy criterion. □

Theorem 2.70. 1. ***Geometric series***:

(a) *If* $|r| < 1$, *then* $\sum\limits_{k=1}^{\infty} r^k$ *converges absolutely to* $\dfrac{r}{1-r}$.

(b) *If* $|r| \geqslant 1$, *then* $\sum\limits_{k=1}^{\infty} r^k$ *does not converge (diverge).*

2. ***Comparison test***: *Let* $\{a_k\}_{k=1}^{\infty}$ *and* $\{b_k\}_{k=1}^{\infty}$ *be sequences of real numbers.*

(a) *If* $\sum\limits_{k=1}^{\infty} a_k$ *converges, and* $0 \leqslant b_k \leqslant a_k$, *then* $\sum\limits_{k=1}^{\infty} b_k$ *converges.*

(b) *If* $\sum\limits_{k=1}^{\infty} a_k$ *diverges, and* $0 \leqslant a_k \leqslant b_k$, *then* $\sum\limits_{k=1}^{\infty} b_k$ *diverges.*

3. ***Integral test***: *If* f *is continuous, non-negative, and monotone decreasing on* $[1,\infty)$, *then* $\sum\limits_{k=1}^{\infty} f(k)$ *converges if and only if the improper integral* $\displaystyle\int_1^{\infty} f(x)dx < \infty$.

4. ***Root test***: *Let* $\{x_k\}_{k=1}^{\infty}$ *be a sequence of real numbers.*

(a) *If* $\limsup\limits_{k\to\infty} \sqrt[k]{|x_k|} < 1$, *then* $\sum\limits_{k=1}^{\infty} x_k$ *converges absolutely.*

(b) *If* $\limsup\limits_{k\to\infty} \sqrt[k]{|x_k|} > 1$, *then* $\sum\limits_{k=1}^{\infty} x_k$ *diverges.*

5. ***Ratio and comparison test***: *Let* $\{a_k\}_{k=1}^{\infty}$ *and* $\{b_k\}_{k=1}^{\infty}$ *be sequences of real numbers, and* $b_k > 0$ *for all* $k \in \mathbb{N}$.

(a) $\limsup\limits_{k\to\infty} \dfrac{|a_k|}{b_k} < \infty$, $\sum\limits_{k=1}^{\infty} b_k$ *is convergent, then* $\sum\limits_{k=1}^{\infty} a_k$ *converges absolutely.*

(b) $\liminf\limits_{k\to\infty} \dfrac{a_k}{b_k} > 0$, $\sum\limits_{k=1}^{\infty} b_k$ *is divergent, then* $\sum\limits_{k=1}^{\infty} a_k$ *diverges.*

6. ***Dirichlet test***: *Let* $\{a_k\}_{n=1}^{\infty}$, $\{p_k\}_{n=1}^{\infty}$ *be sequences of real numbers such that*

(a) *the sequence of partial sums of the series* $\sum\limits_{k=1}^{\infty} a_k$ *is bounded; that is, there exists* $M \in \mathbb{R}$ *such that* $\Big|\sum\limits_{k=1}^{n} a_k\Big| \leqslant M$ *for all* $n \in \mathbb{N}$.

(b) $\{p_k\}_{k=1}^{\infty}$ *is a decreasing sequence, and* $\lim\limits_{k\to\infty} p_k = 0$.

Then $\sum\limits_{k=1}^{\infty} a_k p_k$ *converges.*

Proof. Note that 1 follows from the fact that $\sum\limits_{k=1}^{n} r^k = \dfrac{r - r^{n+1}}{1-r}$ if $r \neq 1$, the comparison test follows from Proposition 2.64, and the integral test follows from the fact that

$$\int_1^{n+1} f(x)\,dx \leqslant S_n \equiv \sum_{k=1}^{n} f(k) \leqslant f(1) + \int_1^n f(x)\,dx$$

with an application of the comparison test, we only prove 4, 5 and 6.

4. Let $r = \limsup\limits_{k\to\infty} \sqrt[k]{|x_k|}$.

   (a) Suppose that $0 \leqslant r < 1$. By Proposition 1.98, there exists $N > 0$ such that $\sqrt[k]{|x_k|} < \dfrac{r+1}{2} \Big(= r + \dfrac{1-r}{2}\Big)$ for all $k \geqslant N$. This implies that

   $$|x_k| \leqslant \Big(\frac{r+1}{2}\Big)^k \qquad \forall\, k \geqslant N\,.$$

   Since $\Big|\dfrac{r+1}{2}\Big| < 1$, the geometric series $\sum\limits_{k=1}^{\infty} \big(\dfrac{r+1}{2}\big)^k$ converges; thus the comparison test implies that the series $\sum\limits_{k=1}^{\infty} |x_k|$ converges.

   (b) Suppose that $r > 1$. By Proposition 1.98 there exist $n_1 < n_2 < \cdots < n_j < \cdots$ such that

   $$\sqrt[k]{|x_k|} > \frac{r+1}{2} \qquad \forall\, k = n_1, n_2, \cdots,$$

   The statement above then implies that $\lim\limits_{k\to\infty} x_k$, if exists, cannot be zero; thus the n-th term test shows that $\sum\limits_{k=1}^{\infty} x_k$ diverges.

5. (a) Suppose that $\limsup\limits_{k\to\infty} \dfrac{|a_k|}{b_k} = c < \infty$. By Proposition 1.98 there exists $N > 0$ such that $\dfrac{|a_k|}{b_k} < c + 1$ for all $k \geqslant N$. This implies that $|a_k| < (c+1)b_k$ for all $k \geqslant N$; thus the convergence of $\sum\limits_{k=1}^{\infty} b_k$ and the comparison test imply that the series $\sum\limits_{k=1}^{\infty} |a_k|$ converges.

   (b) Suppose that $\liminf\limits_{k\to\infty} \dfrac{a_k}{b_k} = c > 0$. By Proposition 1.98 there exists $N > 0$ such that $\dfrac{a_k}{b_k} > \dfrac{c}{2}$ for all $k \geqslant N$. This implies that $a_k > \dfrac{c}{2} b_k$ for all $k \geqslant N$; thus the divergence of $\sum\limits_{k=1}^{\infty} b_k$ and the comparison test imply that the series $\sum\limits_{k=1}^{\infty} |a_k|$ diverges.

6. Let $\varepsilon > 0$ be given. Since $\{p_n\}_{n=1}^\infty$ is decreasing and $\lim\limits_{n\to\infty} p_n = 0$, there exists $N > 0$ such that
$$0 \leqslant p_n < \frac{\varepsilon}{2M+1} \qquad \text{whenever} \quad n \geqslant N\,.$$
Define $S_n = \sum\limits_{k=1}^n a_k$. Then if $n \geqslant N$ and $\ell \geqslant 0$,
$$\begin{aligned}\Big|\sum_{k=n}^{n+\ell} a_k p_k\Big| &= \big|a_n p_n + a_{n+1}p_{n+1} + a_{n+2}p_{n+2} + \cdots + a_{n+\ell-1}p_{n+\ell-1} + a_{n+\ell}p_{n+\ell}\big| \\ &= \big|(S_n - S_{n-1})p_n + (S_{n+1} - S_n)p_{n+1} + (S_{n+2} - S_{n+1})p_{n+2} + \cdots\cdots \\ &\quad + (S_{n+\ell-1} - S_{n+\ell-2})p_{n+\ell-1} + (S_{n+\ell} - S_{n+\ell-1})p_{n+\ell}\big| \\ &= \big|-S_{n-1}p_n + S_n(p_n - p_{n+1}) + S_{n+1}(p_{n+1} - p_{n+2}) + \cdots\cdots \\ &\quad + S_{n+\ell-1}(p_{n+\ell-1} - p_{n+\ell}) + S_{n+\ell}p_{n+\ell}\big| \\ &\leqslant |S_{n-1}p_n| + |S_n(p_n - p_{n+1})| + |S_{n+1}(p_{n+1} - p_{n+2})| + \cdots\cdots \\ &\quad + |S_{n+\ell-1}(p_{n+\ell-1} - p_{n+\ell})| + |S_{n+\ell}p_{n+\ell}| \\ &\leqslant Mp_n + M(p_n - p_{n+1}) + M(p_{n+1} - p_{n+2}) + \cdots\cdots \\ &\quad + M(p_{n+\ell-1} - p_{n+\ell}) + Mp_{n+\ell} \\ &= 2Mp_n < \frac{2M\varepsilon}{2M+1} < \varepsilon\,.\end{aligned}$$
The convergence of $\sum\limits_{k=1}^\infty a_k p_k$ follows from the Cauchy criterion (Theorem 2.65). □

Corollary 2.71. *1. The p-series* $\sum\limits_{k=1}^\infty \frac{1}{k^p}$ *converges if and only if* $p > 1$.

2. *The alternating series* $\sum\limits_{k=1}^\infty (-1)^k a_k$ *converges if* $\{a_k\}_{k=1}^\infty$ *is a decreasing convergent sequence with limit* 0.

Remark 2.72. It can be shown (and the proof is left as an exercise) that
$$\liminf_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} \leqslant \liminf_{k\to\infty} \sqrt[k]{|x_k|} \leqslant \limsup_{k\to\infty} \sqrt[k]{|x_k|} \leqslant \limsup_{k\to\infty} \frac{|x_{k+1}|}{|x_k|}\,.$$
As a consequence, by the root test we obtain

1. if $\limsup\limits_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} < 1$, the series $\sum\limits_{k=1}^\infty x_k$ converges absolutely, and
2. if $\liminf\limits_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} > 1$, the series $\sum\limits_{k=1}^\infty x_k$ diverges.

This is called the ***ratio test***.

Example 2.73. The series $\sum\limits_{k=1}^{\infty} \frac{\sin k}{k^p}$ converges for $p > 0$ since

1. $\sum\limits_{k=1}^{n} \sin k = \dfrac{\cos\frac{1}{2} - \cos\frac{2k+1}{2}}{2\sin\frac{1}{2}}$; $\Big($thus $\Big|\sum\limits_{k=1}^{n} \sin k\Big| \leqslant \dfrac{1}{\sin\frac{1}{2}}\Big)$.
2. $\big\{\frac{1}{n^p}\big\}_{n=1}^{\infty}$ is decreasing and $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$.

We remark here that $\sum\limits_{k=1}^{\infty} \frac{\sin k}{k} = \frac{\pi - 1}{2}$. In fact, $\sum\limits_{k=1}^{\infty} \frac{\sin(kx)}{k}$ is the "Fourier series" of the function $\frac{\pi - x}{2}$.

Example 2.74. Let $\{x_k\}_{k=1}^{\infty}$ be a real sequence defined by

$$x_k = \begin{cases} 2^{-k} & \text{if } k \text{ is odd}\,, \\ 4^{-k} & \text{if } k \text{ is even}\,, \end{cases}$$

or $x_k = (3 + (-1)^k)^{-k}$. Then $\sqrt[k]{|x_k|} = (3 + (-1)^k)^{-1}$ which shows that

$$\liminf_{k\to\infty} \sqrt[k]{|x_k|} = \frac{1}{4} \qquad \text{and} \qquad \limsup_{k\to\infty} \sqrt[k]{|x_k|} = \frac{1}{2}\,.$$

Therefore, the root test implies that the series $\sum\limits_{k=1}^{\infty} x_k$ converges absolutely.

We can also compute the limit superior and limit inferior of $\frac{|x_{k+1}|}{|x_k|}$. Define

$$y_k = \frac{|x_{k+1}|}{|x_k|} = \frac{(3 + (-1)^{k+1})^{-k-1}}{(3 + (-1)^k)^{-k}} = \frac{1}{3 - (-1)^k}\Big(\frac{3 - (-1)^k}{3 + (-1)^k}\Big)^{-k}$$

and observe that $\lim\limits_{k\to\infty} y_{2k} = \infty$ and $\lim\limits_{k\to\infty} y_{2k+1} = 0$. Since $y_k \in [0, \infty)$, we conclude that 0 is the smallest cluster point of $\{y_k\}_{k=1}^{\infty}$ and ∞ is the largest "cluster point" of $\{y_k\}_{k=1}^{\infty}$. This shows that

$$\liminf_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} = 0 \qquad \text{and} \qquad \limsup_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} = \infty\,.$$

We note that in this case even if the series $\sum\limits_{k=1}^{\infty} x_k$ converges absolutely, $\limsup\limits_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} > 1$. Therefore, the condition $\limsup\limits_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} > 1$ cannot be used to guarantee the divergence of the series $\sum\limits_{k=1}^{\infty} x_k$.

Chapter 3

Elementary Point-Set Topology

3.1 Limit Points and Interior Points of Sets

Definition 3.1. Let (M,d) be a metric space, and A be a subset of M.

1. A point $x \in M$ is called a ***limit point*** of A if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in A such that $\{x_n\}_{n=1}^{\infty}$ converges to x.

2. The ***closure*** of A is the collection of all limit points of A is denoted by $\bar{A}$ or $\text{cl}(A)$.

3. A point $x \in M$ is called an ***interior point*** of A if there exists $r > 0$ such that the r-ball about x is contained in A; that is, $B(x,r) \subseteq A$.

4. The ***interior*** of A is the collection of all interior points of A and is denoted by $\mathring{A}$ or $\text{int}(A)$.

5. A point $x \in M$ is called an ***exterior point*** of A if x is an interior point of $A^{\complement}$, and the collection of all exterior points of A is called the ***exterior*** of A.

Example 3.2. For $a, b \in \mathbb{R}$ and $a < b$, consider the interval I in $(R, |\cdot|)$ with end-points a and b. Then $\bar{\mathrm{I}} = [a,b]$ and $\mathring{\mathrm{I}} = (a,b)$.

Remark 3.3. By the definition of the convergence of sequences in metric spaces, we have the following equivalent definition: A point $x \in M$ is called a limit point of A if for every $\varepsilon > 0$, $B(x,\varepsilon)$ contains points in A; that is, $\forall\, \varepsilon > 0, B(x,\varepsilon) \cap A \neq \varnothing$.

Remark 3.4. 1. If $x \in A$, then x is a limit point of A. In other words, $A \subseteq \bar{A}$.

2. If $x \in \mathring{A}$, then $x \in A$; thus $\mathring{A} \subseteq A$.

Theorem 3.5. *Let (M, d) be a metric space, and A, B be subsets of M. If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$ and $\mathring{A} \subseteq \mathring{B}$.*

Proof. 1. Let $x \in \bar{A}$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in A with limit x. Since $A \subseteq B$, $\{x_n\}_{n=1}^{\infty}$ is a sequence in B with limit x; thus $x \in \bar{B}$.

2. Let $x \in \mathring{A}$. Then there exists $r > 0$ such that $B(x, r) \subseteq A$. Since $A \subseteq B$, $B(x, r) \subseteq B$; thus $x \in \mathring{B}$. □

Proposition 3.6. *Let (M, d) be a metric space, and A be a subset of M. Then*

$$x \in \bar{A} \qquad \text{if and only if} \qquad d(x, A) \equiv \inf\left\{d(x, y) \,\middle|\, y \in A\right\} = 0.$$

Proof. "$\Rightarrow$" Suppose that $x \in \bar{A}$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in A with limit x. By the definition of $d(x, A)$,

$$0 \leqslant d(x, A) = \inf\left\{d(x, y) \,\middle|\, y \in A\right\} \leqslant d(x, x_n) \qquad \forall\, n \in \mathbb{N}\,;$$

thus the Sandwich lemma (Lemma 1.34) and Remark 2.43 imply that $d(x, A) = 0$.

"$\Leftarrow$" Suppose $d(x, A) = 0$. By the definition of $d(x, A)$, for all $n \in \mathbb{N}$ there exists $x_n \in A$ such that $d(x, x_n) < d(x, A) + \frac{1}{n} = \frac{1}{n}$. Therefore, we obtain a sequence $\{x_n\}_{n=1}^{\infty}$ in A such that $\lim\limits_{n\to\infty} x_n = x$; thus $x \in \bar{A}$. □

Remark 3.7. Let (M, d) be a metric space. The function $d(x, A)$ defined in the example above does not satisfy that

$$d(x, y) \leqslant d(x, A) + d(y, A) \qquad \forall\, x, y \in M\,, A \subseteq M\,.$$

However, if $d(A, B) = \inf\left\{d(a, b) \,\middle|\, a \in A, b \in B\right\}$, then

$$d(A, B) \leqslant d(x, A) + d(x, B) \qquad \forall\, x \in M\,.$$

The proof of the inequality above is left as an exercise.

Definition 3.8. Let (M, d) be a metric space. A subset A of M is said to be ***dense***（稠密）in another subset B if $A \subseteq B \subseteq \bar{A}$.

Remark 3.9. When A is dense in B, it means that every point in B can be the limit of a sequence in A.

Example 3.10. The rational numbers $\mathbb{Q}$ is dense in the real number system $\mathbb{R}$.

Definition 3.11. Let (M,d) be a metric space, and A be a subset of M. The ***boundary*** of A, denoted by $\mathrm{bd}(A)$ or ∂A, is the intersection of $\bar{A}$ and $\overline{A^\complement}$ ($\partial A = \bar{A} \cap \overline{A^\complement}$).

Remark 3.12. 1. By the definition of limit points of sets, we find that

$$\begin{aligned} x \in \partial A &\Leftrightarrow \exists \{x_n\}_{n=1}^\infty \subseteq A \text{ and } \{y_n\}_{n=1}^\infty \subseteq A^\complement \ni x_n \to x \text{ and } y_n \to x \text{ as } n \to \infty \\ &\Leftrightarrow \forall\, \varepsilon > 0, B(x,\varepsilon) \cap A \neq \varnothing \text{ and } B(x,\varepsilon) \cap A^\complement \neq \varnothing\,. \end{aligned}$$

2. $\partial A = \partial(A^\complement)$.

Proposition 3.13. *Let (M,d) be a metric space, and A be a subset of M. Then $\partial A = \bar{A} \backslash \mathring{A}$.*

Proof. If $x \in \partial A$, then $x \in \bar{A} \cap \overline{A^\complement}$; thus for all $\varepsilon > 0$, $B(x,\varepsilon) \cap A^\complement \neq \varnothing$. Therefore, $x \notin \mathring{A}$ which implies that $\partial A \subseteq \bar{A} \backslash \mathring{A}$.

On the other hand, if $x \in \bar{A} \backslash \mathring{A}$, then $x \notin \mathring{A}$; thus for all $\varepsilon > 0$, $B(x,\varepsilon) \nsubseteq A$. As a consequence, for all $\varepsilon > 0$, $B(x,\varepsilon) \cap A^\complement \neq \varnothing$; thus $x \in \overline{A^\complement}$ and this further implies that $x \in \bar{A} \cap \overline{A^\complement} = \partial A$. □

Remark 3.14. 1. If $A \subseteq B$, then in general $\partial A \nsubseteq \partial B$. For example, let $A = \mathbb{Q} \cap [0,1]$ and $B = [0,1]$. Then $A \subseteq B$ but $\partial A = [0,1]$, $\partial B = \{0,1\}$.

2. It is not always true that $\partial A = \partial(\mathrm{int}(A))$. For example, take $A = [0,1] \cup \{2\}$. Then $\partial A = \{0,1,2\}, \mathrm{int}(A) = (0,1), \partial(\mathrm{int}(A)) = \{0,1\}$, so $\partial A \neq \partial(\mathrm{int}(A))$.

3.2 Closed Sets and Open Sets

3.2.1 Closed sets

Definition 3.15. Let (M,d) be a metric space. A subset F of M is said to be ***closed*** (in M) if F contains all its limit points; that is, $F \supseteq \bar{F}$. In other words, F is closed if every convergent sequence $\{x_k\}_{k=1}^\infty \subseteq F$ converges to a limit in F.

Remark 3.16. Let (M,d) be a metric space, and A be a subset of M.

1. By the definition of the closure of sets, $A \subseteq \bar{A}$; thus A is closed if and only if $A = \bar{A}$.

2. By Remark 3.3, A is closed if and only if for all $x \in A^\complement$ there exists $r > 0$ such that $B(x,r) \subseteq A^\complement$.

3. By the definition of the exterior points and 2 above, A is closed if and only if every point of $A^\complement$ is an interior point of $A^\complement$ (or equivalently, $A^\complement = \text{int}(A^\complement)$).

4. If B is a closed set and $A \subseteq B$, then Theorem 3.5 implies that $\bar{A} \subseteq B$.

Example 3.17. Let $a, b \in \mathbb{R}$. The interval $[a, b]$, $(-\infty, a]$, $[b, \infty)$ in $\mathbb{R}$ are closed. This is why $[a, b]$, $(-\infty, a]$ and $[b, \infty)$ are called "closed" intervals in $\mathbb{R}$.

The set $(0, 1] \subseteq \mathbb{R}$ is not closed because it does not contain 0, a limit point of $(0, 1]$.

In general, we have the following

Proposition 3.18. *Let (M, d) be a metric space, $x \in M$ and $r \geqslant 0$.*

1. *The set $B(x, r)^\complement$ is closed.*

2. *The set $\{y \in M \,|\, d(x, y) \leqslant r\}$ is closed.*

Proof. 1. Suppose the contrary that there exists a sequence $\{y_n\}_{n=1}^\infty \subseteq B(x, r)^\complement$ which converges to some $y \in B(x, r)$. Note that $d(x, y) < r$; thus there exists $N > 0$ such that

$$d(y_n, y) < \varepsilon = r - d(x, y) \qquad \text{whenever} \quad n \geqslant N\,.$$

By the triangle inequality, for $n \geqslant N$ we have

$$d(y_n, x) \leqslant d(y_n, y) + d(y, x) < r - d(x, y) + d(x, y) = r$$

which implies that $y_n \in B(x, r)$ for $n \geqslant N$, a contradiction.

2. Let $A = \{y \in M \,|\, d(x, y) \leqslant r\}$. Suppose the contrary that there exists a sequence $\{y_n\}_{n=1}^\infty \subseteq A$ which converges to some $y \in A^\complement$. Since $d(x, y) > r$, there exists $N > 0$ such that

$$d(y_n, y) < \varepsilon = d(x, y) - r \qquad \text{whenever} \quad n \geqslant N\,.$$

By the triangle inequality, for $n \geqslant N$ we have

$$d(y_n, x) \geqslant d(x, y) - d(y, y_n) > d(x, y) - \big(d(x, y) - r\big) = r$$

which implies that $y_n \notin A$ for $n \geqslant N$, a contradiction. □

Remark 3.19. When $r = 0$, the set $\{y \in M \,|\, d(x, y) \leqslant r\}$ contains only one point x; thus every set consisting of one single point in M is closed.

Definition 3.20. Let (M, d) be a metric space. For each $x \in M$ and $r > 0$, the set

$$B[x, r] \equiv \{y \in M \mid d(x, y) \leqslant r\}$$

is called the closed r-ball about x or the closed ball centered at x with radius r.

Proposition 3.21. *Let (M, d) be a metric space.*

1. *The union of finitely many closed sets is closed.*
2. *The intersection of arbitrary family of closed sets is closed.*
3. *The universal set M and the empty set $\varnothing$ are closed.*

Proof. 1. Let $F_1, \cdots, F_k$ be closed sets in M, $F = \bigcup_{j=1}^{k} F_j$, and $\{x_n\}_{n=1}^{\infty} \subseteq F$ be a convergent sequence with limit $x \in M$. Then there exists $1 \leqslant j_0 \leqslant k$ such that

$$\#\{n \in \mathbb{N} \mid x_n \in F_{j_0}\} = \infty\,;$$

thus $\{n \in \mathbb{N} \mid x_n \in F_{j_0}\} = \{n_1, n_2, \cdots, n_k, \cdots\}$, where $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. By Proposition 2.56, the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x. Since F_{j_0} is closed, $x \in F_{j_0}$; thus $x \in F$. Therefore, every convergent sequence in F converges to a limit in F which shows that F is closed.

2. Let $\mathscr{F} = \{F_\alpha \mid F_\alpha \text{ closed in } M,\, \alpha \in I\}$ be a family of closed sets, $F \equiv \bigcap_{\alpha \in I} F_\alpha$, and $\{x_n\}_{n=1}^{\infty} \subseteq F$ be a convergent sequence with limit $x \in M$. Then for each $\alpha \in I$, $\{x_n\}_{n=1}^{\infty}$ in F_α; thus the closedness of F_α implies that $x \in F_\alpha$ for each $\alpha \in I$. Therefore, $x \in F$ which shows that F is closed.

3. Since every consequence in M converges to a limit in M, M must be closed. Since there is no sequence in $\varnothing$, it holds that $\varnothing$ is closed. □

Alternative proof of 1 *and* 2. 1. Let $F_1, \cdots, F_k$ be closed sets, $F = \bigcup_{j=1}^{k} F_j$, and $x \in F^{\complement}$. By De Morgan's law,

$$F^{\complement} = M \backslash F = M \backslash \bigcup_{j=1}^{k} F_j = \bigcap_{j=1}^{k} (M \backslash F_j) = \bigcap_{j=1}^{k} F_j^{\complement}\,,$$

so $x \in F_j^{\complement}$ for all $1 \leqslant j \leqslant k$. By Remark 3.16, for each $1 \leqslant j \leqslant k$ there exists $r_j > 0$ such that

$$B(x, r_j) \subseteq F_j^{\complement}\,.$$

Define $r = \min\{r_1, r_2, \cdots, r_j\}$. Then $r > 0$ and $B(x,r) \subseteq B(x,r_j) \subseteq F_j^{\complement}$ for all $1 \leqslant j \leqslant k$; thus

$$B(x,r) \subseteq \bigcap_{j=1}^{k} F_j^{\complement} \subseteq \Big(\bigcup_{j=1}^{k} F_j\Big)^{\complement} = F^{\complement}.$$

2. Let $\mathscr{F} = \big\{F_\alpha \,\big|\, F_\alpha \text{ closed in } M,\ \alpha \in I\big\}$ be a family of closed sets, $F \equiv \bigcap_{\alpha\in I} F_\alpha$, and $x \in F$. By De Morgan's law,

$$F^{\complement} = M \backslash \bigcap_{\alpha\in I} F_\alpha = \bigcup_{\alpha\in I}(M\backslash F_\alpha) = \bigcup_{\alpha\in I} F_\alpha^{\complement}$$

so $x \in F_\beta^{\complement}$ for some $\beta \in I$. By Remark 3.16, there exists $r > 0$ such that

$$B(x,r) \subseteq F_\beta^{\complement} \subseteq \bigcup_{\alpha\in I} F_\alpha^{\complement} = \Big(\bigcap_{\alpha\in I} F_\alpha\Big)^{\complement} = F^{\complement}.$$ ▫

Corollary 3.22. *Every set consisting of finitely many points of a metric space is closed.*

Example 3.23. Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, $A \subseteq \mathcal{V}$ be closed, and $B \subseteq \mathcal{V}$ be finite ($\#(B) < \infty$). Then $A + B$ is closed.

Proof. Let $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ be a convergent sequence in $A+B$ with limit $\boldsymbol{x}$. Then $\boldsymbol{x}_n = \boldsymbol{a}_n + \boldsymbol{b}_n$ for some $\boldsymbol{a}_n \in A$ and $\boldsymbol{b}_n \in B$. Since $\#(B) < \infty$, there exists a point $\boldsymbol{b} \in B$ such that

$$\#\{n \in \mathbb{N} \,|\, \boldsymbol{b}_n = \boldsymbol{b}\} = \infty.$$

Let $\{n \in \mathbb{N} \,|\, \boldsymbol{b}_n = b\} = \{n_1, n_2, \cdots, n_k, \cdots\}$, where $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Then $\{\boldsymbol{x}_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{\boldsymbol{x}_n\}$. By Proposition 2.56, $\{\boldsymbol{x}_{n_k}\}_{k=1}^{\infty}$ converges to $\boldsymbol{x}$; thus the fact that $\boldsymbol{a}_{n_k} = \boldsymbol{x}_{n_k} - \boldsymbol{b}$ shows that $\{\boldsymbol{a}_{n_k}\}_{k=1}^{\infty}$ converges to a limit $\boldsymbol{a}$ and $\boldsymbol{x} = \boldsymbol{a} + \boldsymbol{b}$. The closedness of A further implies that $\boldsymbol{a} \in A$; thus $\boldsymbol{x} \in A + B$.

In fact, $A + B = \bigcup_{\boldsymbol{b}\in B} (\boldsymbol{b} + A)$. It should be clear that $\boldsymbol{b} + A$ is closed if A is closed; thus we conclude that $A + B$ is open by Proposition 3.21. ▫

Theorem 3.24. *Let (M, d) be a metric space, and A be a subset of M. Then $\bar{A}$ is closed.*

Proof. Let $x \in \bar{A}^{\complement}$ be given. Remark 3.3 implies that there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \cap A = \varnothing$ or equivalently, $A \subseteq B(x,\varepsilon)^{\complement}$. By Proposition 3.18, $B(x,\varepsilon)^{\complement}$ is closed; thus 4 of Remark 3.16 implies that $\bar{A} \subseteq B(x,\varepsilon)^{\complement}$. Therefore, $B(x,\varepsilon) \subseteq \bar{A}^{\complement}$; thus we established that every point in $\bar{A}^{\complement}$ is an interior point of $\bar{A}^{\complement}$. 4 of Remark 3.16 then shows that $\bar{A}$ is closed. ▫

Remark 3.25. Let (M, d) be a metric space, and A be a subset of M.

1. By the definition of the boundary of sets, Proposition 3.21 and Theorem 3.24 imply that ∂A is closed.
2. By 4 of Remark 3.16, every closed set containing A contains $\bar{A}$; thus the closure of A is the smallest closed set containing A; that is, $\bar{A} = \bigcap_{\substack{A \subseteq F \\ F \text{ closed}}} F$.

Definition 3.26. Let (M, d) be a metric space. A subset A of M is said to be complete if the metric space (A, d) is complete. In other words, A is complete if every Cauchy sequence in A converges to a limit in A.

Theorem 3.27. *Let (M, d) be a complete metric space, and A be a subset of M. Then A is complete if and only if A is closed in M.*

Proof. "$\Rightarrow$" Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in A with limit x. Then Proposition 2.58 implies that $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence in (M, d). Since $\{x_k\}_{k=1}^{\infty} \subseteq A$, we find that $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence in (A, d); thus the completeness of (A, d) implies that $\{x_k\}_{k=1}^{\infty}$ converges to $y \in A$. By Proposition 2.42, the limit is unique; thus $x = y$ which implies that $x \in A$. Therefore, A is closed.

"$\Leftarrow$" Let $\{x_k\}_{k=1}^{\infty}$ be a Cauchy sequence in A. Then

$$\forall \varepsilon > 0, \exists N > 0 \ni d(x_n, x_m) < \varepsilon \quad \text{whenever} \quad n, m \geqslant N .$$

Therefore, $\{x_k\}_{k=1}^{\infty}$ is a Cauchy in (M, d). Since (M, d) is complete, there exists $x \in M$ such that $x_k \to x$ as $k \to \infty$. By the closedness of A, we must have $x \in A$; thus every Cauchy sequence in A converges to a limit in A. □

3.2.2 Open sets

Definition 3.28. Let (M, d) be a metric space. A set $U \subseteq M$ is said to be ***open*** (in M) if every point in U is an interior point of U; that is, $U \subseteq \mathring{U}$. In other words,

$$U \text{ is open (in } M) \quad \Leftrightarrow \quad \forall\, x \in U,\ \exists\, r > 0 \ni B(x, r) \subseteq U .$$

Remark 3.29. Let (M, d) be a metric space, and A be a subset of M.

1. By the definition of the interior of sets, $\mathring{A} \subseteq A$; thus A is open if and only if $A = \mathring{A}$.

2. By 3 of Remark 3.16, A is open if and only if $A^{\complement}$ is closed and A is closed if and only if $A^{\complement}$ is open.

3. The statement above does **NOT** implies that a set in a metric space is either open or closed. There are still sets which is neither open nor closed.

4. If B is an open set and $B \subseteq A$, then Theorem 3.5 implies that $B \subseteq \mathring{A}$.

Example 3.30. The interval (a, b) in $\mathbb{R}$ is open since $\text{int}((a, b)) = (a, b)$. This is why (a, b) is called an open interval in Calculus.

Example 3.31. The set $A = \{(a, b) \in \mathbb{R}^2 \,|\, 0 < a < 1\}$ is open: given $x = (a, b) \in A$, take $r = \min\{1 - a, a\}$, then $B(x, r) \subseteq A$.

On the other hand, the set $A = \{(a, b) \in \mathbb{R}^2 \,|\, 0 < a \leqslant 1\}$ is not open: let $x = (1, 0)$, then for each $r > 0$, $B(x, r) \nsubseteq A$ since the point $\big(1 + \frac{r}{2}, 0\big) \in B(x, r)$ but $\big(1 + \frac{r}{2}, 0\big) \notin A$.

The following proposition is a direct consequence of Proposition 3.18 and Remark 3.29.

Proposition 3.32. *Every r-ball in a metric space is open.*

Alternative proof. Let (M, d) be a metric space, and $B(x, r)$ be an r-ball in M. We would like to show that for each $y \in B(x, r)$, there exists $\delta > 0$ such that $B(y, \delta) \subseteq B(x, r)$. Let $\delta = r - d(x, y)$. Then $\delta > 0$ and if $z \in B(y, \delta)$, we have

$$d(z, x) \leqslant d(z, y) + d(y, x) < \delta + d(y, x) = r\,;$$

thus $z \in B(x, r)$. □

Proposition 3.33. *Let (M, d) be a metric space.*

1. *The intersection of finitely many open sets is open.*

2. *The union of arbitrary family of open sets is open.*

3. *The empty set $\varnothing$ and the universal set M are open.*

Proof. 1. Let $U_1, \cdots, U_k$ be open sets, and $U = \bigcap_{j=1}^{k} U_j$. Then by De Morgan's law,

$$U^{\complement} = M \backslash U = M \backslash \bigcap_{j=1}^{k} U_j = \bigcup_{j=1}^{k} (M \backslash U_j) = \bigcup_{j=1}^{k} U_j^{\complement}\,.$$

Since U_j is open, $U_j^{\complement}$ is closed. By Proposition 3.21, $\bigcup_{j=1}^{k} U_j^{\complement}$ is closed.

2. Let $\mathscr{F} = \{U_\alpha \,|\, U_\alpha \text{ open in } M,\ \alpha \in I\}$ be a family of open sets, and $U \equiv \bigcup\limits_{\alpha\in I} U_\alpha$. Then by De Morgan's law,
$$U^\complement = M \backslash \bigcup_{\alpha\in I} U_\alpha = \bigcap_{\alpha\in I}(M\backslash U_\alpha) = \bigcup_{\alpha\in I} U_\alpha^\complement$$
which implies that $U^\complement$ is the intersection of a family of closed sets $\{U_\alpha^\complement\}_{\alpha\in I}$. By Proposition 3.21 we conclude that $U^\complement$ is closed or equivalently, U is open. □

Alternative proof of 1 *and* 2.

1. Let $U_1, U_2, \cdots, U_k$ be open sets in M, and $U \equiv \bigcap\limits_{i=1}^{k} U_i$. If $y \in U$, then $y \in U_i$ for all $1 \leqslant i \leqslant k$. Since U_i is open, there exist $\delta_i > 0$ such that $B(y, \delta_i) \subseteq U_i$. Let $\delta = \min\{\delta_1, \cdots, \delta_k\}$. Next we show that $B(y,\delta) \subseteq U$ to conclude that U is open.

   Let $z \in B(y,\delta)$. Then $d(y,z) < \delta \leqslant \delta_i$ for all $1 \leqslant i \leqslant k$. Therefore, $z \in B(y,\delta_i)$ for all $1 \leqslant i \leqslant k$ which shows that $z \in U_i$ for all $1 \leqslant i \leqslant k$; thus $z \in \bigcap\limits_{i=1}^{k} U_i \equiv U$.

2. Let $\mathscr{F} = \{U_\alpha \,|\, U_\alpha \text{ open in } M,\ \alpha \in I\}$ be a family of open sets, and $U \equiv \bigcup\limits_{\alpha\in I} U_\alpha$. If $y \in U$, then $y \in U_\beta$ for some $\beta \in I$. Since U_β is open, there exists $\delta > 0$ such that $B(y,\delta) \subseteq U_\beta$; thus $B(y,\delta) \subseteq \bigcup\limits_{\alpha\in I} U_\alpha \equiv U$. □

Remark 3.34. Infinite intersection of open sets need not be open:

1. Take $A_k = \big(-\frac{1}{k}, \frac{1}{k}\big)$, then $\bigcap\limits_{k=1}^{\infty} A_k = \{0\}$ which is not open.

2. Let $U_k = (-2 - \frac{1}{k}, 2 + \frac{1}{k}) \subseteq \mathbb{R}$. Then $A = \bigcap\limits_{k=1}^{\infty} U_k \supseteq [-2,2]$. Moreover, if $x \notin [-2,2]$, then $\exists\, k \in \mathbb{N} \ni x \notin U_k$ $\big($If $x > 2$, $\frac{1}{k} < \frac{x-2}{2}$. If $x < -2$, $\frac{1}{k} < \frac{-x-2}{2}\big)$. Therefore, $\bigcap\limits_{k=1}^{\infty} U_k = [-2,2]$.

Example 3.35. Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space. If A, B are subsets of $\mathcal{V}$ and A is open, the set $A + B$ is open.

Proof. Let $\boldsymbol{y} \in A + B$. Then $\boldsymbol{y} = \boldsymbol{a} + \boldsymbol{b}$ for some $\boldsymbol{a} \in A$, $\boldsymbol{b} \in B$. Since A is open, there exists $\delta > 0$ such that $B(\boldsymbol{a}, \delta) \subseteq A$.

We next show that $B(\boldsymbol{y},\delta) \subseteq A+B$. Let $\boldsymbol{z} \in B(\boldsymbol{y},\delta)$. Then $\|\boldsymbol{z}-\boldsymbol{y}\| < \delta$. Since $\boldsymbol{z} = \boldsymbol{b} + (\boldsymbol{z}-\boldsymbol{b})$, if we can show that $\boldsymbol{z}-\boldsymbol{b} \in A$, then $\boldsymbol{z} \in A+B$. Nevertheless, we have

$$\|(\boldsymbol{z}-\boldsymbol{b})-\boldsymbol{a}\| = \|\boldsymbol{z}-\boldsymbol{a}-\boldsymbol{b}\| = \|\boldsymbol{z}-\boldsymbol{y}\| < \delta$$

which implies that $\boldsymbol{z}-\boldsymbol{b} \in B(\boldsymbol{a},\delta) \subseteq A$. □

Since $A+B = \bigcup_{\boldsymbol{b}\in B} (\boldsymbol{b}+A)$ and it should be clear that $\boldsymbol{b}+A$ is open if A is open, we conclude by Proposition 3.33 that $A+B$ is open.

Theorem 3.36. *Let (M,d) be a metric space, and A be a subset of M. Then $\mathring{A}$ is open.*

Proof. For each $x \in \mathring{A}$, let $\varepsilon_x > 0$ denote a number such that $B(x,\varepsilon_x) \subseteq A$. We would like to show that $\mathring{A} = \bigcup_{x\in\mathring{A}} B(x,\varepsilon_x)$, and the theorem is then a direct consequence of Proposition 3.33.

1. "$\subseteq$": trivial.
2. "$\supseteq$": Let $y \in \bigcup_{x\in\mathring{A}} B(x,\varepsilon_x)$. By the definition of the union of family of sets, there exists $x \in \mathring{A}$ such that $y \in B(x,\varepsilon_x)$. Let $\delta = \varepsilon_x - d(x,y)$. Then $\delta > 0$ and if $z \in B(y,\delta)$,
$$d(z,x) \leqslant d(z,y) + d(y,x) < \delta + d(y,x) = \varepsilon_x$$
which implies that $B(y,\delta) \subseteq B(x,\varepsilon_x) \subseteq A$. Therefore, $y \in \mathring{A}$. □

Remark 3.37. Let (M,d) be a metric space, and A be a subset of M. By 4 of Remark 3.29, every open set contained inside A is contained inside $\mathring{A}$; thus the interior of A is the largest open set contained inside A; that is, $\mathring{A} = \bigcup_{\substack{A\supseteq U \\ U \text{ open}}} U$.

Remark 3.38. In a metric space (M,d), it is **NOT** always true that $\text{int}\big(B[x,R]\big) = B(x,R)$ or $\text{cl}(B(x,R)) = B[x,R]$. For example, we consider the discrete metric

$$d_0(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = 0. \end{cases}$$

Let $R = 1$, and fix $x \in M \neq \varnothing$. Then $B[x,1] = M$ and $B(x,1) = \{x\}$. Since every set in (M,d_0) is both closed and open, we find that

$$\text{int}(B[x,1]) = M \quad \text{and} \quad \text{cl}(B(x,1)) = \{x\}\,;$$

thus as long as M has more than one point, we have $\text{int}\big(B[x,1]\big) \neq B(x,1)$ and $\text{cl}\big(B(x,1)\big) \neq B[x,1]$. We also note that in (M,d_0) the boundary of every set is empty.

3.3 Compactness（緊緻性）

In this section, we investigate a property similar to the Bolzaon-Weierstrass Property.

Definition 3.39. Let (M, d) be a metric space. A subset $K \subseteq M$ is called ***sequentially compact*** if every sequence in K has a subsequence that converges to a point in K.

Definition 3.40. Let (M, d) be a metric space. A subset A of M is said to be bounded if A is contained in some r-ball. In other words, A is bounded if there exists $x \in M$ and $r > 0$ such that $A \subseteq B(x, r)$.

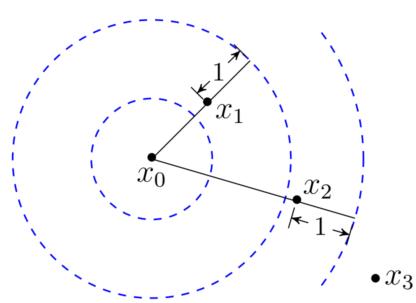
Example 3.41. Each closed and bounded set in $(\mathbb{R}^n, \|\cdot\|_2)$ is sequentially compact. This is a direct consequence of the Bolzano-Weierstrass Theorem (Theorem 2.51) and the definition of the closedness of sets.

Theorem 3.42. *Let (M, d) be a metric space, and $K \subseteq M$ be sequentially compact. Then K is closed and bounded.*

Proof. For closedness, assume that $\{x_k\}_{k=1}^{\infty} \subseteq K$ and $x_k \to x$ as $k \to \infty$. By the definition of sequential compactness, there exists $\{x_{k_j}\}_{j=1}^{\infty}$ converging to a point $y \in K$. By Proposition 2.56, $x = y$; thus $x \in K$.

For boundedness, assume the contrary that for all $x_0 \in M$ and $R > 0$, there exists $y \in K$ such that $d(x_0, y) \geqslant R$. Fix $x_0 \in M$. There exists $x_1 \in K$ such that $d(x_0, x_1) \geqslant 1$. Having x_1, there exists $x_2 \in K$ such that $d(x_2, x_0) \geqslant 1 + d(x_1, x_0)$. Continuing this process, we obtain a sequence $\{x_k\}_{k=1}^{\infty}$ in K such that

$$d(x_k, x_0) \geqslant 1 + d(x_{k-1}, x_0) \quad \forall\, k \in \mathbb{N}.$$



Then any subsequence of $\{x_k\}_{k=1}^{\infty}$ cannot be Cauchy since $d(x_k, x_\ell) \geqslant |k - \ell|$ for all $k, \ell \in \mathbb{N}$; thus $\{x_k\}_{k=1}^{\infty}$ has no convergent subsequence, a contradiction. ◻

Remark 3.43. Example 3.41 and Theorem 3.42 together imply that in $(\mathbb{R}^n, \|\cdot\|_2)$,

$$\text{sequentially compact} \quad \Leftrightarrow \quad \text{closed and bounded}\,.$$

This result is called the ***Heine-Borel*** Theorem.

In fact, if $\mathcal{V}$ is a **finite dimensional** vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} \subseteq \mathcal{V}$ is a basis for $\mathcal{V}$; that is, every $\boldsymbol{x} \in \mathcal{V}$ can be uniquely expressed as

$$\boldsymbol{x} = x^{(1)}\mathbf{e}_1 + x^{(2)}\mathbf{e}_2 + \cdots + x^{(n)}\mathbf{e}_n\,, \qquad x^{(k)} \in \mathbb{F} \text{ for } 1 \leqslant k \leqslant n\,.$$

Define $\|\boldsymbol{x}\|_2 = \Big(\sum\limits_{i=1}^{n} \big|x^{(i)}\big|^2\Big)^{\frac{1}{2}}$ (which is a norm by Example 2.28). Then a subset K of $\mathcal{V}$ is sequentially compact in $(\mathcal{V}, \|\cdot\|_2)$ if and only if K is closed and bounded. Note that by Theorem 3.42 it suffices to show the "if" direction. Let $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ be a sequence in K. Write $\boldsymbol{x}_k = \sum\limits_{i=1}^{n} x_k^{(i)}\mathbf{e}_i$, and define $\boldsymbol{v}_k = \big(x_k^{(1)}, x_k^{(2)}, \cdots, x_k^{(n)}\big)$. Then $\{\boldsymbol{v}_k\}_{k=1}^{\infty}$ is a sequence in $\mathbb{F}^n$. Since $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ is bounded, there exists $M > 0$ such that

$$\|\boldsymbol{x}_k\|_2 \leqslant M \qquad \forall\, k \in \mathbb{N}\,;$$

thus $\|\boldsymbol{v}_k\|_2 \leqslant M$ (here $\|\boldsymbol{v}_k\|_2$ is the usual norm of $\boldsymbol{v}_k$ on $\mathbb{F}^n$) for all $k \in \mathbb{N}$. By the Bolzano-Weierstrass Theorem (Theorem 2.51 and Remark 2.52), there exists a subsequence $\{\boldsymbol{v}_{k_j}\}_{j=1}^{\infty}$ such that $\{\boldsymbol{v}_{k_j}\}_{j=1}^{\infty}$ converges to some $\boldsymbol{v} \in \mathbb{F}^n$. Let $\boldsymbol{v} = \big(x^{(1)}, x^{(2)}, \cdots, x^{(n)}\big)$ and $\boldsymbol{x} = x^{(1)}\mathbf{e}_1 + \cdots + x^{(n)}\mathbf{e}_n$. Then

$$\big\|\boldsymbol{x}_{k_j} - \boldsymbol{x}\big\|_2 = \Big(\sum_{i=1}^{n} \big|x_{k_j}^{(i)} - x^{(i)}\big|^2\Big)^{\frac{1}{2}} = \|\boldsymbol{v}_{k_j} - \boldsymbol{v}\|_2 \to 0 \ \text{ as } \ j \to \infty\,,$$

and the closedness of K implies that $\boldsymbol{x} \in K$, so we establish that K is sequentially compact in $(\mathcal{V}, \|\cdot\|_2)$ if K is closed and bounded.

Example 3.44. Let $A = [0, 1] \cup (2, 3] \subseteq (\mathbb{R}, |\cdot|)$. Since A is not closed, A is not sequentially compact.

Corollary 3.45. *If* $K \subseteq \mathbb{R}$ *is sequentially compact, then* $\inf K \in K$ *and* $\sup K \in K$.

Proof. By Theorem 3.42, K must be closed and bounded. Therefore, $\inf K \in \mathbb{R}$. Then for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that $\inf K \leqslant x_n < \inf K + \dfrac{1}{n}$. Since $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, the Bolzano-Weierstrass property of $\mathbb{R}$ implies that there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and $x \in \mathbb{R}$ such that $\lim\limits_{k\to\infty} x_{n_k} = x$. Note that $x = \inf K$, and by the closedness of K, $x \in K$. The proof of $\sup K \in K$ is similar. □

Definition 3.46. Let (M,d) be a metric space. A subset $A \subseteq M$ is called ***totally bounded*** if for each $r > 0$, there exists $\{x_1, \cdots, x_N\} \subseteq M$ such that

$$A \subseteq \bigcup_{i=1}^{N} B(x_i, r).$$

Remark 3.47. In a general metric space (M,d), a bounded set might not be totally bounded. For example, consider the metric space (M,d) with the discrete metric, and $A \subseteq M$ be a set having infinitely many points. Then A is bounded since $A \subseteq B(x,2)$ for any $x \in M$; however, A is not totally bounded since A cannot be covered by finitely many balls with radius $\frac{1}{2}$.

Proposition 3.48. *Let (M,d) be a metric space, and $A \subseteq M$ be totally bounded. Then A is bounded. In other words, totally bounded sets are bounded.*

Proof. By total boundedness, there exists $\{y_1, \cdots, y_N\} \subseteq M$ such that $A \subseteq \bigcup_{i=1}^{N} B(y_i, 1)$. Let $x_0 = y_1$ and $R = \max\big\{d(x_0, y_2), \cdots, d(x_0, y_N)\big\} + 1$. Then if $z \in A$, $z \in B(y_j, 1)$ for some $j = 1, \cdots, N$, and

$$d(z, x_0) \leqslant d(z, y_j) + d(y_j, x_0) < 1 + d(x_0, y_j) \leqslant R$$

which implies that $A \subseteq B(x_0, R)$. Therefore, A is bounded. □

Example 3.49. Every bounded set in $(\mathbb{R}^n, \|\cdot\|_2)$ is totally bounded **(Check!)**. In particular, the set $\{1\} \times [1,2]$ in $(\mathbb{R}^2, \|\cdot\|_2)$ is totally bounded.

On the other hand, let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2\,, \\ |x_1 - y_1| + |x_2 - y_2| + 1 & \text{if } x_2 \neq y_2\,, \end{cases} \quad \text{where } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

Then $(\mathbb{R}^2, d)$ is also a metric space (exercise). The set $\{1\} \times [1,2]$ is bounded **(Check!)** but not totally bounded. In fact, consider open ball with radius $\frac{1}{2}$:

$$\begin{aligned} y \in B\big(x, \tfrac{1}{2}\big) &\Leftrightarrow \|x - y\| < \tfrac{1}{2} \Leftrightarrow |x_1 - y_1| < \tfrac{1}{2} \text{ and } x_2 = y_2 \\ &\Leftrightarrow y_1 \in \big(x_1 - \tfrac{1}{2}, x_1 + \tfrac{1}{2}\big) \text{ and } x_2 = y_2\,. \end{aligned}$$

In other words,

$$B\big(x, \tfrac{1}{2}\big) = \big(x_1 - \tfrac{1}{2}, x_1 + \tfrac{1}{2}\big) \times \{x_2\}\,;$$

thus one cannot cover $\{1\} \times [1,2]$ by the union of finitely many balls with radius $\frac{1}{2}$.

Example 3.50. Let ℓ^∞ denote the collection of all bounded real sequences (cf. Example 2.20); that is,

$$\ell^\infty = \Big\{ \{x_k\}_{k=1}^\infty \subseteq \mathbb{R} \,\Big|\, \text{for some } M > 0, |x_k| \leqslant M \text{ for all } k \Big\}.$$

The number $\sup\limits_{k\geqslant 1} |x_k| \equiv \sup\{|x_1|, |x_2|, \cdots, |x_k|, \cdots\} < \infty$ is denoted by $\big\|\{x_k\}_{k=1}^\infty\big\|_\infty$ $\Big($for example, if $x_k = \dfrac{(-1)^k}{k}$, then $\big\|\{x_k\}_{k=1}^\infty\big\|_\infty = 1\Big)$. Then $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space (left as an exercise). Define

$$\begin{aligned}
A &= \Big\{ \{x_k\}_{k=1}^\infty \in \ell^\infty \,\Big|\, |x_k| \leqslant \frac{1}{k} \Big\}, \\
B &= \Big\{ \{x_k\}_{k=1}^\infty \in \ell^\infty \,\Big|\, x_k \to 0 \text{ as } k \to \infty \Big\}, \\
C &= \Big\{ \{x_k\}_{k=1}^\infty \in \ell^\infty \,\Big|\, \text{the sequence } \{x_k\}_{k=1}^\infty \text{ converges} \Big\}, \\
D &= \Big\{ \{x_k\}_{k=1}^\infty \in \ell^\infty \,\Big|\, \sup_{k\geqslant 1} |x_k| = 1 \Big\} \quad (\text{the unit sphere in } (\ell^\infty, \|\cdot\|)).
\end{aligned}$$

The closedness of A (which implies the completeness of $(A, \|\cdot\|_\infty)$) is left as an exercise. We show that A is totally bounded.

Let $r > 0$ be given. Then there exists $N > 0$ such that $\dfrac{1}{N} < r$. Define

$$\begin{aligned}
E = \Big\{ \{x_k\}_{k=1}^\infty \,\Big|\, & x_1 = \frac{i_1}{N+1}, x_2 = \frac{i_2}{N+1}, \cdots, x_{N-1} = \frac{i_{N-1}}{N+1} \text{ for some} \\
& i_1, \cdots, i_{N-1} = -N, -N+1, \cdots, N-1, N, \text{ and } x_k = 0 \text{ if } k \geqslant N+1 \Big\}.
\end{aligned}$$

Then

1. $\#E < \infty$. In fact, $\#E = (2N+1)^{N-1} < \infty$.
2. $A \subseteq \bigcup\limits_{\{x_k\}_{k=1}^\infty \in E} B\big(\{x_k\}_{k=1}^\infty, \frac{1}{N}\big) \subseteq \bigcup\limits_{\{x_k\}_{k=1}^\infty \in E} B\big(\{x_k\}_{k=1}^\infty, r\big)$.

Therefore, A is totally bounded.

On the other hand, B and C are not bounded; thus not totally bounded by Proposition 3.48. D is bounded but not totally bounded. In fact, D cannot be covered by the union of finitely many balls with radius $\dfrac{1}{2}$ since each ball with radius $\dfrac{1}{2}$ contains at most one of the points from the subset $\Big\{ \{x_j^{(k)}\}_{j=1}^\infty \Big\}_{k=1}^\infty \subseteq D$, where for each k

$$\{x_j^{(k)}\}_{j=1}^\infty = \{ \underbrace{0, \cdots, 0}_{(k-1) \text{ terms}}, 1, 0, \cdots \};$$

that is, $x_j^{(k)} = \delta_{kj}$, the kronecker delta.

Proposition 3.51. *Let (M,d) be a metric space, and $T \subseteq M$ be totally bounded. If $S \subseteq T$, then S is totally bounded. In other words, subsets of totally bounded sets are totally bounded.*

Proof. Let $r > 0$ be given. By the total boundedness of T, there exists $\{x_1, \cdots, x_N\} \subseteq M$ such that

$$S \subseteq T \subseteq \bigcup_{i=1}^{N} B(x_i, r)\,. \qquad \square$$

Proposition 3.52. *Let (M,d) be a metric space, and $A \subseteq M$. Then A is totally bounded if and only if for all $r > 0$, there exists $\{y_1, \cdots, y_N\} \subseteq A$ such that $A \subseteq \bigcup_{i=1}^{N} B(y_i, r)$.*

Proof. It suffices to show the "only if" part. Let $r > 0$ be given. Since A is totally bounded,

$$\exists\, \{y_1, \cdots, y_N\} \subseteq M \ni A \subseteq \bigcup_{i=1}^{N} B\big(y_i, \frac{r}{2}\big)\,.$$

W.L.O.G., we may assume that for each $i = 1, \cdots, N$, $B\big(y_i, \frac{r}{2}\big) \cap A \neq \varnothing$. Then for each $i = 1, \cdots, N$, there exists $x_i \in B\big(y_i, \frac{r}{2}\big) \cap A$ which implies that

$$A \subseteq \bigcup_{i=1}^{N} B\big(y_i, \frac{r}{2}\big) \subseteq \bigcup_{i=1}^{N} B(x_i, r)$$

since $B\big(y_i, \frac{r}{2}\big) \subseteq B(x_i, r)$ for all $i = 1, \cdots, N$. $\square$

Theorem 3.53. *Let (M,d) be a metric space, and K be a subset of M. Then K is sequentially compact if and only if K is totally bounded and complete.*

Proof. "$\Rightarrow$" Assume that K is sequentially compact. For the total boundedness, suppose the contrary that there is an $r > 0$ such that any finite set $\{y_1, \cdots, y_n\} \subseteq K$, $K \nsubseteq \bigcup_{i=1}^{n} B(y_i, r)$. This implies that we can choose a sequence $\{x_k\}_{k=1}^{\infty} \subseteq K$ such that

$$x_{k+1} \in K \setminus \bigcup_{i=1}^{k} B(x_i, r)\,.$$

Then $\{x_k\}_{k=1}^{\infty}$ is a sequence in K without convergent subsequence since $d(x_k, x_\ell) > r$ for all $k, \ell \in \mathbb{N}$ and $k \neq \ell$.

Next we show that K is complete. Let $\{x_k\}_{k=1}^{\infty} \subseteq K$ be a Cauchy sequence. By sequential compactness of K, there is a subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ converging to a point $x \in K$. By Proposition 2.58, $\{x_k\}_{k=1}^{\infty}$ also converges to x; thus every Cauchy sequence in K converges to a point in K.

"$\Leftarrow$" The proof of this direction is similar to the proof of Theorem 1.79 and we proceed as follows. Let $\{x_k\}_{k=1}^\infty$ be a sequence in $T_0 \equiv K$. Since K is totally bound, there exists $\{y_1^{(1)}, \cdots y_{N_1}^{(1)}\} \subseteq K$ such that

$$T_0 \equiv K \subseteq \bigcup_{i=1}^{N_1} B(y_i^{(1)}, 1)\,.$$

One of these $B(y_i^{(1)}, 1)$'s must contain infinitely many x_k's; that is, there exists $1 \leqslant \ell_1 \leqslant N_1$ such that $\#\{k \in \mathbb{N} \,|\, x_k \in B(y_{\ell_1}^{(1)}, 1)\} = \infty$. Define $T_1 = K \cap B(y_{\ell_1}^{(1)}, 1)$. Then T_1 is also totally bounded by Proposition 3.51, so there exists $\{y_1^{(2)}, \cdots, y_{N_2}^{(2)}\} \subseteq T_1$ such that

$$T_1 \subseteq \bigcup_{i=1}^{N_2} B\big(y_i^{(2)}, \frac{1}{2}\big)\,.$$

Suppose that $\#\{k \in \mathbb{N} \,|\, x_k \in B\big(y_{\ell_2}^{(2)}, \frac{1}{2}\big)\} = \infty$ for some $1 \leqslant \ell_2 \leqslant N_2$. Define $T_2 = T_1 \cap B\big(y_{\ell_2}^{(2)}, \frac{1}{2}\big)$. We continue this process, and obtain that for all $n \in \mathbb{N}$,

(1) there exists $\{y_1^{(n)}, \cdots, y_{N_n}^{(n)}\} \subseteq T_{n-1}$ such that

$$T_{n-1} \subseteq \bigcup_{i=1}^{N_n} B\big(y_i^{(n)}, \frac{1}{n}\big)\,.$$

(2) $T_n = T_{n-1} \cap B(y_{\ell_n}^{(n)}, \frac{1}{n})$, where $1 \leqslant \ell_n \leqslant N_n$ is chosen so that

$$\#\Big\{k \in \mathbb{N} \,\Big|\, x_k \in B\big(y_{\ell_n}^{(n)}, \frac{1}{n}\big)\Big\} = \infty\,. \tag{3.3.1}$$

Pick an $k_1 \in \{k \in \mathbb{N} \,|\, x_k \in B\big(y_{\ell_1}^{(1)}, 1\big)\}$, and $k_j \in \Big\{k \in \mathbb{N} \,\Big|\, x_k \in B\big(y_{\ell_j}^{(j)}, \frac{1}{j}\big)\Big\}$ such that $k_{j+1} > k_j$ for all $j \in \mathbb{N}$. We note such k_j always exists because of (3.3.1). Then $\{x_{k_j}\}_{j=1}^\infty$ is a subsequence of $\{x_k\}_{k=1}^\infty$, and $x_{k_j} \in T_j \subseteq K$ for all $j \in \mathbb{N}$.

Claim: $\{x_{k_j}\}_{j=1}^\infty$ is a Cauchy sequence.

Proof of claim: Let $\varepsilon > 0$ be given, and $N > 0$ be large enough so that $\frac{1}{N} < \frac{\varepsilon}{2}$. Then if $j \geqslant N$, we must have $x_{k_j} \in B\big(y_{\ell_N}^{(N)}, \frac{1}{N}\big)$; thus we conclude that if $n, m \geqslant N$, the triangle inequality implies that

$$d\big(x_{k_n}, x_{k_m}\big) \leqslant d\big(x_{k_n}, y_{\ell_N}^{(N)}\big) + d\big(x_{k_m}, y_{\ell_N}^{(N)}\big) < \frac{1}{N} + \frac{1}{N} < \varepsilon.$$

Since (K, d) is complete, the Cauchy sequence $\{x_{k_j}\}_{j=1}^\infty$ converges to a point in K. ▫

Finally we introduce the concept of compact sets in a metric space in the remaining part of the section. First we need to talk about open cover of sets.

Definition 3.54. Let (M,d) be a metric space, and A be a subset of M. A ***cover*** of A is a collection of sets $\{U_\alpha\}_{\alpha\in I}$ satisfying that $A \subseteq \bigcup_{\alpha\in I} U_\alpha$. It is an ***open cover*** of A if U_α is open for all $\alpha \in I$. A ***subcover*** of a given cover $\{U_\alpha\}_{\alpha\in I}$ is a sub-collection $\{U_\alpha\}_{\alpha\in J}$, where $J \subseteq I$, satisfying that $A \subseteq \bigcup_{\alpha\in I} U_\alpha$. It is a ***finite subcover*** if $\#J < \infty$.

Example 3.55. The collection $\big\{(-k,k) \,\big|\, k\in\mathbb{N}\big\}$ is an open cover of $\mathbb{R}$, and $\big\{(-2k,2k) \,\big|\, k\in\mathbb{N}\big\}$ is a subcover of $\big\{(-k,k) \,\big|\, k\in\mathbb{N}\big\}$.

The so-called compact sets is defined in the following

Definition 3.56. Let (M,d) be a metric space. A subset $K \subseteq M$ is called ***compact*** if every open cover of K possesses a finite subcover; that is, $K \subseteq M$ is compact if

$$\big(\forall \text{ open cover } \{U_\alpha\}_{\alpha\in I} \text{ of } K\big)\big(\exists\, J \subseteq I \wedge \#J < \infty\big)\Big(K \subseteq \bigcup_{\alpha\in J} U_\alpha\Big).$$

Example 3.57. Let $A = \{0\} \cup \Big\{1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\Big\}$, and $\{U_\alpha\}_{\alpha\in I}$ be a given open cover of A. Then $0 \in U_\beta$ for all $\beta \in I$. By the fact that U_β is open, there exists $\varepsilon > 0$ such that $B(0,\varepsilon) \subseteq U_\beta$. Let $N \in \mathbb{N}$ satisfy $\frac{1}{N} < \varepsilon$. Then

$$\Big\{\frac{1}{N}, \frac{1}{N+1}, \cdots\Big\} = \Big\{\frac{1}{k} \,\Big|\, k \geqslant N\Big\} \subseteq U_\beta\,.$$

On the other hand, for each $1 \leqslant k \leqslant N-1$ there exists $\alpha_k \in I$ such that $\frac{1}{k} \in U_{\alpha_k}$. Therefore,

$$A = \Big(\{0\} \cup \Big\{\frac{1}{N}, \frac{1}{N+1}, \cdots\Big\}\Big) \cup \Big\{1, \frac{1}{2}, \cdots, \frac{1}{N-1}\Big\} \subseteq U_\beta \cup \bigcup_{k=1}^{N-1} U_{\alpha_k}\,;$$

thus we obtain a finite subcover for a given open cover. Therefore, A is compact.

In general, if $\{x_k\}_{k=1}^\infty$ is a convergent sequence in a metric space with limit x, then the set $A = \{x_1, x_2, \cdots\} \cup \{x\}$ is compact.

Theorem 3.58. *Let (M,d) be a metric space, and A be a subset of M. Then A is compact if and only if A is sequentially compact.*

Proof. "⇒" Since a compact set must be totally bounded $\big($a finite sub-cover of $\big\{B(x,r)\,\big|\,x\in K\big\}$ suffices the purpose$\big)$, it suffices to show the completeness of K. Let $\{x_k\}_{k=1}^{\infty}$ be a Cauchy sequence in K. Suppose that $\{x_k\}_{k=1}^{\infty}$ does not converge in K. Then Proposition 2.56 and 2.58 imply that every point of K is not a cluster point of $\{x_k\}_{k=1}^{\infty}$; thus

$$\forall\, y\in K, \exists\, \delta_y>0 \ni \#\big\{k\in\mathbb{N}\,|\,x_k\in B(y,\delta_y)\big\}<\infty\,. \tag{3.3.2}$$

The collection $\big\{B(y,\delta_y)\big\}_{y\in K}$ then is an open cover of K; thus possesses a finite sub-cover $\big\{B\big(y_i,\delta_{y_i}\big)\big\}_{i=1}^{N}$. In particular, $\{x_k\}_{k=1}^{\infty}\subseteq\bigcup\limits_{i=1}^{N}B\big(y_i,\delta_{x_i}\big)$ or

$$\#\Big\{k\in\mathbb{N}\,\Big|\,x_k\in\bigcup_{i=1}^{N}B(y_i,\delta_{y_i})\Big\}=\infty$$

which contradicts to (3.3.2).

"⇐" Let $\big\{U_\alpha\big\}_{\alpha\in I}$ be an open cover of K.

Claim: there exists $r>0$ such that for each $x\in K$, $B(x,r)\subseteq U_\alpha$ for some $\alpha\in I$.

Proof of claim: Suppose the contrary that for each $k\in\mathbb{N}$, there exists $x_k\in K$ such that $B\big(x_k,\frac{1}{k}\big)\nsubseteq U_\alpha$ for all $\alpha\in I$. Then $\{x_k\}_{k=1}^{\infty}$ is a sequence in K; thus by the assumption of sequential compactness, there exists a convergent subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ with limit $x\in K$. Since $\{U_\alpha\}_{\alpha\in I}$ is an open cover of K, $x\in U_\beta$ for some $\beta\in I$. Then

(1) there is $r>0$ such that $B(x,r)\subseteq U_\beta$ since U_β is open.

(2) there exists $N>0$ such that $d(x_{k_j},x)<\dfrac{r}{2}$ for all $j\geqslant N$.

Choose $j\geqslant N$ such that $\dfrac{1}{k_j}<\dfrac{r}{2}$. Then $B\big(x_{k_j},\frac{1}{k_j}\big)\subseteq B(x,r)\subseteq U_\beta$, a contradiction.

Having established the claim, by the fact that K is totally bounded (Theorem 3.53) there exists $\{y_1,\cdots,y_N\}\subseteq K$ such that $K\subseteq\bigcup\limits_{i=1}^{N}B(y_i,r)$. For each $1\leqslant i\leqslant N$, the claim above implies that there exists $\alpha_i\in I$ such that $B(y_i,r)\subseteq U_{\alpha_i}$. Then $\bigcup\limits_{i=1}^{N}B(y_i,r)\subseteq\bigcup\limits_{i=1}^{N}U_{\alpha_i}$ which implies that

$$K\subseteq\bigcup_{i=1}^{N}U_{\alpha_i}\,.$$

□

Remark 3.59. For a given open cover $\{U_\alpha\}_{\alpha\in I}$ of a compact set K, the positive number r appearing in the claim above is called a ***Lebesgue number*** for the open cover. It has the property that for each $x \in K$ there exists $\alpha \in I$ such that $B(x,r) \subseteq U_\alpha$.

Definition 3.60. Let (M,d) be a metric space. A subset A of M is called ***pre-compact*** if $\bar{A}$ is compact. Let $U \subseteq M$ be an open set, a subset A of U is said to be ***compactly embedded*** in U, denoted by $A \subset\subset U$, if A is pre-compact and $\bar{A} \subseteq U$.

Remark 3.61. Suppose that A is a pre-compact set in (M,d). If $\{x_k\}_{k=1}^\infty$ be a sequence in A, then $\{x_k\}_{k=1}^\infty$ is a sequence in $\bar{A}$; thus the (sequential) compactness of $\bar{A}$ implies that there exists convergent subsequence $\{x_{k_j}\}_{j=1}^\infty$ (with limit in $\bar{A}$). Therefore, every sequence in a pre-compact set has a convergent subsequence.

Example 3.62. Let (M,d) be a complete metric space, and $A \subseteq M$ be totally bounded. Then $\bar{A}$ is totally bounded (by enlarging the radius of the balls); thus Theorem 3.27 and 3.53 imply that $\bar{A}$ is sequentially compact. In other words, in a complete metric space, totally bounded sets are pre-compact.

Remark 3.63. A generalized version of the Bolzano-Weierstrass property in a general metric space is the following: a metric space is said to satisfy the Bolzano-Weierstrass property if every totally bounded sequence has a convergent subsequence. Then a metric space is complete if and only if it satisfies the Bolzano-Weierstrass property.

3.4 Connectedness（連通性）

Definition 3.64. Let (M,d) be a metric space, and A be a subset of M. Two non-empty open sets U and V are said to separate A if

1. $A \cap U \cap V = \varnothing$; 2. $A \cap U \neq \varnothing$; 3. $A \cap V \neq \varnothing$; 4. $A \subseteq U \cup V$.

We say that A is ***disconnected*** or ***separated*** if such separation exists, and A is ***connected*** if no such separation exists.

Proposition 3.65. *Let (M,d) be a metric space. A subset $A \subseteq M$ is disconnected if and only if $A = A_1 \cup A_2$ with $A_1 \cap \bar{A}_2 = \bar{A}_1 \cap A_2 = \varnothing$ for some non-empty A_1 and A_2.*

Proof. "$\Rightarrow$" Suppose that there exist U, V non-empty open sets such that 1-4 in Definition 3.64 hold. Let $A_1 = A \cap U$ and $A_2 = A \cap V$. By 1, $A_1 \subseteq V^\complement$; thus Theorem 3.5 implies that $\bar{A}_1 \subseteq V^\complement$. This shows that $\bar{A}_1 \cap A_2 = \varnothing$. Similarly, $\bar{A}_2 \cap A_1 = \varnothing$.

"⇐" Let $U = \mathrm{cl}(A_2)^\complement$ and $V = \mathrm{cl}(A_1)^\complement$ be two open sets. Then $V \cap A_1 = U \cap A_2 = \varnothing$; thus

$$A \cap U \cap V = (A_1 \cup A_2) \cap U \cap V = (A_1 \cap U) \cap V = U \cap (A_1 \cap V) = \varnothing .$$

Moreover, since $A_1 \cap \bar{A}_2 = A_2 \cap \bar{A}_1 = \varnothing$, $A_1 \subseteq \mathrm{cl}(A_2)^\complement = U$ and $A_2 \subseteq \mathrm{cl}(A_1)^\complement = V$ so that Property 2 and 3 in Definition 3.64 hold. Finally, since $\bar{A}_1 \subseteq A_2^\complement$ and $\bar{A}_2 \subseteq A_1^\complement$, we have

$$(U \cup V)^\complement = U^\complement \cap V^\complement = \bar{A}_2 \cap \bar{A}_1 \subseteq A_1^\complement \cap A_2^\complement = (A_1 \cup A_2)^\complement = A^\complement$$

which implies that $A \subseteq U \cup V$. Therefore, A is disconnected. □

Proposition 3.65 implies the following alternative definition of connected sets (without defining disconnected sets first):

Definition 3.66. Let (M, d) be a metric space. A subset A of M is said to be ***connected*** if A cannot be represented as the union of two non-empty disjoint sets neither of which contains a limit point of the other.

Corollary 3.67. *Let (M, d) be a metric space. Suppose that a subset $A \subseteq M$ is connected, and $A = A_1 \cup A_2$, where $A_1 \cap \bar{A}_2 = \bar{A}_1 \cap A_2 = \varnothing$. Then A_1 or A_2 is empty.*

Theorem 3.68. *A subset A of the Euclidean space $(\mathbb{R}, |\cdot|)$ is connected if and only if it has the property that if $x, y \in A$ and $x < z < y$, then $z \in A$.*

Proof. "⇒" Suppose that there exist $x, y \in A$, $x < z < y$ but $z \notin A$. Then $A = A_1 \cup A_2$, where

$$A_1 = A \cap (-\infty, z) \quad \text{and} \quad A_2 = A \cap (z, \infty) .$$

Since $x \in A_1$ and $y \in A_2$, A_1 and A_2 are non-empty. Moreover, $\bar{A}_1 \cap A_2 = A_1 \cap \bar{A}_2 = \varnothing$; thus by Proposition 3.65, A is disconnected, a contradiction.

"⇐" Suppose the contrary that A is not connected (disconnected). Then there exist non-empty sets A_1 and A_2 such that $A = A_1 \cup A_2$ with $\bar{A}_1 \cap A_2 = A_1 \cap \bar{A}_2 = \varnothing$. Pick $x \in A_1$ and $y \in A_2$. W.L.O.G., we may assume that $x < y$. Define $z = \sup(A_1 \cap [x, y])$.

Claim: $z \in \bar{A}_1$.

Proof of claim: By definition, for any $n > 0$ there exists $x_n \in A_1 \cap [x, y]$ such that $z - \frac{1}{n} < x_n \leqslant z$. Therefore, $x_n \to z$ as $n \to \infty$ which implies that $z \in \bar{A}_1$.

Since $z \in \bar{A}_1$, $z \notin A_2$. In particular, $x \leqslant z < y$.

(a) If $z \notin A_1$, then $x < z < y$ and $z \notin A$, a contradiction.

(b) If $z \in A_1$, then $z \notin \bar{A}_2$; thus there exists $0 < r < y - z$ such that $(z - r, z + r) \subseteq \text{cl}(A_2)^{\complement}$. Then for all $z_1 \in (z, z + r)$, $z < z_1 < y$ and $z_1 \notin A_2$. Then $x < z_1 < y$ and $z_1 \notin A$, a contradiction. □

Corollary 3.69. *Connected sets in* $(\mathbb{R}, |\cdot|)$ *are intervals.*

3.5 Subspace Topology

Let (M, d) be a metric space, and $N \subseteq M$ be a subset. Then (N, d) is a metric space, and the topology of (N, d) is called the ***subspace topology*** of (N, d).

Remark 3.70. The topology of a metric space is the collection of all open sets of that metric space.

Proposition 3.71. *Let* (M, d) *be a metric space, and* N *be a subset of* M*. A subset* $V \subseteq N$ *is open in* (N, d) *if and only if* $V = U \cap N$ *for some open set* U *in* (M, d)*.*

Proof. Let $B_N(x, r)$ denote the r-ball about x in (N, d); that is,

$$B_N(x, r) \equiv \left\{ y \in N \,\middle|\, d(x, y) < r \right\} \subseteq V\,.$$

Note that $B_N(x, r) = B(x, r) \cap N$, where $B(x, r)$ is the r-ball about x in the metric space (M, d).

"⇒" Let $V \subseteq N$ be open in (N, d). Then for all $x \in M$, there exists $r_x > 0$ such that $B_N(x, r_x) \subseteq V$. In particular,

$$V = \bigcup_{x \in V} B_N(x, r_x) = \bigcup_{x \in V} B(x, r) \cap N\,.$$

Define $U = \bigcup\limits_{x \in V} B(x, r_x)$. Then U is open in (M, d) (by Proposition 3.33), and

$$V = \bigcup_{x \in V} B(x, r_x) \cap N = U \cap N\,.$$

"⇐" Suppose that $V = U \cap N$ for some open set U in (M, d). Let $x \in V$. Then $x \in U$; thus there exists $r > 0$ such that $B(x, r) \subseteq U$. Therefore,

$$B_N(x, r) = B(x, r) \cap N \subseteq U \cap N = V$$

which implies that x is an interior point of V. This shows that V is open in (N, d). □

Corollary 3.72. *Let (M, d) be a metric space, and N be a subset of M. A subset $E \subseteq N$ is closed in (N, d) if and only if $E = F \cap N$ for some closed set F in (M, d).*

Proof. Note that if A, B are subsets of N, then $A = B$ if and only if $N \cap A^\complement = N \cap B^\complement$, where $A^\complement$ denote the set $M \backslash A$. Then for a subset E of N,

$$\begin{aligned} E \text{ is closed in } (N, d) &\Leftrightarrow N \backslash E \text{ is open in } (N, d) \Leftrightarrow N \cap E^\complement \text{ is open in } (M, d) \\ &\Leftrightarrow N \cap E^\complement = N \cap U \text{ for some open set } U \text{ in } (M, d) \\ &\Leftrightarrow N \cap (N \cap E^\complement)^\complement = N \cap (N \cap U)^\complement \text{ for some open set } U \text{ in } (M, d) \\ &\Leftrightarrow N \cap E = N \cap U^\complement \text{ for some open set } U \text{ in } (M, d) \\ &\Leftrightarrow N \cap E = N \cap F \text{ for some closed set } F \text{ in } (M, d) \\ &\Leftrightarrow E = F \cap N \text{ for some closed set } F \text{ in } (M, d). \end{aligned}$$

□

Remark 3.73. Let (M, d) be a metric space, N be a subset of M, and $\{x_n\}_{n=1}^\infty$ be a sequence in N. We note that the convergence of $\{x_n\}_{n=1}^\infty$ in (N, d) implies the convergence of $\{x_n\}_{n=1}^\infty$ in (M, d), but not vice versa. For example, the sequence $\left\{\frac{1}{n}\right\}_{n=1}^\infty$ is convergent in $(\mathbb{R}, |\cdot|)$ but is not convergent in $((0, \infty), d)$, where d is the metric induced from the norm $|\cdot|$. Since the concept of convergence is different in a subspace, we expect that in a subspace the concept of closed will be different. In other words, the concept of closedness (and openness as well) of sets highly depends on the background metric space.

Definition 3.74. Let (M, d) be a metric space, and N be a subset of M. A subset A is said to be $\begin{array}{c}\text{open}\\ \text{closed}\\ \text{compact}\end{array}$ relative to N if $A \cap N$ is $\begin{array}{c}\text{open}\\ \text{closed}\\ \text{compact}\end{array}$ in the metric space (N, d).

Note that base on the definition above, Proposition 3.71 and Corollary 3.72 imply that if A is closed/open in (M, d), then A is relative closed/open in (N, d). However, we note that if $A \subseteq N$ is closed/open in (N, d), A is not necessary closed/open in (M, d).

Example 3.75. Let (M, d) be $(\mathbb{R}, |\cdot|)$, and $N = \mathbb{Q}$. Consider the set $F = [0, 1] \cap \mathbb{Q}$. By Corollary 3.72 F is closed in $(\mathbb{Q}, |\cdot|)$; however, F is not closed in $(\mathbb{R}, |\cdot|)$ since it is not complete (a Cauchy sequence in F might not converge).

Theorem 3.76. *Let (M, d) be a metric space, and N be a subset of M. A subset A of M is closed relative to N if and only if $A^\complement$ is open relative to N.*

Proof. Note that if A, B are subsets of N, then $A = B$ if and only if $N \cap A^{\complement} = N \cap B^{\complement}$, where $A^{\complement}$ denote the set $M \backslash A$. Then for a subset A of M,

$$\begin{aligned} A \text{ is closed relative to } N &\Leftrightarrow A \cap N \text{ is closed in } (N, d) \\ &\Leftrightarrow A \cap N = F \cap N \text{ for some closed set } F \text{ in } (M, d) \\ &\Leftrightarrow N \cap (A \cap N)^{\complement} = N \cap (F \cap N)^{\complement} \text{ for some closed set } F \text{ in } (M, d) \\ &\Leftrightarrow N \cap A^{\complement} = N \cap F^{\complement} \text{ for some closed set } F \text{ in } (M, d) \\ &\Leftrightarrow N \cap A^{\complement} = N \cap U \text{ for some open set } U \text{ in } (M, d) \end{aligned}$$

which, by Proposition 3.71, implies that $A^{\complement}$ is open relative to N. ▫

Theorem 3.77. *Let (M, d) be a metric space, and $K \subseteq N \subseteq M$. Then K is compact in (M, d) if and only if K is compact in (N, d).*

Proof. "⇒" Let $\{V_\alpha\}_{\alpha \in I}$ be an open cover of K in (N, d). By Proposition 3.71 there exists a collection of open sets $\{U_\alpha\}_{\alpha \in I}$ in (M, d) such that $V_\alpha = U_\alpha \cap N$ for all $\alpha \in I$. Since $\{V_\alpha\}_{\alpha \in I}$ is an open cover of K in (N, d), $\{U_\alpha\}_{\alpha \in I}$ is a cover of K in (M, d); thus by the compactness of K in (M, d), there exists $\alpha_1, \cdots, \alpha_N$ such that

$$K \subseteq \bigcup_{j=1}^{N} U_{\alpha_j} .$$

Since $K \subseteq N$ and $V_\alpha = U_\alpha \cap N$, we must have $K \subseteq \bigcup\limits_{j=1}^{N} V_{\alpha_j}$ which shows that there is a finite subcover of K for the open cover $\{V_\alpha\}_{\alpha \in I}$. Therefore, K is compact in (N, d).

"⇐" Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of K in (M, d). Define $V_\alpha = U_\alpha \cap N$. Since $K \subseteq N$, Proposition 3.71 implies that $\{V_\alpha\}_{\alpha \in I}$ is an open cover of K in (N, d); thus the compactness of K in (N, d) implies that there exists $\alpha_1, \cdots, \alpha_N$ such that

$$K \subseteq \bigcup_{j=1}^{N} V_{\alpha_j} .$$

Since $V_\alpha \subseteq U_\alpha$ for all $\alpha \in I$, $K \subseteq \bigcup\limits_{j=1}^{N} U_{\alpha_j}$ which shows that there is a finite subcover of K for the open cover $\{U_\alpha\}_{\alpha \in I}$. Therefore,$K$ is compact in (M, d). ▫

Alternative proof - sketch. Let $\{x_k\}_{k=1}^{\infty} \subseteq K$ be a sequence. By sequential compactness of K in either (M, d) or (N, d), there exists $\{x_{k_j}\}_{j=1}^{\infty}$ and $x \in K$ such that $x_{k_j} \to x$ as $j \to \infty$.

As long as the metric d used in different space are identical, the concept of convergence of a sequence are the same; thus (sequential) compactness in (M,d) or (N,d) are the same. ▫

Example 3.78. Let (M,d) be $(\mathbb{R},|\cdot|)$, and $N=\mathbb{Q}$. Then $F=[0,1]\cap\mathbb{Q}$ is not compact in $(\mathbb{Q},|\cdot|)$ since F is not complete. We can also apply Theorem 3.77 to see this: if $F\subseteq\mathbb{Q}$ is compact in $(\mathbb{Q},|\cdot|)$, then F is compact in $(\mathbb{R},|\cdot|)$ which is clearly not the case since F is not even closed in $(\mathbb{R},|\cdot|)$.

Remark 3.79. Let (M,d) be a metric space. By Proposition 3.65 a subset $A\subseteq M$ is disconnected if and only if there exist two subsets U_1, U_2 of A, open relative to A, such that $A=U_1\cup U_2$ and $U_1\cap U_2=\varnothing$ (one choice of (U_1,U_2) is $U_1=A\backslash\bar{A}_1$ and $U_2=A\backslash\bar{A}_2$, where A_1 and A_2 are given by Proposition 3.65). Note that U_1 and U_2 are also closed relative to A.

Given the observation above, if A is a connected set and E is a subset of A such that E is closed and open relative to A, then $E=\varnothing$ or $E=A$.

3.6 Exercises

In the exercise section of this chapter, we first introduce the concepts of accumulation points, isolated points and derived set of a set as follows.

Definition 3.80. Let (M,d) be a normed vector space, and A be a subset of M.

1. A point $x\in M$ is called an ***accumulation point*** of A if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $A\backslash\{x\}$ such that $\{x_n\}_{n=1}^{\infty}$ converges to x.

2. A point $x\in A$ is called an ***isolated point*** (孤立點) (of A) if there exists no sequence in $A\backslash\{x\}$ that converges to x.

3. The ***derived set*** of A is the collection of all accumulation points of A, and is denoted by A'.

§3.1 Limit Points and Interior Points of Sets

Problem 3.1. Let (M,d) be a metric space, and A be a subset of M.

1. Show that the collection of all isolated points of A is $A\backslash A'$.

2. Show that $A' = \bar{A} \backslash (A \backslash A')$. In other words, the derived set consists of all limit points that are not isolated points. Also show that $\bar{A} \backslash A' = A \backslash A'$.

Problem 3.2. Let A and B be subsets of a metric space (M, d). Show that

1. $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.

2. $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.

3. $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$. Find examples of that $\text{cl}(A \cap B) \subsetneq \text{cl}(A) \cap \text{cl}(B)$.

Problem 3.3. Let A and B be subsets of a metric space (M, d). Show that

1. $\text{int}(\text{int}(A)) = \text{int}(A)$.

2. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.

3. $\text{int}(A \cup B) \supseteq \text{int}(A) \cup \text{int}(B)$. Find examples of that $\text{int}(A \cup B) \supsetneq \text{int}(A) \cup \text{int}(B)$.

Problem 3.4. Let (M, d) be a metric space, and A be a subset of M. Show that

$$\partial A = \big(A \cap \text{cl}(M \backslash A)\big) \cup \big(\text{cl}(A) \backslash A\big).$$

Problem 3.5. Recall that in a metric space (M, d), a subset A is said to be dense in S if subsets satisfy $A \subseteq S \subseteq \text{cl}(A)$. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$.

1. Show that if A is dense in S and if S is dense in T, then A is dense in T.

2. Show that if A is dense in S and $B \subseteq S$ is open, then $B \subseteq \text{cl}(A \cap B)$.

Problem 3.6. Let A and B be subsets of a metric space (M, d). Show that

1. $\partial(\partial A) \subseteq \partial(A)$. Find examples of that $\partial(\partial A) \subsetneq \partial A$. Also show that $\partial(\partial A) = \partial A$ if A is closed.

2. $\partial(A \cup B) \subseteq \partial A \cup \partial B \subseteq \partial(A \cup B) \cup A \cup B$. Find examples of that equalities do not hold.

3. If $\text{cl}(A) \cap \text{cl}(B) = \varnothing$, then $\partial(A \cup B) = \partial A \cup \partial B$.

4. $\partial(A \cap B) \subseteq \partial A \cup \partial B$. Find examples of the equalities do not hold.

5. $\partial(\partial(\partial A)) = \partial(\partial A)$.

§3.2 Closed Sets and Open Sets

Problem 3.7. Let (M, d) be a metric space, and A be a subset of M. Show that $A \supseteq A'$ if and only if A is closed.

Problem 3.8. Show that the derived set of a set (in a metric space) is closed.

Problem 3.9. Let $A \subseteq \mathbb{R}^n$. Define the sequence of sets $A^{(m)}$ as follows: $A^{(0)} = A$ and $A^{(m+1)} =$ the derived set of $A^{(m)}$ for $m \in \mathbb{N}$. Complete the following.

1. Prove that each $A^{(m)}$ for $m \in \mathbb{N}$ is a closed set; thus $A^{(1)} \supseteq A^{(2)} \supseteq \cdots$.

2. Show that if there exists some $m \in \mathbb{N}$ such that $A^{(m)}$ is a countable set, then A is countable.

3. For any given $m \in \mathbb{N}$, is there a set A such that $A^{(m)} \neq \varnothing$ but $A^{(m+1)} = \varnothing$?

4. Let A be uncountable. Then each $A^{(m)}$ is an uncountable set. Is it possible that $\bigcap\limits_{m=1}^{\infty} A^{(m)} = \varnothing$?

5. Let $A = \Big\{\dfrac{1}{m} + \dfrac{1}{k} \,\Big|\, m - 1 > k(k-1), m, k \in \mathbb{N}\Big\}$. Find $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$.

Problem 3.10. Recall that a cluster point x of a sequence $\{x_n\}_{n=1}^{\infty}$ satisfies that

$$\forall\, \varepsilon > 0, \#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(x, \varepsilon)\big\} = \infty\,.$$

Show that the collection of cluster points of a sequence (in a metric space) is closed.

Problem 3.11. Determine whether the following sets are open or not.

1. $\bigcup\limits_{n=1}^{\infty} [-2 + \frac{1}{n}, 2 + \frac{1}{n}]\,.$

2. $\bigcup\limits_{n=1}^{\infty} [-2 - \frac{1}{n}, 2 - \frac{1}{n}]\,.$

3. $\bigcup\limits_{n=1}^{\infty} [-2 + \frac{1}{n}, 2 - \frac{1}{n}]\,.$

4. $\bigcup\limits_{n=1}^{\infty} [-2 - \frac{1}{n}, 2 + \frac{1}{n}]\,.$

Problem 3.12. Let $(\mathcal{V}, \|\cdot\|)$ ba a normed vector space, and C be a non-empty convex set in $\mathcal{V}$.

1. Show that $\bar{C}$ is convex.
2. Show that if $\boldsymbol{x} \in \mathring{C}$ and $\boldsymbol{y} \in \bar{C}$, then $(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} \in \mathring{C}$ for all $\lambda \in (0,1)$. This result is sometimes called the ***line segment principle***.
3. Show that $\mathring{C}$ is convex (you may need the conclusion in 2 to prove this).
4. Show that $\text{cl}(\mathring{C}) = \text{cl}(C)$.
5. Show that $\text{int}(\bar{C}) = \text{int}(C)$.

Hint: 2. Prove by contradiction.
3 and 4. Use the line segment principle.
5. Show that $\boldsymbol{x} \in \text{int}(\bar{C})$ can be written as $(1-\lambda)\boldsymbol{y} + \lambda\boldsymbol{z}$ for some $\boldsymbol{y} \in \mathring{C}$ and $\boldsymbol{z} \in B(\boldsymbol{x}, \varepsilon) \subseteq \bar{C}$.

Problem 3.13. Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space. Show that for all $\boldsymbol{x} \in \mathcal{V}$ and $r > 0$,

$$\text{int}\big(B[\boldsymbol{x}, r]\big) = B(\boldsymbol{x}, r)\,.$$

Is the identity above true in general metric space?

Problem 3.14. Let $\mathcal{M}_{n\times n}$ denote the collection of all $n \times n$ square real matrices, and $(\mathcal{M}_{n\times n}, \|\cdot\|_{p,q})$ be a normed space with norm $\|\cdot\|_{p,q}$ given in Problem **??**. Show that the set

$$\text{GL}(n) \equiv \big\{A \in \mathcal{M}_{n\times n} \,\big|\, \det(A) \neq 0\big\}$$

is an open set in $\mathcal{M}_{n\times n}$. The set $\text{GL}(n)$ is called the general linear group.

Problem 3.15. Show that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint open intervals; that is, if $U \subseteq \mathbb{R}$ is open, then

$$U = \bigcup_{k\in\mathcal{I}} (a_k, b_k)\,,$$

where $\mathcal{I}$ is countable, and $(a_k, b_k) \cap (a_\ell, b_\ell) = \varnothing$ if $k \neq \ell$.
Hint: For each point $x \in U$, define $L_x = \big\{y \in \mathbb{R} \,\big|\, (y, x) \subseteq U\big\}$ and $R_x = \big\{y \in \mathbb{R} \,\big|\, (x, y) \subseteq U\big\}$. Define $I_x = (\inf L_x, \sup R_x)$. Show that $I_x = I_y$ if $(x, y) \in U$ and if $(x, y) \nsubseteq U$ then $I_x \cap I_y = \varnothing$

Problem 3.16. Let (M,d) be a metric space. A set $A \subseteq M$ is said to be ***perfect*** if $A = A'$ (so that there is no isolated points). The Cantor set is constructed by the following procedure: let $E_0 = [0,1]$. Remove the segment $\big(\frac{1}{3}, \frac{2}{3}\big)$, and let E_1 be the union of the intervals

$$[0, \frac{1}{3}], [\frac{2}{3}, 1].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$[0, \frac{1}{9}], [\frac{2}{9}, \frac{3}{9}], [\frac{6}{9}, \frac{7}{9}], [\frac{8}{9}, 1].$$

Continuing in this way, we obtain a sequence of closed set E_k such that

(a) $E_1 \supseteq E_2 \supseteq E_2 \supseteq \cdots$;

(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set $C = \bigcap\limits_{n=1}^{\infty} E_n$ is called the ***Cantor set***.

1. Show that C is a perfect set.

2. Show that C is uncountable.

3. Find $\text{int}(C)$.

§3.3 Compactness

Problem 3.17. Let $\mathcal{V}$ be a vector fields over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} \subseteq \mathcal{V}$ is a basis for $\mathcal{V}$; that is, every $\boldsymbol{x} \in \mathcal{V}$ can be uniquely expressed as

$$\boldsymbol{x} = x^{(1)}\mathbf{e}_1 + x^{(2)}\mathbf{e}_2 + \cdots + x^{(n)}\mathbf{e}_n = \sum_{i=1}^{n} x^{(i)}\mathbf{e}_i\,.$$

Define $\|\boldsymbol{x}\|_2 = \Big(\sum\limits_{i=1}^{n} \big|x^{(i)}\big|^2\Big)^{\frac{1}{2}}$.

1. Show that $\|\cdot\|_2$ is a norm on $\mathcal{V}$.

2. Show that K is compact in $(\mathcal{V}, \|\cdot\|_2)$ if and only if K is closed and bounded.

Problem 3.18. Let (M,d) be a metric space.

1. Show that a closed subset of a compact set is compact.

2. Show that the union of a finite number of sequentially compact subsets of M is compact.

3. Show that the intersection of an arbitrary collection of sequentially compact subsets of M is sequentially compact.

Problem 3.19. A metric space (M, d) is said to be ***separable*** if there is a countable subset A which is dense in M. Show that every sequentially compact set is separable.
Hint: Consider the total boundedness using balls with radius $\dfrac{1}{n}$ for $n \in \mathbb{N}$.

Problem 3.20. Given $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ a bounded sequence, define

$$A = \left\{x \in \mathbb{R} \,\middle|\, \text{there exists a subsequence } \left\{a_{k_j}\right\}_{j=1}^{\infty} \text{ such that } \lim_{j\to\infty} a_{k_j} = x\right\}.$$

Show that A is a non-empty sequentially compact set in $\mathbb{R}$. Furthermore , $\limsup\limits_{k\to\infty} a_k = \sup A$ and $\liminf\limits_{k\to\infty} a_k = \inf A$.

Problem 3.21. Let (M, d) be a metric space.

1. Show that if M is complete and A is a totally bounded subset of M, then $\mathrm{cl}(A)$ is sequentially compact.

2. Show that M is complete if and only if every totally bounded sequence has a convergent subsequence.

Problem 3.22. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x, y) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2\,, \\ |x_1 - y_1| + |x_2 - y_2| + 1 & \text{if } x_2 \neq y_2\,. \end{cases} \quad \text{where } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

Problem **??** shows that d is a metric on $\mathbb{R}^2$. Consider the metric space $(\mathbb{R}^2, d)$.

1. Find $B(x, r)$ with $r < 1$, $r = 1$ and $r > 1$.

2. Show that the set $\{c\} \times [a, b] \subseteq (\mathbb{R}^2, d)$ is closed and bounded.

3. Examine whether the set $\{c\} \times [a, b] \subseteq (\mathbb{R}^2, d)$ is sequentially compact or not.

Problem 3.23. Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in a metric space, and $x_k \to x$ as $k \to \infty$. Show that the set $A \equiv \{x_1, x_2, \cdots, \} \cup \{x\}$ is sequentially compact.

Problem 3.24. 1. Show the so-called "***Finite Intersection Property***":

> Let (M, d) be a metric space, and K be a subset of M. Then K is compact if and if for any family of closed subsets $\{F_\alpha\}_{\alpha \in I}$, we have
> $$K \cap \bigcap_{\alpha \in I} F_\alpha \neq \varnothing$$
> whenever $K \cap \bigcap_{\alpha \in J} F_\alpha \neq \varnothing$ for all $J \subseteq I$ satisfying $\# J < \infty$.

2. Show the so-called "***Nested Set Properpty***":

> Let (M, d) be a metric space. If $\{K_n\}_{n=1}^{\infty}$ is a sequence of non-empty compact sets in M such that $K_j \supseteq K_{j+1}$ for all $j \in \mathbb{N}$, then there exists at least one point in $\bigcap_{j=1}^{\infty} K_j$; that is,
> $$\bigcap_{j=1}^{\infty} K_j \neq \varnothing \,.$$

Problem 3.25. Let (M, d) be a metric space, and M itself is a sequentially compact set. Show that if $\{F_k\}_{k=1}^{\infty}$ is a sequence of closed sets such that $\text{int}(F_k) = \varnothing$, then $M \backslash \bigcup_{k=1}^{\infty} F_k \neq \varnothing$.

Problem 3.26. Let ℓ^2 be the collection of all sequences $\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ such that $\sum_{k=1}^{\infty} |x_k|^2 < \infty$. In other words,

$$\ell^2 = \Big\{ \{x_k\}_{k=1}^{\infty} \,\Big|\, x_k \in \mathbb{R} \text{ for all } k \in \mathbb{N}, \sum_{k=1}^{\infty} |x_k|^2 < \infty \Big\} \,.$$

Define $\|\cdot\|_2 : \ell^2 \to \mathbb{R}$ by

$$\big\|\{x_k\}_{k=1}^{\infty}\big\|_2 = \Big(\sum_{k=1}^{\infty} |x_k|^2 \Big)^{\frac{1}{2}} \,.$$

1. Show that $\|\cdot\|_2$ is a norm on ℓ^2. The normed space $(\ell^2, \|\cdot\|)$ usually is denoted by ℓ^2.

2. Show that $\|\cdot\|_2$ is induced by an inner product.

3. Show that $(\ell^2, \|\cdot\|_2)$ is complete.

4. Let $A = \{\boldsymbol{x} \in \ell^2 \,|\, \|\boldsymbol{x}\|_2 \leqslant 1\}$. Is A sequentially compact or not?

Problem 3.27. Let A, B be two non-empty subsets in $\mathbb{R}^n$. Define

$$d(A, B) = \inf\{\|x - y\|_2 \,|\, x \in A, y \in B\}$$

to be the distance between A and B. When $A = \{x\}$ is a point, we write $d(A, B)$ as $d(x, B)$ (which is consistent with the one given in Proposition 3.6).

(1) Prove that $d(A, B) = \inf\{d(x, B) \,|\, x \in A\}$.

(2) Show that $\big|d(x_1, B) - d(x_2, B)\big| \leqslant \|x_1 - x_2\|_2$ for all $x_1, x_2 \in \mathbb{R}^n$.

(3) Define $B_\varepsilon = \{x \in \mathbb{R}^n \,|\, d(x, B) < \varepsilon\}$ be the collection of all points whose distance from B is less than ε. Show that B_ε is open and $\bigcap\limits_{\varepsilon > 0} B_\varepsilon = \mathrm{cl}(B)$.

(4) If A is sequentially compact, show that there exists $x \in A$ such that $d(A, B) = d(x, B)$.

(5) If A is closed and B is sequentially compact, show that there exists $x \in A$ and $y \in B$ such that $d(A, B) = d(x, y)$.

(6) If A and B are both closed, does the conclusion of (5) hold?

Problem 3.28. Let $\mathcal{K}(n)$ denote the collection of all non-empty sequentially compact sets in $\mathbb{R}^n$. Define the Hausdorff distance of $K_1, K_2 \in \mathcal{K}(n)$ by

$$d^H(K_1, K_2) = \max\left\{\sup_{x \in K_2} d(x, K_1), \sup_{x \in K_1} d(x, K_2)\right\},$$

in which $d(x, K)$ is the distance between x and K given in Problem 3.27. Show that $(\mathcal{K}(n), d^H)$ is a metric space.

Problem 3.29. Let $M = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant 1\}$ with the standard metric $\|\cdot\|_2$. Show that $A \subseteq M$ is sequentially compact if and only if A is closed.

Problem 3.30. 1. Let $\{x_k\}_{k=1}^\infty \subseteq \mathbb{R}$ be a sequence in $(\mathbb{R}, |\cdot|)$ that converges to x and let $A_k = \{x_k, x_{k+1}, \cdots\}$. Show that $\{x\} = \bigcap\limits_{k=1}^{\infty} \overline{A_k}$. Is this true in any metric space?

2. Suppose that $\{K_j\}_{j=1}^{\infty}$ is a sequence of comapct non-empty sets satisfying the nested set property; that is, $K_j \supseteq K_{j+1}$, and $\text{diam}(K_j) \to 0$ as $j \to \infty$, where

$$\text{diam}(K_j) = \sup \left\{ d(x,y) \,\middle|\, x, y \in K_j \right\}.$$

Show that there is exactly one point in $\bigcap\limits_{j=1}^{\infty} K_j$.

Problem 3.31. Let (M,d) be a metric space, and A be a subset of M satisfying that every sequence in A has a convergent subsequence (with limit in M). Show that A is pre-compact. Remark: Together with Remark 3.61, we conclude that a subset A is pre-compact if and only if A has the property that "every sequence in A has a convergent subsequence".

§3.4 Connectedness

Problem 3.32. Let (M,d) be a metric space, and $A \subseteq M$. Show that A is disconnected (not connected) if and only if there exist non-empty closed set F_1 and F_2 such that

1. $A \cap F_1 \cap F_2 = \varnothing$; 2. $A \cap F_1 \neq \varnothing$; 3. $A \cap F_2 \neq \varnothing$; 4. $A \subseteq F_1 \cup F_2$.

Problem 3.33. Prove that if A is connected in a metric space (M,d) and $A \subseteq B \subseteq \bar{A}$, then B is connected.

Problem 3.34. Let (M,d) be a metric space, and $A \subseteq M$ be a subset. Suppose that A is connected and contain more than one point. Show that $A \subseteq A'$.

Problem 3.35. Show that the Cantor set C defined in Problem 3.16 is totally disconnected; that is, if $x, y \in C$, and $x \neq y$, then $x \in U$ and $y \in V$ for some open sets U, V separate C.

Problem 3.36. Let F_k be a nest of connected compact sets (that is, $F_{k+1} \subseteq F_k$ and F_k is connected for all $k \in \mathbb{N}$). Show that $\bigcap\limits_{k=1}^{\infty} F_k$ is connected. Give an example to show that compactness is an essential condition and we cannot just assume that F_k is a nest of closed connected sets.

Problem 3.37. Let $\{A_k\}_{k=1}^{\infty}$ be a family of connected subsets of M, and suppose that A is a connected subset of M such that $A_k \cap A \neq \varnothing$ for all $k \in \mathbb{N}$. Show that the union $\big(\bigcup\limits_{k \in \mathbb{N}} A_k\big) \cup A$ is also connected.

Problem 3.38. Let $A, B \subseteq M$ and A is connected. Suppose that $A \cap B \neq \varnothing$ and $A \cap B^{\complement} \neq \varnothing$. Show that $A \cap \partial B \neq \varnothing$.

Problem 3.39. Let (M,d) be a metric space and A be a non-empty subset of M. A maximal connected subset of A is called a ***connected component*** of A.

1. Let $a \in A$. Show that there is a unique connected components of A containing a.

2. Show that any two distinct connected components of A are disjoint. Therefore, A is the disjoint union of its connected components.

3. Show that every connected component of A is a closed subset of A.

4. If A is open, prove that every connected component of A is also open. Therefore, when $M = \mathbb{R}^n$, show that A has at most countable infinite connected components.

5. Find the connected components of the set of rational numbers or the set of irrational numbers in $\mathbb{R}$.

Problem 3.40 (True or False). Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1. Every open set in a metric space is a countable union of closed sets.

2. Let $A \subseteq \mathbb{R}$ be bounded from above, then $\sup A \in A'$.

3. An infinite union of distinct closed sets cannot be closed.

4. An interior point of a subset A of a metric space (M,d) is an accumulation point of that set.

5. Let (M,d) be a metric space, and $A \subseteq M$. Then $(A')' = A'$.

6. There exists a metric space in which some unbounded Cauchy sequence exists.

7. Every metric defined in $\mathbb{R}^n$ is induced from some "norm" in $\mathbb{R}^n$.

8. There exists a non-zero dimensional normed vector space in which some compact non-zero dimensional linear subspace exists.

9. There exists a set $A \subseteq (0,1]$ which is compact in $(0,1]$ (in the sense of subspace topology), but A is not compact in $\mathbb{R}$.

10. Let $A \subseteq \mathbb{R}^n$ be a non-empty set. Then a subset B of A is compact in A if and only if B is closed and bounded in A.

Problem 3.41. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Determine which of the following statements are true.

1. $\text{int}A = A \backslash \partial A$.
2. $\text{cl}(A) = M \backslash \text{int}(M \backslash A)$.
3. $\text{int}(\text{cl}(A)) = \text{int}(A)$.
4. $\text{cl}(\text{int}(A)) = A$.
5. $\partial(\text{cl}(A)) = \partial A$.
6. If A is open, then $\partial A \subseteq M \backslash A$.
7. If A is open, then $A = \text{cl}(A) \backslash \partial A$. How about if A is not open?

Chapter 4

Continuous Maps

4.1 Continuity

Definition 4.1. Let (M, d) be a metric space, and A be a subset of M.

1. A point $x \in M$ is called an ***accumulation point*** of A if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $A \backslash \{x\}$ such that $\{x_n\}_{n=1}^{\infty}$ converges to x.

2. The ***derived set*** of A is the collection of all accumulation points of A, and is denoted by A'.

Remark 4.2. 1. A point $x \in A'$ if and only if $(\forall\, \varepsilon > 0)\big(B(x, \varepsilon) \cap (A \backslash \{x\}) \neq \varnothing\big)$. Therefore, $x \notin A'$ if and only if $(\exists\, \varepsilon > 0)\big(B(x, \varepsilon) \cap A \subseteq \{x\}\big)$.

2. A point $x \in M$ is an accumulation point x of A if and only if

$$(\forall\, \varepsilon > 0)\big(\#\big\{y \in M \,\big|\, y \in B(x, \varepsilon) \cap A\big\} = \infty\big)\,.$$

   Therefore, accumulation points of a set can be viewed as a generalization of cluster points of a sequence.

3. A subset A of M is closed if and only if $A \supseteq A'$. In fact, $\bar{A} = A \cup A'$.

4. The derived set A' of a subset A of M is closed.

5. A point $x \in A \backslash A'$ satisfies that there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap A = \{x\}$. Such kind of points are called ***isolated points*** of A.

Definition 4.3. Let (M, d) and (N, ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be a map. For a given $c \in A'$, we say that the limit of f at c exists if for every sequence $\{x_k\}_{k=1}^{\infty} \subseteq A \backslash \{c\}$ converging to c, the sequence $\big\{f(x_k)\big\}_{k=1}^{\infty}$ converges（所有在定義域中取值不是 c 但收斂到 c 的數列，其函數值所形成的數列收斂）.

Similar to Proposition 2.42, the limit of f at x_0, if it exists, is unique.

Proposition 4.4. *Let* (M, d) *and* (N, ρ) *be metric spaces,* A *be a subset of* M*,* $c \in A'$*, and* $f : A \to N$ *be a map. If the limit of* f *at* c *exists, then the limit is unique in the sense that if* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are sequences in* $A \backslash \{c\}$ *and* $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = c$*, then* $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} f(y_n)$.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in $A\backslash\{c\}$ so that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = c$. Then $\lim\limits_{n\to\infty} f(x_n) = a$ and $\lim\limits_{n\to\infty} f(y_n) = b$ both exist. Define a new sequence

$$z_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd}\,, \\ y_{\frac{n}{2}} & \text{if } n \text{ is even}\,, \end{cases}$$

or $\{z_n\}_{n=1}^{\infty} = \{x_1, y_1, x_2, y_2, \cdots\}$. Then $z_n \to c$ as $n \to \infty$; thus $\lim\limits_{n\to\infty} f(z_n)$ exists. Since $\big\{f(x_n)\big\}_{n=1}^{\infty}$ and $\big\{f(y_n)\big\}_{n=1}^{\infty}$ are both subsequences of $\big\{f(z_n)\big\}_{n=1}^{\infty}$, by Proposition 2.56 we conclude that $a = b$. ▫

Notation: When the limit of f at c exists, we use $\lim\limits_{x\to c} f(x)$ to denote the common limit of $\lim\limits_{k\to\infty} f(x_k)$ if $\{x_k\}_{k=1}^{\infty} A\backslash\{c\}$ converges to c.

Proposition 4.5. *Let* (M, d) *and* (N, ρ) *be metric spaces,* A *be a subset of* M*,* $c \in A'$*, and* $f : A \to N$ *be a map. Then* $\lim\limits_{x\to c} f(x) = b$ *if and only if*

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\big(0 < d(x, c) < \delta \text{ and } x \in A \Rightarrow \rho(f(x), b) < \varepsilon\big)\,.$$

Proof. "⇒" Assume the contrary that there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x_\delta \in A$ with

$$0 < d(x_\delta, c) < \delta \quad \text{and} \quad \rho(f(x_\delta), b) \geqslant \varepsilon\,.$$

In particular, for each $k \in \mathbb{N}$, we can find $x_k \in A\backslash\{c\}$ such that

$$0 < d(x_k, c) < \frac{1}{k} \quad \text{and} \quad \rho(f(x_k), b) \geqslant \varepsilon\,.$$

Then $x_k \to c$ as $k \to \infty$ but $f(x_k) \not\to b$ as $k \to \infty$, a contradiction.

"⇐" Let $\{x_k\}_{k=1}^{\infty} \subseteq A\backslash\{c\}$ be such that $x_k \to c$ as $k \to \infty$, and $\varepsilon > 0$ be given. By assumption,

$$\exists\, \delta > 0 \ni \rho(f(x), b) < \varepsilon \quad \text{whenever} \quad 0 < d(x, c) < \delta \text{ and } x \in A\,.$$

Since $x_k \to c$ as $k \to \infty$, there exists $N > 0$ such that $d(x_k, c) < \delta$ if $k \geqslant N$. Therefore,

$$\rho(f(x_k), b) < \varepsilon \quad \forall\, k \geqslant N$$

which implies that $\lim\limits_{k\to\infty} f(x_k) = b$. □

Remark 4.6. The positive number δ in the proposition above usually depends on ε, as well as the point c. Therefore, we also write $\delta = \delta(c, \varepsilon)$ to emphasize the dependence of c and ε.

Remark 4.7. Let $(M, d) = (N, \rho) = (\mathbb{R}, |\cdot|)$, $A = (a, b)$, and $f : A \to N$. We write $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ for the limit $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to b} f(x)$, respectively, if the later exist.

Definition 4.8. Let (M, d) and (N, ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be a map. For a given $c \in A$, f is said to be continuous at c if either $c \in A\backslash A'$ or $\lim\limits_{x\to c} f(x) = f(c)$.

Proposition 4.9. *Let (M, d) and (N, ρ) be metric spaces, A be a subset of M, $c \in A$ and $f : A \to N$ be a map. Then the following three statements are equivalent.*

1. *f is continuous at c.*
2. *For every convergent sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ with limit c, $\lim\limits_{n\to\infty} f(x_n) = f(c)$.*
3. *For each $\varepsilon > 0$, there exists $\delta = \delta(c, \varepsilon) > 0$ such that*

$$\rho(f(x), f(c)) < \varepsilon \quad \textit{whenever} \quad x \in B_M(c, \delta) \cap A\,.$$

In logical notation,

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\big(x \in B_M(c, \delta) \cap A \Rightarrow f(x) \in B_N(f(c), \varepsilon)\big)\,,$$

where $B_M(\cdot, \cdot)$ and $B_N(\cdot, \cdot)$ denote balls in (M, d) and (N, ρ), respectively.

Proof. "1 ⇒ 3" Note that $A = (A \cap A') \cup (A\backslash A')$.

Case 1: If $c \in A \cap A'$, then f is continuous at c if and only if

$$\forall\, \varepsilon > 0, \exists\, \delta = \delta(c, \varepsilon) > 0 \ni \rho(f(x), f(c)) < \varepsilon \ \text{ whenever } \ x \in B_M(c, \delta) \cap A\backslash\{c\}\,.$$

Since $\rho(f(c), f(c)) = 0 < \varepsilon$, we find that the statement above is equivalent to that

$$\forall\, \varepsilon > 0, \exists\, \delta = \delta(c, \varepsilon) > 0 \ni \rho(f(x), f(c)) < \varepsilon \quad \text{whenever} \quad x \in B_M(c, \delta) \cap A\,.$$

Case 2: Let $c \in A \backslash A'$. Then there exists $\delta > 0$ such that $B_M(c, \delta) \cap A = \{c\}$. Therefore,

$$x \in B_M(c, \delta) \cap A \Rightarrow \rho(f(x), f(c)) < \varepsilon$$

no matter what $\varepsilon > 0$ is given.

"$3 \Rightarrow 2$" The proof of this direction is almost identical as the proof of the direction "$\Leftarrow$" of Proposition 4.5. Let $\{x_k\}_{k=1}^{\infty} \subseteq A\backslash$ be such that $x_k \to c$ as $k \to \infty$, and $\varepsilon > 0$ be given. By assumption,

$$\exists\, \delta > 0 \ni \rho(f(x), b) < \varepsilon \quad \text{whenever} \quad d(x, c) < \delta \text{ and } x \in A\,.$$

Since $x_k \to c$ as $k \to \infty$, there exists $N > 0$ such that $d(x_k, c) < \delta$ if $k \geqslant N$. Therefore,

$$\rho(f(x_k), b) < \varepsilon \quad \forall\, k \geqslant N$$

which implies that $\lim\limits_{k\to\infty} f(x_k) = f(c)$.

"$2 \Rightarrow 1$" If $c \in A\backslash A'$, f is continuous at c; thus it suffices to show that f is continuous at c $\big($or equivalently, $\lim\limits_{x\to c} f(x) = f(c)\big)$ for $c \in A \cap A'$ when 2 is true.

Let $\{x_k\}_{k=1}^{\infty} \subseteq A\backslash\{c\}$ be a sequence with limit c. By assumption, $\lim\limits_{k\to\infty} f(x_k) = f(c)$; thus we establish that for every convergent sequence $\{x_k\}_{k=1}^{\infty} \subseteq A\backslash\{c\}$ with limit c, the sequence $\{f(x_k)\}_{k=1}^{\infty}$ converges to $f(c)$; thus $\lim\limits_{x\to c} f(x) = f(c)$. ▫

Remark 4.10. We remark here that Proposition 4.9 implies that f is continuous at $c \in A$ if and only if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni f(B(c, \delta) \cap A) \subseteq B(f(c), \varepsilon)\,.$$

Example 4.11. Let $X = \mathscr{C}([a, b]; \mathbb{R})$, the collection of all real-valued continuous functions defined on $[a, b]$, and $\|\cdot\|_X$ be the norm given by $\|f\|_X = \max\limits_{x\in[a,b]} \big|f(x)\big|$. Note that $(X, \|\cdot\|_X)$ is a normed vector space (Example 2.21). Define $\mathrm{I} : X \to \mathbb{R}$ by

$$\mathrm{I}(f) = \int_a^b \big|f(x)\big|^2\, dx\,.$$

In the following we show that I is continuous at any points on X. Let $f \in X$ and $\varepsilon > 0$ be given. Choose $\delta = \min\Big\{\varepsilon, \dfrac{\varepsilon}{2(b-a)\big(2\|f\|_X + \varepsilon\big)}\Big\}$. Then $0 < \delta \leqslant \varepsilon$ and if $g \in X$ satisfies

$\|f-g\|_X < \delta$, we must have

$$\begin{aligned}(b-a)\big[2\|f\|_X + \|f-g\|_X\big]\|f-g\|_X &\leqslant (b-a)\big(2\|f\|_X+\delta\big)\delta \\ &\leqslant (b-a)\big(2\|f\|_X+\varepsilon\big)\frac{\varepsilon}{2(b-a)\big(2\|f\|_X+\varepsilon\big)} = \frac{\varepsilon}{2} < \varepsilon\,.\end{aligned}$$

Therefore, if $g \in X$ and $\|g-f\|_X < \delta$,

$$\begin{aligned}\big|\mathrm{I}(g)-\mathrm{I}(f)\big| = \Big|\int_a^b \big[|g(x)|^2 - |f(x)|^2\big]\,dx\Big| &\leqslant \int_a^b \big|g(x)-f(x)\big|\big|g(x)+f(x)\big|\,dx \\ \leqslant (b-a)\big(\|f\|_X+\|g\|_X\big)\|f-g\|_X &\leqslant (b-a)\big(2\|f\|_X+\|f-g\|_X\big)\|f-g\|_X < \varepsilon\,;\end{aligned}$$

thus I is continuous on X.

Definition 4.12. Let (M,d) and (N,ρ) be metric spaces, and A be a subset of M. A map $f : A \to N$ is said to be continuous on the set $B \subseteq A$ if f is continuous at each point of B.

Remark 4.13. The Dirichlet function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \cap \mathbb{Q}\,, \\ 1 & \text{if } x \in [0,1] \cap \mathbb{Q}^{\complement}\,. \end{cases}$$

is not continuous at any point of $[0,1]$; however, the restriction of f to $B = [0,1] \cap \mathbb{Q}$ (or $B = [0,1] \cap \mathbb{Q}^{\complement}$), denoted by $f\!\upharpoonright_B$, is continuous on B. Therefore, f is continuous on B is different from that $f\!\upharpoonright_B$ is continuous on B.

Theorem 4.14. *Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. Then the following assertions are equivalent:*

1. *f is continuous on A.*

2. *For each open set $V \subseteq N$, $f^{-1}(V) \subseteq A$ is open relative to A; that is, $f^{-1}(V) = U \cap A$ for some U open in M.*

3. *For each closed set $E \subseteq N$, $f^{-1}(E) \subseteq A$ is closed relative to A; that is, $f^{-1}(E) = F \cap A$ for some F closed in M.*

Proof. It should be clear that $2 \Leftrightarrow 3$ (left as an exercise); thus we show that $1 \Leftrightarrow 2$. Before proceeding, we recall that $B \subseteq f^{-1}(f(B))$ for all $B \subseteq A$ and $f(f^{-1}(B)) \subseteq B$ for all $B \subseteq N$.

"1 ⇒ 2" Let $a \in f^{-1}(V)$. Then $f(a) \in V$. Since V is open in (N,ρ), there exists $\varepsilon_{f(a)} > 0$ such that $B_N(f(a),\varepsilon_{f(a)}) \subseteq V$. By continuity of f (and Remark 4.10), there exists $\delta_a > 0$ such that

$$f(B_M(a,\delta_a) \cap A) \subseteq B_N\big(f(a),\varepsilon_{f(a)}\big)\,.$$

Therefore, by Proposition 0.11, for each $a \in f^{-1}(V)$, there exists $\delta_a > 0$ such that

$$B_M(a,\delta_a) \cap A \subseteq f^{-1}\big(f(B_M(a,\delta_a) \cap A)\big) \subseteq f^{-1}\big(B_N\big(f(a),\varepsilon_{f(a)}\big)\big) \subseteq f^{-1}(V)\,. \quad (4.1.1)$$

Let $U = \bigcup\limits_{a \in f^{-1}(V)} B_M(a,\delta_a)$. Then U is open (since it is the union of arbitrarily many open balls), and

(a) $U \supseteq f^{-1}(V)$ since U contains every center of balls whose union forms U;

(b) $U \cap A \subseteq f^{-1}(V)$ by (4.1.1).

Therefore, $U \cap A = f^{-1}(V)$.

"2 ⇒ 1" Let $a \in A$ and $\varepsilon > 0$ be given. Define $V = B_N(f(a),\varepsilon)$. By assumption there exists an open set U in (M,d) such that $f^{-1}(V) = U \cap A$. Since $a \in f^{-1}(V)$, $a \in U$; thus by the openness of U, there exists $\delta > 0$ such that $B_M(a,\delta) \subseteq U$. Therefore, by Proposition 0.11 we have

$$f(B_M(a,\delta) \cap A) \subseteq f(U \cap A) = f(f^{-1}(V)) \subseteq V = B_N(f(a),\varepsilon)$$

which implies that f is continuous at a. Therefore, f is continuous at a for all $a \in A$; thus f is continuous on A. □

Example 4.15. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Then $\big\{x \in \mathbb{R}^n \,\big|\, \|f(x)\|_2 < 1\big\}$ is open since

$$\big\{x \in \mathbb{R}^n \,\big|\, \|f(x)\|_2 < 1\big\} = f^{-1}\big(B(0,1)\big)\,.$$

Example 4.16. Let $f : \mathcal{M}_{n\times n} \to \mathbb{R}$ be defined by $f(A) = \det(A)$. Then the set

$$\mathrm{GL}(n) \equiv \big\{A \in \mathcal{M}_{n\times n} \,\big|\, \det(A) \neq 0\big\}$$

is open in $(\mathcal{M}_{n\times n}, \|\cdot\|_{p,q})$ for all $1 \leqslant p,q \leqslant \infty$ (if one can show that the determinant function is continuous).

Remark 4.17. For a function f of two variables or more, it is important to distinguish the continuity of f and the continuity in each variable (by holding all other variables fixed). For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 1 & \text{if either } x = 0 \text{ or } y = 0, \\ 0 & \text{if } x \neq 0 \text{ and } y \neq 0. \end{cases}$$

Observe that $f(0, 0) = 1$, but f is not continuous at $(0, 0)$. In fact, for any $\delta > 0$, $f(x, y) = 0$ for infinitely many values of $(x, y) \in B((0, 0), \delta)$; that is, $|f(x, y) - f(0, 0)| = 1$ for such values. However if we consider the function $g(x) = f(x, 0) = 1$ or the function $h(y) = f(0, y) = 1$, then g, h are continuous. Note also that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exists but $\lim\limits_{x\to 0}(\lim\limits_{y\to 0} f(x, y)) = \lim\limits_{x\to 0} 0 = 0$.

4.2 Operations on Continuous Maps

Definition 4.18. Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space over field $\mathbb{F}$, A be a subset of M, and $f, g : A \to \mathcal{V}$ be maps, $h : A \to \mathbb{F}$ be a function. The maps $f + g$, $f - g$ and hf, mapping from A to $\mathcal{V}$, are defined by

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \qquad && \forall\, x \in A\,, \\ (f - g)(x) &= f(x) - g(x) && \forall\, x \in A\,, \\ (hf)(x) &= h(x)f(x) && \forall\, x \in A\,. \end{aligned}$$

The map $\dfrac{f}{h} : A\backslash\{x \in A \,|\, h(x) = 0\} \to \mathcal{V}$ is defined by

$$\Big(\frac{f}{h}\Big)(x) = \frac{f(x)}{h(x)} \qquad \forall\, x \in A\backslash\{x \in A \,|\, h(x) = 0\}\,.$$

Proposition 4.19. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space over field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), A be a subset of M, and $f, g : A \to \mathcal{V}$ be maps, $h : A \to \mathbb{F}$ be a function. Suppose that $x_0 \in A'$, and $\lim\limits_{x\to x_0} f(x) = a$, $\lim\limits_{x\to x_0} g(x) = b$, $\lim\limits_{x\to x_0} h(x) = c$. Then*

$$\begin{aligned} \lim_{x\to x_0} (f + g)(x) &= a + b\,, \\ \lim_{x\to x_0} (f - g)(x) &= a - b\,, \\ \lim_{x\to x_0} (hf)(x) &= ca\,, \\ \lim_{x\to x_0} \Big(\frac{f}{h}\Big) &= \frac{a}{c} \quad \text{if } c \neq 0\,. \end{aligned}$$

Corollary 4.20. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a normed vector space over field $\mathbb{F}$ ($\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$), A be a subset of M, and $f,g:A\to\mathcal{V}$ be maps, $h:A\to\mathbb{F}$ be a function. Suppose that f,g,h are continuous at $x_0\in A$. Then the maps $f+g$, $f-g$ and hf are continuous at x_0, and $\dfrac{f}{h}$ is continuous at x_0 if $h(x_0)\neq 0$.*

Corollary 4.21. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a normed vector space over field $\mathbb{F}$ ($\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$), $A\subseteq M$, and $f,g:A\to\mathcal{V}$ be continuous maps, $h:A\to\mathbb{F}$ be a continuous function. Then the maps $f+g$, $f-g$ and hf are continuous on A, and $\dfrac{f}{h}$ is continuous on $A\backslash\{x\in A\,|\,h(x)=0\}$.*

Definition 4.22. Let (M,d), (N,ρ) and (P,δ) be metric spaces, A be a subset of M, B be a subset of N, and $f:A\to N$, $g:B\to P$ be maps such that $f(A)\subseteq B$. The composite function $g\circ f:A\to P$ is the map defined by

$$(g\circ f)(x)=g\big(f(x)\big)\qquad\forall\,x\in A\,.$$

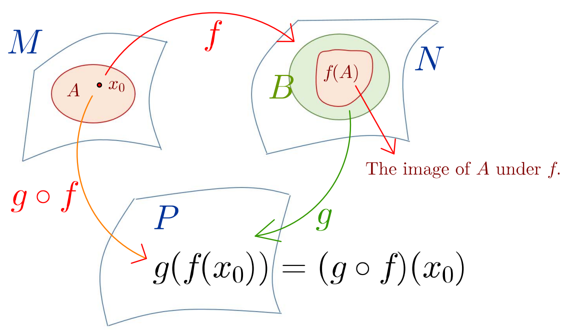


Figure 4.1: The composition of functions

Theorem 4.23. *Let (M,d), (N,ρ) and (P,δ) be metric spaces, A be a subset of M, B be a subset of N, and $f:A\to N$, $g:B\to P$ be maps such that $f(A)\subseteq B$. Suppose that $a\in A\cap A'$ and $\lim\limits_{x\to a}f(x)=b$, and g is continuous at b. Then $\lim\limits_{x\to a}(g\circ f)(x)=g(b)$.*

Proof. Let $\{x_n\}_{n=1}^{\infty}\subseteq A\backslash\{a\}$ be a convergent sequence with limit a. Then $\{f(x_n\}_{n=1}^{\infty}$ is a convergent sequence with limit b; thus the continuity of g at b implies that $\big\{g(f(x_n))\big\}_{n=1}^{\infty}$ converges to $g(b)$. □

Corollary 4.24. *Let (M,d), (N,ρ) and (P,δ) be metric spaces, A be a subset of M, B be a subset of N, and $f:A\to N$, $g:B\to P$ be maps such that $f(A)\subseteq B$.*

1. *If f is continuous at a and g is continuous at $f(a)$, then $g \circ f : A \to P$ is continuous at a.*

2. *If f is continuous on A and g is continuous on B, then $g \circ f : A \to P$ is continuous on A.*

Alternative Proof of 2 *in Corollary* 4.24. Let $\mathcal{W}$ be an open set in (P, r). By Theorem 4.14, there exists $\mathcal{V}$ open in (N, ρ) such that $g^{-1}(\mathcal{W}) = \mathcal{V} \cap B$. Since $\mathcal{V}$ is open in (N, ρ), by Theorem 4.14 again there exists $\mathcal{U}$ open in (M, d) such that $f^{-1}(\mathcal{V}) = \mathcal{U} \cap A$. Then

$$(g \circ f)^{-1}(\mathcal{W}) = f^{-1}\big(g^{-1}(\mathcal{W})\big) = f^{-1}(\mathcal{V} \cap B) = f^{-1}(\mathcal{V}) \cap f^{-1}(B) = \mathcal{U} \cap A \cap f^{-1}(B)\,,$$

while the fact that $f(A) \subseteq B$ further implies that

$$(g \circ f)^{-1}(\mathcal{W}) = \mathcal{U} \cap A\,.$$

Therefore, by Theorem 4.14 we find that $(g \circ f)$ is continuous on A. □

4.3 Images under Continuous Maps

4.3.1 Image of compact sets

Theorem 4.25. *Let (M, d) and (N, ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be a continuous map.*

1. *If $K \subseteq A$ is compact, then $f(K)$ is compact in (N, ρ).*

2. *Moreover, if $(N, \rho) = (\mathbb{R}, |\cdot|)$, then there exist $x_0, x_1 \in K$ such that*

$$f(x_0) = \inf f(K) = \inf \big\{ f(x) \,\big|\, x \in K \big\} \quad \textit{and} \quad f(x_1) = \sup f(K) = \sup \big\{ f(x) \,\big|\, x \in K \big\}\,.$$

Proof. 1. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $f(K)$. Then there exists $\{x_n\}_{n=1}^{\infty} \subseteq K$ such that $y_n = f(x_n)$. Since K is sequentially compact, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with limit $x \in K$. Let $y = f(x) \in f(K)$. By the continuity of f,

$$\lim_{k\to\infty} \rho(y_{n_k}, y) = \lim_{k\to\infty} \rho\big(f(x_{n_k}), f(x)\big) = 0$$

which implies that the sequence $\{y_{n_k}\}_{k=1}^{\infty}$ converges to $y \in f(K)$. Therefore, $f(K)$ is sequentially compact.

2. By 1, $f(K)$ is sequentially compact. Corollary 3.45 then implies that $\inf f(K) \in f(K)$ and $\sup f(K) \in f(K)$. □

Alternative Proof of Part 1. Let $\{V_\alpha\}_{\alpha \in I}$ be an open cover of $f(K)$. Since V_α is open, by Theorem 4.14 there exists U_α open in (M, d) such that $f^{-1}(V_\alpha) = U_\alpha \cap A$. Since $f(K) \subseteq \bigcup\limits_{\alpha \in I} V_\alpha$,

$$K \subseteq f^{-1}(f(K)) \subseteq \bigcup_{\alpha \in I} f^{-1}(V_\alpha) = A \cap \bigcup_{\alpha \in I} U_\alpha$$

which implies that $\{U_\alpha\}_{\alpha \in I}$ is an open cover of K. Therefore,

$$\exists\, J \subseteq I, \#J < \infty \ni K \subseteq A \cap \bigcup_{\alpha \in J} U_\alpha = \bigcup_{\alpha \in J} f^{-1}(V_\alpha)\,;$$

thus $f(K) \subseteq \bigcup\limits_{\alpha \in J} f(f^{-1}(V_\alpha)) \subseteq \bigcup\limits_{\alpha \in J} V_\alpha$. □

Corollary 4.26 (Extreme Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f attains its maximum and minimum in $[a, b]$; that is, there are $x_0 \in [a, b]$ and $x_1 \in [a, b]$ such that*

$$f(x_0) = \inf\big\{f(x)\,\big|\,x \in [a, b]\big\} \quad \textit{and} \quad f(x_1) = \sup\big\{f(x)\,\big|\,x \in [a, b]\big\}\,. \tag{4.3.1}$$

Proof. The Heine-Borel Theorem shows that $[a, b]$ is a compact set in $\mathbb{R}$; thus Theorem 4.25 implies that $f([a, b])$ must be compact in $\mathbb{R}$. By Corollary 3.45,

$$\inf f([a, b]) \in f([a, b]) \quad \text{and} \quad \sup f([a, b]) \in f([a, b])$$

that further imply (4.3.1). □

Remark 4.27. If f attains its maximum (or minimum) on a set B, we use $\max\big\{f(x)\,\big|\,x \in B\big\}$ $\big($or $\min\big\{f(x)\,\big|\,x \in B\big\}\big)$ to denote $\sup\big\{f(x)\,\big|\,x \in B\big\}$ $\big($or $\inf\big\{f(x)\,\big|\,x \in B\big\}\big)$. Therefore, (4.3.1) can be rewritten as

$$f(x_0) = \min\big\{f(x)\,\big|\,x \in [a, b]\big\} \quad \text{and} \quad f(x_1) = \max\big\{f(x)\,\big|\,x \in [a, b]\big\}\,.$$

Remark 4.28. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$. Then f is continuous. Note that $\{0\} \subseteq \mathbb{R}$ is compact, but $f^{-1}(\{0\}) = \mathbb{R}$ is not compact. In other words, the pre-image of a compact set under a continuous map might not be compact.

Example 4.29. Recall that two norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ on a vector space $\mathcal{V}$ are called equivalent if there are positive constants c and C such that

$$c\|x\| \leqslant |\!|\!|x|\!|\!| \leqslant C\|x\| \qquad \forall\, x \in \mathcal{V}\,.$$

We note that equivalent norms on a vector space $\mathcal{V}$ induce the same topology; that is, if $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are equivalent norms on $\mathcal{V}$, then U is open in the normed space $(\mathcal{V}, \|\cdot\|)$ if and only if U is open in the normed space $(\mathcal{V}, |\!|\!|\cdot|\!|\!|)$. In fact, let U be an open set in $(\mathcal{V}, \|\cdot\|)$. Then for any $x \in U$, there exists $r > 0$ such that

$$B_{\|\cdot\|}(x, r) \equiv \big\{y \in \mathcal{V} \,\big|\, \|x - y\| < r\big\} \subseteq U\,,$$

here we use the norm in the subscript to indicate that the distance in this ball is measured by this norm. As in the proof of Theorem 2.39, the ball $B_{|\!|\!|\cdot|\!|\!|}(x, cr) \subseteq B_{\|\cdot\|}(x, r)$. Therefore, U is open in $(\mathcal{V}, |\!|\!|\cdot|\!|\!|)$. Similarly, if U is open in $(\mathcal{V}, |\!|\!|\cdot|\!|\!|)$, then the inequality $|\!|\!|x|\!|\!| \leqslant C_2\|x\|$ implies that U is open in $(\mathcal{V}, \|\cdot\|)$.

In fact, for a vector space $\mathcal{V}$ with two equivalent norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$, we have

1. $\{x_k\}_{k=1}^{\infty}$ converges in $(\mathcal{V}, \|\cdot\|)$ if and only if $\{x_k\}_{k=1}^{\infty}$ converges in $(\mathcal{V}, |\!|\!|\cdot|\!|\!|)$.
2. F is (totally) bounded in $(\mathcal{V}, \|\cdot\|)$ if and only if F is a (totally) bounded subset in $(\mathcal{V}, |\!|\!|\cdot|\!|\!|)$.
3. F is closed in $(\mathcal{V}, \|\cdot\|)$ if and only if F is closed in $(\mathcal{V}, |\!|\!|\cdot|\!|\!|)$.
4. U is open in $(\mathcal{V}, \|\cdot\|)$ if and only if U is open in $(\mathcal{V}, |\!|\!|\cdot|\!|\!|)$.
5. K is compact in $(\mathcal{V}, \|\cdot\|)$ if and only if K is compact in $(\mathcal{V}, |\!|\!|\cdot|\!|\!|)$.

In the following, we prove the following

Claim: Any two norms on a **finite dimensional** vector space $\mathcal{V}$ over field $\mathbb{R}$ (or $\mathbb{C}$) are equivalent.

Proof of claim: Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be a basis of $\mathcal{V}$. Then each $\boldsymbol{x} \in \mathcal{V}$ can be uniquely expressed as $\boldsymbol{x} = \sum\limits_{i=1}^{n} x^{(i)}\mathbf{e}_i$ for some $x^{(i)} \in \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Define the norm

$$\|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x^{(i)}|^2}$$

as given in Example 2.28. It suffices to show that any norm $\|\cdot\|$ on $\mathcal{V}$ is equivalent to $\|\cdot\|_2$. In fact, if $C_1\|\boldsymbol{x}\| \leqslant \|\boldsymbol{x}\|_2 \leqslant C_2\|\boldsymbol{x}\|$ and $C_3|\!|\!|\boldsymbol{x}|\!|\!| \leqslant \|\boldsymbol{x}\|_2 \leqslant C_4|\!|\!|\boldsymbol{x}|\!|\!|$ for all $\boldsymbol{x} \in \mathcal{V}$, then

$$\frac{C_1}{C_4}\|\boldsymbol{x}\| \leqslant |\!|\!|\boldsymbol{x}|\!|\!| \leqslant \frac{C_2}{C_3}\|\boldsymbol{x}\| \qquad \forall\, \boldsymbol{x} \in \mathcal{V}\,.$$

Before proceeding, we first recall that a subset K is (sequentially) compact in $(\mathcal{V}, \|\cdot\|_2)$ if and only if K is closed and bounded (see Remark 3.43). By the triangle inequality and the Cauchy-Schwarz inequality,

$$\|\boldsymbol{x}\| \leqslant \sum_{i=1}^{n} |x^{(i)}|\|\mathbf{e}_i\| \leqslant \|\boldsymbol{x}\|_2 \sqrt{\sum_{i=1}^{n} \|\mathbf{e}_i\|^2}\,; \tag{4.3.2}$$

thus letting $C = \sqrt{\sum\limits_{i=1}^{n} \|\mathbf{e}_i\|^2}$ we have $\|\boldsymbol{x}\| \leqslant C\|\boldsymbol{x}\|_2$.

On the other hand, define $f : \mathcal{V} \to \mathbb{R}$ by

$$f(\boldsymbol{x}) = \|\boldsymbol{x}\| = \Big\|\sum_{i=1}^{n} x^{(i)}\mathbf{e}_i\Big\|\,.$$

Because of (4.3.2), f is continuous on $(\mathcal{V}, \|\cdot\|_2)$. In fact, for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$,

$$\big|f(\boldsymbol{x}) - f(\boldsymbol{y})\big| = \big|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|\big| \leqslant \|\boldsymbol{x} - \boldsymbol{y}\| \leqslant C\|\boldsymbol{x} - \boldsymbol{y}\|_2$$

which guarantees the continuity of f on $(\mathcal{V}, \|\cdot\|_2)$. Let $K = \big\{\boldsymbol{x} \in \mathcal{V} \,\big|\, \|\boldsymbol{x}\|_2 = 1\big\}$. Then K is sequentially compact in $(\mathcal{V}, \|\cdot\|_2)$ since K is closed and bounded in $(\mathcal{V}, \|\cdot\|_2)$; thus by Theorem 4.25 f attains its minimum on K at some point $\boldsymbol{a} \in K$. Moreover, $f(\boldsymbol{a}) > 0$ (since if $f(\boldsymbol{a}) = 0$, $\boldsymbol{a} = \mathbf{0} \notin K$). Then for all $\boldsymbol{x} \in \mathcal{V}\backslash\{\mathbf{0}\}$, $\dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2} \in K$; thus

$$f\Big(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}\Big) \geqslant f(\boldsymbol{a}) \qquad \forall\, \boldsymbol{x} \in \mathcal{V}\backslash\{\mathbf{0}\}\,.$$

The inequality above further implies that $f(\boldsymbol{a})\|\boldsymbol{x}\|_2 \leqslant f(\boldsymbol{x}) = \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathcal{V}$; thus letting $c = f(\boldsymbol{a})$ we have $c\|\boldsymbol{x}\|_2 \leqslant \|\boldsymbol{x}\|$.

Having established that every two norms on a finite dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$), for a finite dimensional normed vector space $(\mathcal{V}, \|\cdot\|)$ over $\mathbb{R}$ (or $\mathbb{C}$) we have the following results:

1. A subset K of $\mathcal{V}$ is compact if and only if K is closed and bounded (because of Remark 3.43).

2. Every bounded sequence in $(\mathcal{V}, \|\cdot\|)$ has a convergent subsequence (for a given bounded sequence, consider the bounded and closed ball $B[0,R]$ for $R \gg 1$ and make use of the sequentially compactness of $B[0,R]$).

3. $(\mathcal{V}, \|\cdot\|)$ is a Banach space; that is, $(\mathcal{V}, \|\cdot\|)$ is complete (every Cauchy sequence is bounded; thus possessing a convergent subsequence so that the convergence of the Cauchy sequence is guaranteed by Proposition 2.58).

Example 4.30. The determinant function $f : \mathcal{M}_{n\times n} \to \mathbb{R}$ defined by $f(A) = \det(A)$ is continuous on $(\mathcal{M}_{n\times n}, \|\cdot\|)$ for any norm $\|\cdot\|$ (thus the set $\mathrm{GL}(n)$ defined in Example 4.16 is open). To see this, we note that $\mathcal{M}_{n\times n}$ is finite dimensional vector space over $\mathbb{R}$; thus the norm $\|\cdot\|$ is equivalent to the norm

$$|||[a_{ij}]||| = \sum_{i,j=1}^{n} |a_{ij}|\,.$$

Clearly f is continuous on $(\mathcal{M}_{n\times n}, |||\cdot|||)$ since $f(A)$ is the sum of product of entries of A and $|||B-A||| \to 0$ if and only if $b_{ij} \to a_{ij}$ for all $1 \leqslant i,j \leqslant n$. Since $|||B-A||| \to 0$ if and only if $\|B-A\| \to 0$, we conclude that f is continuous on $(\mathcal{M}_{n\times n}, \|\cdot\|)$.

Corollary 4.31. *Let (M,d) be a metric space, K be a compact subset of M, and $f : K \to \mathbb{R}$ be continuous. Then the set*

$$\big\{x \in K \,\big|\, f(x) \text{ is the maximum of } f \text{ on } K\big\}$$

is a non-empty compact set.

Proof. Note that $f(K)$ is compact in $(\mathbb{R}, |\cdot|)$; hence $f(K)$ is closed and bounded so that $\mathrm{M} = \sup f(K)$ exists and $\mathrm{M} \in f(K)$. Then the set defined above is $f^{-1}(\{\mathrm{M}\})$. Moreover,

1. $f^{-1}(\{\mathrm{M}\})$ is non-empty by Theorem 4.25;

2. $f^{-1}(\{\mathrm{M}\})$ is a subset of K; thus By Proposition 3.51 implies that $f^{-1}(\{\mathrm{M}\})$ is totally bounded;

3. By Theorem 4.14, $f^{-1}(\{\mathrm{M}\})$ is closed since $\{\mathrm{M}\}$ is a closed set in $(\mathbb{R}, |\cdot|)$; thus Theorem 3.27 implies that $f^{-1}(\{\mathrm{M}\})$ is complete.

Therefore, Theorem 3.53 shows that $f^{-1}(\{\mathrm{M}\})$ is compact. □

4.3.2 Image of connected sets

Theorem 4.32. *Let (M,d) and (N,ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be a continuous map. If $C \subseteq A$ is connected, then $f(C)$ is connected in (N,ρ).*

Proof. Suppose that there are two open sets V_1 and V_2 in (N,ρ) such that

(a) $f(C) \cap V_1 \cap V_2 = \varnothing$; (b) $f(C) \cap V_1 \neq \varnothing$; (c) $f(C) \cap V_2 \neq \varnothing$; (d) $f(C) \subseteq V_1 \cup V_2$.

By Theorem 4.14, there are U_1 and U_2 open in (M,d) such that $f^{-1}(V_1) = U_1 \cap A$ and $f^{-1}(V_2) = U_2 \cap A$. By (d),

$$C \subseteq f^{-1}(f(C)) \subseteq f^{-1}(V_1) \cup f^{-1}(V_2) = (U_1 \cup U_2) \cap A \subseteq U_1 \cup U_2 \,.$$

Moreover, by (a) we find that

$$\begin{aligned} C \cap U_1 \cap U_2 &= C \cap (U_1 \cap A) \cap (U_2 \cap A) = C \cap f^{-1}(V_1) \cap f^{-1}(V_2) \\ &\subseteq f^{-1}(f(C) \cap V_1 \cap V_2) = \varnothing \end{aligned}$$

which implies $C \cap U_1 \cap U_2 = \varnothing$. Finally, (b) implies that for some $x \in C$, $f(x) \in V_1$. Therefore, $x \in f^{-1}(V_1) = U_1 \cap A$ which shows that $x \in U_1$; thus $C \cap U_1 \neq \varnothing$. Similarly, $C \cap U_2 \neq \varnothing$. Therefore, C is disconnected which is a contradiction. □

Corollary 4.33 (Intermediate Value Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a) \neq f(b)$, then for all d in between $f(a)$ and $f(b)$, there exists $c \in (a,b)$ such that $f(c) = d$.*

Proof. The closed interval $[a,b]$ is connected by Theorem 3.68, so Theorem 4.32 implies that $f([a,b])$ must be connected in $\mathbb{R}$. By Theorem 3.68 again, if d is in between $f(a)$ and $f(b)$, then d belongs to $f([a,b])$. Therefore, for some $c \in (a,b)$ we have $f(c) = d$. □

Remark 4.34. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then f is continuous. Note that $C = \{1\}$ is connected, but $f^{-1}(C) = \{1,-1\}$ is not connected. In other words, the pre-image of a connected set under a continuous map might not be connected.

Example 4.35. Let $f : [0,1] \to [0,1]$ be continuous. Then there exists $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

Proof. Let $g(x) = x - f(x)$.

Case 1: $g(0) = 0$ or $g(1) = 0$. Then $x_0 = 0$ or $x_0 = 1$ satisfies $f(x_0) = x_0$.

Case 2: $g(0) \neq 0$ and $g(1) \neq 0$. Then $g(0) < 0$ and $g(1) > 0$; thus by the continuity of $g : [0,1] \to \mathbb{R}$, there exists $x_0 \in [0,1]$ such that $g(x_0) = 0$ which implies the existence of $x_0 \in (0,1)$ satisfying $f(x_0) = x_0$. □

Remark 4.36. Such an x_0 in Example 4.35 is called a ***fixed-point*** of f.

4.4 Uniform Continuity（均勻連續）

Definition 4.37. Let (M,d) and (N,ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be a map. For a set $B \subseteq A$, f is said to be ***uniformly continuous on*** B if for any two sequences $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq B$ with the property that $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$, one has $\lim\limits_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) = 0$. In logic notation, f is uniformly continuous on B if

$$\big(\forall \{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq B\big)\Big(\lim_{n\to\infty} d(x_n,y_n) = 0 \Rightarrow \lim_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) = 0\Big).$$

Proposition 4.38. *Let (M,d) and (N,ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be a map. If f is uniformly continuous on A, then f is continuous on A.*

Proof. Let $a \in A \cap A'$, and $\{x_k\}_{k=1}^\infty \subseteq A$ be a sequence such that $x_k \to a$ as $k \to \infty$. Let $\{y_k\}_{k=1}^\infty$ be a constant sequence with value a; that is, $y_k = a$ for all $k \in \mathbb{N}$. Then $\{y_k\}_{k=1}^\infty \subseteq A$ and $d(x_k,y_k) \to 0$ as $k \to \infty$. By the uniform continuity of f on A,

$$\lim_{k\to\infty} \rho\big(f(x_k), f(a)\big) = \lim_{k\to\infty} \rho\big(f(x_k), f(y_k)\big) = 0$$

which implies that f is continuous on a. □

Example 4.39. Let $f : [0,1] \to \mathbb{R}$ be the Dirichlet function; that is,

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{Q}^\complement. \end{cases}$$

and $B = \mathbb{Q} \cap [0,1]$. Then f is continuous **nowhere** in $[0,1]$, but f is uniformly continuous on B. However, the proposition above guarantees that if f is uniformly continuous on $[0,1]$, then f must be continuous on $[0,1]$ (Check why the proof of Proposition 4.38 does not go through if B is a proper subset of $[0,1]$).

Example 4.40. The function $f(x) = |x|$ is uniformly continuous on $\mathbb{R}$. In fact, by the triangle inequality,

$$\big|f(x) - f(y)\big| = \big||x| - |y|\big| \leqslant |x - y|\,;$$

thus if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences in $\mathbb{R}$ and $\lim\limits_{n\to\infty} |x_n - y_n| = 0$, by the Sandwich lemma we must have $\lim\limits_{n\to\infty} \big|f(x_n) - f(y_n)\big| = 0$.

Example 4.41. The function $f : (0,\infty) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[a,\infty)$ for all $a > 0$. To see this, let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in $[a,\infty)$ such that $\lim\limits_{n\to\infty} |x_n - y_n| = 0$. Then

$$|f(x_n) - f(y_n)| = \Big|\frac{1}{x_n} - \frac{1}{y_n}\Big| = \frac{|x_n - y_n|}{|x_n y_n|} \leqslant \frac{|x_n - y_n|}{a^2} \to 0 \quad \text{as} \quad n \to \infty$$

which implies that f is uniformly continuous on $[a,\infty)$ if $a > 0$.

However, f is not uniformly continuous on $(0,\infty)$. Let $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{1}{2n}$. Then

$$|x_n - y_n| = \frac{1}{2n} \to 0 \;\text{ as }\; n \to \infty \quad \text{but} \quad |f(x_n) - f(y_n)| = n \geqslant 1\,.$$

Example 4.42. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then f is continuous in $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$. Let $x_n = n$ and $y_n = n + \dfrac{1}{2n}$. Then

$$\big|f(x_n) - f(y_n)\big| = \Big|n^2 - (n + \frac{1}{2n})^2\Big| = \Big|n^2 - n^2 - 1 - \frac{1}{4n^2}\Big| = 1 + \frac{1}{4n^2} \not\to 0 \;\text{ as }\; n \to \infty\,.$$

Example 4.43. The function $f(x) = \sin(x^2)$ is not uniform continuous on $\mathbb{R}$. Define $x_n = 2n\sqrt{\pi} + \dfrac{\sqrt{\pi}}{8n}$ and $y_n = 2n\sqrt{\pi} - \dfrac{\sqrt{\pi}}{8n}$. Then $\lim\limits_{n\to\infty} |x_n - y_n| = 0$ while

$$\big|\sin(x_n^2) - \sin(y_n^2)\big| = \Big|\sin\big(4n^2\pi + \frac{\pi}{2} + \frac{\pi}{64n^2}\big) - \sin\big(4n^2\pi - \frac{\pi}{2} + \frac{\pi}{64n^2}\big)\Big| = 2\cos\frac{\pi}{64n^2}\,;$$

thus $\lim\limits_{n\to\infty} \big|\sin(x_n^2) - \sin(y_n^2)\big| = 1 \neq 0$.

Example 4.44. The function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \sin\dfrac{1}{x}$ is not uniformly continuous on $(0,1)$.

Let $x_n = \big(2n\pi + \frac{\pi}{2}\big)^{-1}$ and $y_n = \big(2n\pi - \frac{\pi}{2}\big)^{-1}$. Then

$$\Big|\sin\frac{1}{x_n} - \sin\frac{1}{y_n}\Big| = 2\,,$$

while $|x_n - y_n| = \dfrac{\pi}{4n^2\pi^2 - \frac{\pi^2}{4}} = \dfrac{1}{(4n^2 - \frac{1}{4})\pi} \to 0$ as $n \to \infty$.

Theorem 4.45. *Let (M,d) and (N,ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be a map. For a set $B \subseteq A$, f is uniformly continuous on B if and only if*

$$\forall\,\varepsilon > 0, \exists\,\delta > 0 \ni \rho\big(f(x), f(y)\big) < \varepsilon \quad \textit{whenever} \quad d(x,y) < \delta \textit{ and } x, y \in B\,.$$

Proof. "$\Leftarrow$" Let $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ be sequences in B such that $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$, and $\varepsilon > 0$ be given. By assumption, there exists $\delta > 0$ such that

$$\rho\big(f(x), f(y)\big) < \varepsilon \quad \text{whenever} \quad d(x,y) < \delta \text{ and } x, y \in B\,.$$

Since $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$, there exists $N > 0$ such that

$$d(x_n, y_n) < \delta \quad \text{whenever} \quad n \geqslant N\,;$$

thus

$$\rho\big(f(x_n), f(y_n)\big) < \varepsilon \quad \text{whenever} \quad n \geqslant N\,.$$

"$\Rightarrow$" Suppose the contrary that there exists $\varepsilon > 0$ such that for all $\delta = \dfrac{1}{n} > 0$, there exist two points x_n and $y_n \in B$ such that

$$d(x_n, y_n) < \frac{1}{n} \quad \text{but} \quad \rho\big(f(x_n), f(y_n)\big) \geqslant \varepsilon\,.$$

These points form two sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ in B such that $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$, while the limit of $\rho\big(f(x_n), f(y_n)\big)$, if exists, does not converges to zero as $n \to \infty$. As a consequence, f is not uniformly continuous on B, a contradiction. □

Remark 4.46. The theorem above provides another way (the blue color part) of defining the uniform continuity of a function over a subset of its domain. Moreover, according to this alternative definition, $f : A \to N$ is uniformly continuous on $B \subseteq A$ if

$$(\forall\,\varepsilon > 0)(\exists\,\delta > 0)\Big(\operatorname{diam}\big(f\big(B_M\big(b, \frac{\delta}{2}\big) \cap B\big)\big) < \varepsilon\Big)\,;$$

that is, the diameter of the image, under f, of subsets of B whose diameter is not greater than δ is not greater than ε（在 B 中直徑不超過 δ 的子集合被函數 f 映過去之後，在對應域中的直徑不會超過 ε）. The statement above is the same as

$$(\forall\,\varepsilon > 0)(\exists\,\delta > 0)(\forall\,b \in M)(\exists\,c \in M)\Big(f\big(B_M\big(b, \frac{\delta}{2}\big) \cap B\big) \subseteq B_N\big(c, \frac{\varepsilon}{2}\big)\Big)\,.$$

Remark 4.47. For a given function f, let $\delta(f, c, \varepsilon)$ denote the supremum of all $\delta(c, \varepsilon)$ mentioned in Remark 4.6. Then the uniform continuity of a function $f : A \to N$ is equivalent to that

$$\delta_f(\varepsilon) \equiv \inf_{c \in A} \delta(f, c, \varepsilon) > 0 \qquad \forall\, \varepsilon > 0\,.$$

Remark 4.48. Let (M, d) and (N, ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be a map. For a set $B \subseteq A$, the following four statements are equivalent:

(1) f is **not** uniformly continuous on B.

(2) $\exists\, \{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq B \ni \lim\limits_{n\to\infty} d(x_n, y_n) = 0$ and $\limsup\limits_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0$.

(3) $\exists\, \{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq B \ni \lim\limits_{n\to\infty} d(x_n, y_n) = 0$ and $\lim\limits_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0$.

(4) $\exists\, \varepsilon > 0 \ni \forall\, n > 0, \exists\, x_n, y_n \in B$ and $d(x_n, y_n) < \dfrac{1}{n} \ni \rho\big(f(x_n), f(y_n)\big) \geqslant \varepsilon$.

Theorem 4.49. *Let (M, d) and (N, ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be a map. If K is a compact subset of A and f is continuous on K, then f is uniformly continuous on K.*

Proof. Assume the contrary that f is not uniformly continuous on K. Then ((3) of Remark 4.48 implies that) there are sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in K such that

$$\lim_{n\to\infty} d(x_n, y_n) = 0 \quad \text{but} \quad \lim_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0\,.$$

Since K is (sequentially) compact, there exist convergent subsequences $\{x_{n_k}\}_{k=1}^\infty$ and $\{y_{n_k}\}_{k=1}^\infty$ with limits $x, y \in K$. On the other hand, $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$, we must have $x = y$; thus by the continuity of f (on K),

$$0 = \rho\big(f(x), f(x)\big) = \lim_{k\to\infty} \rho\big(f(x_{n_k}), f(y_{n_k})\big) = \lim_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0\,,$$

a contradiction. ◻

Alternative proof. Let $\varepsilon > 0$ be given. Since f is continuous on K,

$$\forall\, a \in K, \exists\, \delta = \delta(a) > 0 \ni \rho\big(f(x), f(a)\big) < \frac{\varepsilon}{2} \text{ whenever } x \in B(a, \delta) \cap A\,.$$

Then $\Big\{B\big(a, \dfrac{\delta(a)}{2}\big)\Big\}_{a\in K}$ is an open cover of K; thus the compactness of K implies that

$$\exists\, \{a_1, \cdots, a_N\} \subseteq K \ni K \subseteq \bigcup_{i=1}^N B\big(a_i, \frac{\delta_i}{2}\big)\,,$$

where $\delta_i = \delta(a_i)$. Let $\delta = \dfrac{1}{2}\min\{\delta_1, \cdots, \delta_N\}$. Then $\delta > 0$, and if $x_1, x_2 \in K$ and $d(x_1, x_2) < \delta$, there must be $j = 1, \cdots, N$ such that $x_1, x_2 \in B(a_j, \delta_j)$. In fact, since $x_1 \in B\big(a_j, \dfrac{\delta_j}{2}\big)$ for some $j = 1, \cdots, N$, then

$$d(x_2, a_j) \leqslant d(x_1, x_2) + d(x_1, a_j) < \delta + \frac{\delta_j}{2} \leqslant \delta_j\,.$$

Therefore, $x_1, x_2 \in B(a_j, \delta_j) \cap A$ for some $j = 1, \cdots, N$; thus

$$\rho\big(f(x_1), f(x_2)\big) \leqslant \rho\big(f(x_1), f(a_j)\big) + \rho\big(f(x_2), f(a_j)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$ □

Lemma 4.50. *Let (M, d) and (N, ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be uniformly continuous. If $\{x_k\}_{k=1}^\infty \subseteq A$ is a Cauchy sequence, so is $\big\{f(x_k)\big\}_{k=1}^\infty$.*

Proof. Let $\{x_k\}_{k=1}^\infty$ be a Cauchy sequence in (M, d), and $\varepsilon > 0$ be given. Since $f : A \to N$ is uniformly continuous,

$$\exists\, \delta > 0 \ni \rho\big(f(x), f(y)\big) < \varepsilon \quad \text{whenever} \quad d(x, y) < \delta \text{ and } x, y \in A\,.$$

For this particular δ, there exists $N > 0$ such that $d(x_k, x_\ell) < \delta$ whenever $k, \ell \geqslant N$. Therefore,

$$\rho(f(x_k), f(x_\ell)\big) < \varepsilon \quad \text{whenever} \quad k, \ell \geqslant N\,.$$ □

Corollary 4.51. *Let (M, d) and (N, ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be uniformly continuous. If N is complete, then f has a unique extension to a continuous function on $\bar{A}$; that is, there exists $g : \bar{A} \to N$ such that*

(1) *g is uniformly continuous on $\bar{A}$;*

(2) *$g(x) = f(x)$ for all $x \in A$;*

(3) *if $h : \bar{A} \to N$ is a continuous map satisfying $h(x) = f(x)$ for all $x \in A$, then $h = g$.*

Proof. Let $x \in \bar{A}\backslash A$. Then there exists $\{x_k\}_{k=1}^\infty \subseteq A$ such that $x_k \to x$ as $k \to \infty$. Since $\{x_k\}_{k=1}^\infty$ is Cauchy, by Lemma 4.50 $\big\{f(x_k)\big\}_{k=1}^\infty$ is a Cauchy sequence in (N, ρ); thus is convergent. Moreover, if $\{z_k\}_{k=1}^\infty \subseteq A$ is another sequence converging to x, we must have $d(x_k, z_k) \to 0$ as $k \to \infty$; thus $\rho(f(x_k), f(z_k)) \to 0$ as $k \to \infty$, so the limit of $\big\{f(x_k)\big\}_{k=1}^\infty$ and $\big\{f(z_k)\big\}_{k=1}^\infty$ must be the same.

Define $g : \bar{A} \to N$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A\,, \\ \lim\limits_{k\to\infty} f(x_k) & \text{if } x \in \bar{A}\backslash A, \text{ and } \{x_k\}_{k=1}^\infty \subseteq A \text{ converging to } x \text{ as } k \to \infty\,. \end{cases}$$

Then the argument above shows that g is well-defined, and (2) holds.

Let $\varepsilon > 0$ be given. Since $f : A \to N$ is uniformly continuous,

$$\exists\, \delta > 0 \ni \rho\big(f(x), f(y)\big) < \frac{\varepsilon}{3} \text{ whenever } d(x,y) < 2\delta \text{ and } x, y \in A\,.$$

Suppose that $x, y \in \bar{A}$ such that $d(x,y) < \delta$. Let $\{x_k\}_{k=1}^{\infty}, \{y_k\}_{k=1}^{\infty} \subseteq A$ be sequences converging to x and y, respectively. Then there exists $N > 0$ such that

$$d(x_k, x) < \frac{\delta}{2}\,, d(y_k, y) < \frac{\delta}{2} \text{ and } \rho\big(f(x_k), g(x)\big) < \frac{\varepsilon}{3}\,, \rho\big(f(y_k), g(y)\big) < \frac{\varepsilon}{3} \quad \forall\, k \geqslant N\,.$$

In particular, due to the triangle inequality,

$$d(x_N, y_N) \leqslant d(x_N, x) + d(x,y) + d(y, y_N) < \frac{\delta}{2} + \delta + \frac{\delta}{2} = 2\delta\,;$$

thus $\rho\big(f(x_N), f(y_N)\big) < \frac{\varepsilon}{3}$. As a consequence,

$$\rho\big(g(x), g(y)\big) \leqslant \rho\big(g(x), f(x_N)\big) + \rho\big(f(x_N), f(y_N)\big) + \rho\big(f(y_N), f(y)\big) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

which establishes (1).

Finally, suppose that $h : \bar{A} \to N$ is a continuous map satisfying $h = f$ on A, and $a \in A$. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in A with limit a. By Proposition 4.38, g is continuous on $\bar{A}$; thus Proposition 4.9 implies that

$$g(a) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} h(x_k) = h(a)\,,$$

so (3) is also concluded. □

Chapter 5

Differentiation of Maps

5.1 Bounded Linear Maps

Definition 5.1. Let X, Y be vector spaces over a common scalar field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A map L from X to Y is said to be ***linear*** if $L(c\boldsymbol{x}_1 + \boldsymbol{x}_2) = cL(\boldsymbol{x}_1) + L(\boldsymbol{x}_2)$ for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ and $c \in \mathbb{F}$. We often write Lx instead of $L(x)$, and the collection of all linear maps from X to Y is denoted by $\mathscr{L}(X, Y)$.

Suppose further that X and Y are normed spaces equipped with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. A linear map $L : X \to Y$ is said to be bounded if

$$\sup_{\|\boldsymbol{x}\|_X = 1} \|L\boldsymbol{x}\|_Y < \infty .$$

The collection of all bounded linear maps from X to Y is denoted by $\mathscr{B}(X, Y)$, and the number $\sup\limits_{\|\boldsymbol{x}\|_X = 1} \|L\boldsymbol{x}\|_Y$ is often denoted by $\|L\|_{\mathscr{B}(X,Y)}$.

Example 5.2. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be given by $L\boldsymbol{x} = A\boldsymbol{x}$, where A is an $m \times n$ matrix. Then Example 2.19 shows that $\|L\|_{\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)}$ is the square root of the largest eigenvalue of $A^{\mathrm{T}}A$ which is certainly a finite number. Therefore, any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is bounded.

Example 5.3. Recall that $X = \mathscr{C}([a, b]; \mathbb{R})$ and $\|\cdot\|_X = \|\cdot\|_2$ given in Example 2.21 which makes a normed vector space $(X, \|\cdot\|_X)$. Let $\phi \in X$ be given. Define $F : X \to \mathbb{R}$ by

$$F(f) = \int_a^b f(x)\phi(x)\, dx .$$

Then clearly $F \in \mathscr{L}(X, \mathbb{R})$ (the proof is left as an exercise). Moreover, the Cauchy-Schwarz inequality implies that

$$\big|F(f)\big| \leqslant \|f\|_2 \|\phi\|_2 \,;$$

thus

$$\sup_{\|f\|_\infty = 1} F(f) \leqslant \|\phi\|_2 < \infty \,.$$

Therefore, $F \in \mathscr{B}(X, \mathbb{R})$.

Example 5.4. Let $(X, \|\cdot\|_X) = (Y, \|\cdot\|_Y) = (\mathbb{C}, |\cdot|)$, and we consider the bounded linear maps $\mathscr{B}(X, Y)$.

1. Treat $\mathbb{C}$ as a vector space over $\mathbb{C}$. Then $\{1\}$ is a basis of $\mathbb{C}$ and a linear map $L \in \mathscr{L}(\mathbb{C}, \mathbb{C})$ is determined by $L1$ and we have $Lz = zL1$ (thus any linear map from $\mathbb{C}$ to $\mathbb{C}$ is a multiple of a complex number). Moreover,

   $$\sup_{|z|=1} |Lz| = |L1| < \infty$$

   which shows that $L \in \mathscr{B}(\mathbb{C}, \mathbb{C})$.

2. Treat $\mathbb{C}$ as a vector space over $\mathbb{R}$. Then $\{1, i\}$ is a basis of $\mathbb{C}$ and a linear map $L \in \mathscr{L}(\mathbb{C}, \mathbb{C})$ is determined by $L1$ and Li. In fact, if $L1 = a + bi$ and $Li = c + di$ for some $a, b, c, d \in \mathbb{R}$, then for $x, y \in \mathbb{R}$,

   $$L(x + yi) = xL1 + yLi = x(a + bi) + y(c + di) = (ax + cy) + (bx + dy)i \,.$$

   Treating $x + yi$ as a vector $(x, y) \in \mathbb{R}^2$, the map L maps $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Since

   $$\sup_{|x+yi|=1} |L(x + yi)| = \sup_{|x+yi|=1} \big|(ax + cy) + (bx + dy)i\big| = \sup_{\|(x,y)\|_2 = 1} \Big\| \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Big\|_2 \,,$$

   the norm of L is the same as the 2-norm of the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Therefore,

   $$\|L\|_{\mathscr{B}(\mathbb{C},\mathbb{C})} = \text{ the square root of the largest eigenvalue of } \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} < \infty \,;$$

   thus $L \in \mathscr{B}(\mathbb{C}, \mathbb{C})$. Since $\mathbb{C}$ over $\mathbb{R}$ is identical to $\mathbb{R}^2$ over $\mathbb{R}$ (from the discussion above), we always treat $\mathbb{C}$ as a vector space over $\mathbb{C}$.

Remark 5.5. By treating $\mathbb{C}$ as a vector space over $\mathbb{C}$ (which, we emphasize again, will always be the case), there is only one linear map from $\mathbb{C}$ to $\mathbb{R}$, the trivial linear map (which sends any vectors to the zero vector). This result is left as an exercise.

Proposition 5.6. *Let* $(X,\|\cdot\|_X)$ *and* $(Y,\|\cdot\|_Y)$ *be normed spaces over a common scalar field* $\mathbb{F}$, *where* $\mathbb{F}=\mathbb{R}$ *or* $\mathbb{C}$, *and* $L\in\mathscr{B}(X,Y)$. *Then*

$$\|L\|_{\mathscr{B}(X,Y)}=\sup_{x\neq 0}\frac{\|L\boldsymbol{x}\|_Y}{\|\boldsymbol{x}\|_X}=\inf\big\{M>0\,\big|\,\|L\boldsymbol{x}\|_Y\leqslant M\|\boldsymbol{x}\|_X\big\}\,.$$

In particular, the first equality implies that

$$\|L\boldsymbol{x}\|_Y\leqslant\|L\|_{\mathscr{B}(X,Y)}\|\boldsymbol{x}\|_X\qquad\forall\,\boldsymbol{x}\in X\,.$$

Proposition 5.7. *Let* $(X,\|\cdot\|_X)$ *and* $(Y,\|\cdot\|_Y)$ *be normed spaces over a common scalar field* $\mathbb{F}$, *where* $\mathbb{F}=\mathbb{R}$ *or* $\mathbb{C}$, *and* $L\in\mathscr{L}(X,Y)$. *Then* L *is continuous on* X *if and only if* $L\in\mathscr{B}(X,Y)$.

Proof. "⇒" Since L is continuous at $0\in X$, there exists $\delta>0$ such that

$$\|L\boldsymbol{x}\|_Y=\|L\boldsymbol{x}-L\mathbf{0}\|_Y<1\qquad\text{whenever}\quad\|\boldsymbol{x}\|_X<\delta\,.$$

Then $\big\|L\big(\frac{\delta}{2}\boldsymbol{x}\big)\big\|_Y\leqslant 1$ whenever $\big\|\frac{\delta}{2}\boldsymbol{x}\big\|_X<\delta$; thus by the linearity of L and properties of norms,

$$\|L\boldsymbol{x}\|_Y\leqslant\frac{2}{\delta}\qquad\text{whenever}\quad\|\boldsymbol{x}\|_X<2\,.$$

Therefore, $\sup\limits_{\|\boldsymbol{x}\|_X=1}\|L\boldsymbol{x}\|_Y\leqslant\frac{2}{\delta}$ which implies that $L\in\mathscr{B}(X,Y)$.

"⇐" If $L\in\mathscr{B}(X,Y)$, then $M=\|L\|_{\mathscr{B}(X,Y)}<\infty$, and

$$\|L\boldsymbol{x}_1-L\boldsymbol{x}_2\|_Y=\|L(\boldsymbol{x}_1-\boldsymbol{x}_2)\|_Y\leqslant M\|\boldsymbol{x}_1-\boldsymbol{x}_2\|_X$$

which shows that L is uniformly continuous on X. □

Proposition 5.8. *Let* $(X,\|\cdot\|_X)$ *and* $(Y,\|\cdot\|_Y)$ *be normed spaces over a common scalar field* $\mathbb{F}$, *where* $\mathbb{F}=\mathbb{R}$ *or* $\mathbb{C}$. *Then* $\big(\mathscr{B}(X,Y),\|\cdot\|_{\mathscr{B}(X,Y)}\big)$ *is a normed space. Moreover, if* $(Y,\|\cdot\|_Y)$ *is a Banach space, so is* $\big(\mathscr{B}(X,Y),\|\cdot\|_{\mathscr{B}(X,Y)}\big)$.

Proof. That $\big(\mathscr{B}(X,Y),\|\cdot\|_{\mathscr{B}(X,Y)}\big)$ is a normed space is left as an exercise. Now suppose that $\big(Y,\|\cdot\|_Y\big)$ is a Banach space. Let $\{L_k\}_{k=1}^{\infty}\subseteq\mathscr{B}(X,Y)$ be a Cauchy sequence. Then by Proposition 5.6, for each $\boldsymbol{x}\in X$ we have

$$\|L_k\boldsymbol{x}-L_\ell\boldsymbol{x}\|_Y=\|(L_k-L_\ell)\boldsymbol{x}\|_Y\leqslant\|L_k-L_\ell\|_{\mathscr{B}(X,Y)}\|\boldsymbol{x}\|_X\to 0\quad\text{as}\quad k,\ell\to\infty\,.$$

Therefore, for each $\boldsymbol{x} \in X$ the sequence $\{L_k\boldsymbol{x}\}_{k=1}^{\infty}$ is Cauchy in Y; thus convergent. Suppose that $\lim\limits_{k\to\infty} L_k\boldsymbol{x} = \boldsymbol{y}$. We then establish a map $\boldsymbol{x} \mapsto \boldsymbol{y}$ which we denoted by L; that is, $L\boldsymbol{x} = \boldsymbol{y}$. Then L is linear since if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ and $c \in \mathbb{F}$,

$$L(c\boldsymbol{x}_1 + \boldsymbol{x}_2) = \lim_{k\to\infty} L_k(c\boldsymbol{x}_1 + \boldsymbol{x}_2) = \lim_{k\to\infty} \big(cL_k\boldsymbol{x}_1 + L_k\boldsymbol{x}_2\big) = cL\boldsymbol{x}_1 + L\boldsymbol{x}_2 \,.$$

Moreover, since $\{L_k\}_{k=1}^{\infty}$ is a Cauchy sequence, by Proposition 2.58 there exists $M > 0$ such that $\|L_k\|_{\mathscr{B}(X,Y)} \leqslant M$ for all $k \in \mathbb{N}$. For each $\boldsymbol{x} \in X$ there exists $N = N_x > 0$ such that

$$\|L_k\boldsymbol{x} - L\boldsymbol{x}\|_Y < 1 \quad \text{ whenever } \quad k \geqslant N_x \,.$$

Therefore, for $k \geqslant N_x$,

$$\|L\boldsymbol{x}\|_Y < \|L_k\boldsymbol{x}\|_Y + 1 \leqslant \|L_k\|_{\mathscr{B}(X,Y)}\|\boldsymbol{x}\|_X + 1 \leqslant M\|\boldsymbol{x}\|_X + 1$$

which implies that $\sup\limits_{\|\boldsymbol{x}\|_X=1} \|L\boldsymbol{x}\|_Y \leqslant M + 1$; thus $L \in \mathscr{B}(X,Y)$.

Finally, we show that $\lim\limits_{k\to\infty} \|L_k - L\|_{\mathscr{B}(X,Y)} = 0$. Let $\varepsilon > 0$ be given. Since $\{L_k\}_{k=1}^{\infty}$ is a Cauchy sequence, there exists $N > 0$ such that $\|L_k - L_\ell\|_{\mathscr{B}(X,Y)} < \dfrac{\varepsilon}{2}$ whenever $k, \ell \geqslant N$. Then if $k \geqslant N$, for every $\boldsymbol{x} \in X$ we have

$$\|L_k\boldsymbol{x} - L\boldsymbol{x}\|_Y = \lim_{\ell\to\infty} \|L_k\boldsymbol{x} - L_\ell\boldsymbol{x}\|_Y \leqslant \limsup_{\ell\to\infty} \|L_k - L_\ell\|_{\mathscr{B}(X,Y)}\|\boldsymbol{x}\|_X \leqslant \frac{\varepsilon}{2}\|\boldsymbol{x}\|_X \,;$$

thus $\|L_k - L\|_{\mathscr{B}(X,Y)} < \varepsilon$ whenever $k \geqslant N$. □

Proposition 5.9. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be normed spaces over a common scalar field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $L \in \mathscr{B}(X,Y)$, $K \in \mathscr{B}(Y,Z)$. Then $K \circ L \in \mathscr{B}(X,Z)$, and*

$$\|K \circ L\|_{\mathscr{B}(X,Z)} \leqslant \|K\|_{\mathscr{B}(Y,Z)}\|L\|_{\mathscr{B}(X,Y)} \,.$$

We often write $K \circ L$ as KL if K and L are linear.

Proof. By the properties of the norm of a bounded linear map,

$$\|K \circ L(x)\|_Z = \|K(Lx)\|_Z \leqslant \|K\|_{\mathscr{B}(Y,Z)}\|Lx\|_Y \leqslant \|K\|_{\mathscr{B}(Y,Z)}\|L\|_{\mathscr{B}(X,Y)}\|x\|_X \,.$$ □

From now on, when the domain X and the target Y of a linear map L is clear, we use $\|L\|$ instead of $\|L\|_{\mathscr{B}(X,Y)}$ to simplify the notation.

Theorem 5.10. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces over a common scalar field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and X be finite dimensional. Then every linear map from X to Y is bounded; that is, $\mathscr{L}(X,Y) = \mathscr{B}(X,Y)$.*

Proof. Let $\{\mathbf{e}_k\}_{k=1}^n$ be a basis of X (so that $\dim(X) = n$). Then every $x \in X$ can be expressed as a unique linear combination of $\mathbf{e}_k$'s; that is, for all $\boldsymbol{x} \in X$, there exist unique n numbers $c_1 = c_1(\boldsymbol{x}), \cdots, c_n = c_n(\boldsymbol{x}) \in \mathbb{F}$ such that

$$\boldsymbol{x} = c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n .$$

Define an inner product $\langle \cdot, \cdot \rangle$ on X by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{k=1}^{n} c_k(\boldsymbol{x})\overline{c_k(\boldsymbol{y})}$$

and let $\|\cdot\|_2$ be the norm induced by this inner product; that is, $\|\boldsymbol{x}\|_2 = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. That $\langle \cdot, \cdot \rangle$ is indeed an inner product on X is left as an exercise.

Having define $\langle \cdot, \cdot \rangle$, these coefficients c_k's in fact are determined by $c_k(\boldsymbol{x}) = \langle \boldsymbol{x}, \mathbf{e}_k \rangle$, and, by Example 4.29 and the Cauchy-Schwarz inequality, satisfy

$$\big|c_k(\boldsymbol{x})\big| \leqslant \|\boldsymbol{x}\|_2 \|\mathbf{e}_k\|_2 \leqslant C\|\boldsymbol{x}\|_X \qquad \forall\, 1 \leqslant k \leqslant n$$

for some constant $C > 0$. As a consequence, if L is a linear map from X to Y, then

$$\begin{aligned} \|L\boldsymbol{x}\|_Y = \big\|L(c_1(\boldsymbol{x})\mathbf{e}_1 + \cdots + c_n(\boldsymbol{x})\mathbf{e}_n)\big\|_Y &\leqslant |c_1(\boldsymbol{x})|\|L\mathbf{e}_1\|_Y + \cdots |c_n(\boldsymbol{x})|\|L\mathbf{e}_n\|_Y \\ &\leqslant C\big(\|L\mathbf{e}_1\|_Y + \|L\mathbf{e}_2\|_Y + \cdots + \|L\mathbf{e}_n\|_Y\big)\|\boldsymbol{x}\|_X \leqslant M\|\boldsymbol{x}\|_X \end{aligned}$$

for some constant $M > 0$; thus $\|L\|_{\mathscr{B}(X,Y)} \leqslant M < \infty$ which shows that $L \in \mathscr{B}(X,Y)$. □

Theorem 5.11. *Let $\mathrm{GL}(n)$ be the set of all invertible linear maps on $\mathbb{R}^n$; that is,*

$$\mathrm{GL}(n) = \big\{ L \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n) \,\big|\, L \text{ is one-to-one (and onto)} \big\} .$$

1. *If $L \in \mathrm{GL}(n)$ and $K \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\|K - L\|\|L^{-1}\| < 1$, then $K \in \mathrm{GL}(n)$.*

2. *$\mathrm{GL}(n)$ is an open set of $\mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$.*

3. *The mapping $L \mapsto L^{-1}$ is continuous on $\mathrm{GL}(n)$.*

Proof. 1. Let $\|L^{-1}\| = \dfrac{1}{\alpha}$ and $\|K - L\| = \beta$. Then $\beta < \alpha$; thus for every $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned}\alpha\|\boldsymbol{x}\|_{\mathbb{R}^n} &= \alpha\|L^{-1}L\boldsymbol{x}\|_{\mathbb{R}^n} \leqslant \alpha\|L^{-1}\|\|L\boldsymbol{x}\|_{\mathbb{R}^n} = \|L\boldsymbol{x}\|_{\mathbb{R}^n} \leqslant \|(L-K)\boldsymbol{x}\|_{\mathbb{R}^n} + \|K\boldsymbol{x}\|_{\mathbb{R}^n} \\ &\leqslant \beta\|\boldsymbol{x}\|_{\mathbb{R}^n} + \|K\boldsymbol{x}\|_{\mathbb{R}^n}\,.\end{aligned}$$

As a consequence, $(\alpha - \beta)\|\boldsymbol{x}\|_{\mathbb{R}^n} \leqslant \|K\boldsymbol{x}\|_{\mathbb{R}^n}$ and this implies that $K : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one hence invertible.

2. By 1, we find that if $\|K - L\| < \dfrac{1}{\|L^{-1}\|}$, then $K \in \mathrm{GL}(n)$. Then $B\Big(L, \dfrac{1}{\|L^{-1}\|}\Big) \subseteq \mathrm{GL}(n)$ if $L \in \mathrm{GL}(n)$. Therefore, $\mathrm{GL}(n)$ is open.

3. Let $L \in \mathrm{GL}(n)$ and $\varepsilon > 0$ be given. Choose $\delta = \min\Big\{\dfrac{1}{2\|L^{-1}\|}, \dfrac{\varepsilon}{2\|L^{-1}\|^2}\Big\}$. Then $\delta > 0$, and $K \in \mathrm{GL}(n)$ whenever $\|K - L\| < \delta$. Since $K^{-1} - L^{-1} = K^{-1}(L - K)L^{-1}$, we find that if $\|K - L\| < \delta$,

$$\|K^{-1}\| - \|L^{-1}\| \leqslant \|K^{-1} - L^{-1}\| \leqslant \|K^{-1}\|\|K - L\|\|L^{-1}\| < \frac{1}{2}\|K^{-1}\|$$

which implies that $\|K^{-1}\| < 2\|L^{-1}\|$. Therefore, if $\|K - L\| < \delta$,

$$\|K^{-1} - L^{-1}\| \leqslant \|K^{-1}\|\|K - L\|\|L^{-1}\| < 2\|L^{-1}\|^2\delta \leqslant \varepsilon\,.$$ □

5.2 Definition of Derivatives

Definition 5.12. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces over a common scalar field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A map $f : A \subseteq X \to Y$ is said to be ***differentiable*** at $a \in A$ if there exists a map in $\mathscr{B}(X, Y)$, denoted by $(Df)(a)$ and called the ***derivative*** of f at a, such that

$$\lim_{x \to a} \frac{\big\|f(x) - f(a) - (Df)(a)(x - a)\big\|_Y}{\|x - a\|_X} = 0\,,$$

where $(Df)(a)(x - a)$ denotes the value of the bounded linear map $(Df)(a)$ applied to the vector $x - a \in X$ (so $(Df)(a)(x - a) \in Y$). In other words, f is differentiable at $a \in A$ if there exists $L \in \mathscr{B}(X, Y)$ such that

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \|f(x) - f(a) - L(x - a)\|_Y \leqslant \varepsilon\|x - a\|_X \text{ whenever } x \in B(a, \delta) \cap A\,.$$

If f is differentiable at each point of A, we say that f is differentiable on A.

Remark 5.13. If f is differentiable on A, then for each $x \in A$, $(Df)(x)$ is a bounded linear map from X to Y, but Df in general is not linear in x.

Remark 5.14. The condition
$$\lim_{x\to a} \frac{\|f(x)-f(a)-(Df)(a)(x-a)\|_Y}{\|x-a\|_X} = 0\,,$$
is sometimes written as
$$f(x) = f(a) + (Df)(a)(x-a) + \mathcal{o}(\|x-a\|_X) \quad \text{as} \quad x \to a\,,$$
where $f = g + \mathcal{o}(h)$ as $x \to a$ is a short-hand notation for $\lim\limits_{x\to a} \dfrac{\|f-g\|}{h} = 0$.

Remark 5.15. Let $a \in A$ and v be a unit vector in $(X, \|\cdot\|_X)$ such that $a + tv \in A$ for all $t \in [0,1]$. If $f : A \to \mathbb{R}$ is differentiable at a, then
$$\lim_{t\to 0^+} \frac{\big|f(a+tv)-f(a)-(Df)(a)(tv)\big|}{\|(a+tv)-a\|_X} = 0\,.$$
Since $(Df)(a)(tv) = t(Df)(a)(v)$ and $\|tv\|_X = t$ (since $t > 0$), the identity above implies that
$$\lim_{t\to 0^+} \Big|\frac{f(a+tv)-f(a)}{t} - (Df)(a)(v)\Big| = 0$$
or equivalently,
$$\lim_{t\to 0^+} \frac{f(a+tv)-f(a)}{t} = (Df)(a)(v)\,.$$
In Calculus, the limit on the left-hand side is the ***directional derivative of*** f ***at*** a ***in direction*** v and is usually denoted by $(D_v f)(a)$; thus the quantity $(Df)(a)(v)$ is a generalization of the directional derivative.

Example 5.16. Let $f : (a,b) \to \mathbb{R}$ be differentiable at $c \in (a,b)$. Then there exists $L \in \mathscr{B}(\mathbb{R},\mathbb{R})$ such that
$$\lim_{x\to c} \frac{\big|f(x)-f(c)-L(x-c)\big|}{|x-c|} = 0\,.$$
Since $L \in \mathscr{B}(\mathbb{R},\mathbb{R})$, there exists a real number m such that $L(x) = mx$ for all $x \in \mathbb{R}$; thus the identity above implies that
$$\lim_{x\to c} \frac{f(x)-f(c)-m(x-c)}{x-c} = 0$$
or equivalently,
$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = m\,.$$
In other words, a function $f : (a,b) \to \mathbb{R}$ is differentiable at $c \in (a,b)$ if and only if the limit $\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c}$ exists. The limit is usually denoted by $f'(c)$, and we identify the linear map L with the real number $f'(c)$ using the relation $L(x) = f'(c)x$.

Example 5.17. Let $f:(0,\infty)\to\mathbb{R}$ be given by $f(x)=\dfrac{1}{x}$. Then f is differentiable at any $a\in(0,\infty)$ since $(Df)(a):\mathbb{R}\to\mathbb{R}$ is the linear map given by

$$(Df)(a)(x)=-\frac{1}{a^2}\cdot x\,.$$

To see this, we observe that

$$\lim_{x\to a}\frac{\left|\frac{1}{x}-\frac{1}{a}-\frac{-1}{a^2}(x-a)\right|}{|x-a|}=\lim_{x\to a}\frac{\left|\frac{a^2-xa+x^2-xa}{xa^2}\right|}{|x-a|}=\lim_{x\to a}\frac{a^2-2xa+x^2}{xa^2|x-a|}$$
$$=\lim_{x\to a}\frac{|x-a|}{xa^2}=0\,.$$

Example 5.18. Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two normed spaces. Then every bounded linear map $L:X\to Y$ is differentiable. In fact, $(DL)(a)=L$ for all $a\in X$ since

$$\lim_{x\to a}\frac{\|Lx-La-L(x-a)\|_Y}{\|x-a\|_X}=0\,.$$

Example 5.19. Recall that $\mathcal{M}_{n\times n}$ denotes the collection of all $n\times n$ real matrices. Equip it with 2-norm and let $f:\mathcal{M}_{n\times n}\to\mathcal{M}_{n\times n}$ be given by $f(L)=L^2$. Then for $K,L\in\mathcal{M}_{n\times n}$,

$$f(K)-f(L)=K^2-L^2=L(K-L)+(K-L)L+(K-L)^2\,.$$

This motivates us to define $(Df)(L)$ by $(Df)(L)(H)=LH+HL$ so that

$$\|f(K)-f(L)-(Df)(L)(K-L)\|_2\leqslant\|K-L\|_2^2\,;$$

which shows

$$\lim_{K\to L}\frac{\|f(K)-f(L)-(Df)(L)(K-L)\|_2}{\|K-L\|_2}=0\,.$$

Therefore, f is differentiable at every $L\in\mathcal{M}_{n\times n}$.

Example 5.20. Let $f:\mathrm{GL}(n)\to\mathrm{GL}(n)$ be given by $f(L)=L^{-1}$, where $\mathrm{GL}(n)$ is defined in Theorem 5.11. Then f is differentiable at any "point" $L\in\mathrm{GL}(n)$ with derivative $(Df)(L)\in\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))$ given by

$$(Df)(L)(K)=-L^{-1}KL^{-1}\qquad\text{for all } K\in\mathrm{GL}(n)\,.$$

To see this, for $K,L\in\mathrm{GL}(n)$,

$$\begin{aligned}\left\|f(K)-f(L)+L^{-1}(K-L)L^{-1}\right\|_{\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))}&=\left\|(L^{-1}-K^{-1})(K-L)L^{-1}\right\|_{\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))}\\&\leqslant\|L^{-1}-K^{-1}\|_{\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))}\|K-L\|_{\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))}\|L^{-1}\|_{\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))}\,;\end{aligned}$$

thus if $K \neq L$,

$$\frac{\big\|f(K) - f(L) + L^{-1}(K-L)L^{-1}\big\|_{\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))}}{\|K-L\|_{\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))}} \leqslant \|L^{-1}\|_{\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))}\|L^{-1}-K^{-1}\|_{\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))}.$$

By the fact that the map $L \mapsto L^{-1}$ is continuous on $\mathrm{GL}(n)$ (Theorem 5.11), we find that

$$\lim_{K\to L} \frac{\big\|f(K) - f(L) + L^{-1}(K-L)L^{-1}\big\|_{\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))}}{\|K-L\|_{\mathscr{B}(\mathrm{GL}(n),\mathrm{GL}(n))}} = 0\,.$$

Example 5.21. Recall the setting in Example 4.11 that $X = \mathscr{C}([a,b];\mathbb{R})$, $\|\cdot\|_X = \|\cdot\|_2$, and $\mathrm{I} : X \to \mathbb{R}$ given by

$$\mathrm{I}(f) = \int_a^b \big|f(x)\big|^2\,dx\,.$$

Then I is differentiable at every $f \in X$ since if $(D\mathrm{I})(f)(h) \equiv 2\int_a^b f(x)h(x)\,dx$, Example 5.3 shows that $(D\mathrm{I})(f) \in \mathscr{B}(X,\mathbb{R})$ and

$$\begin{aligned}\big|\mathrm{I}(g) - \mathrm{I}(f) - (D\mathrm{I})(f)(g-f)\big| &= \Big|\int_a^b \Big[|g(x)|^2 - |f(x)|^2 - 2f(x)\big(g(x)-f(x)\big)\Big]\,dx\Big| \\ &= \int_a^b \big[g(x)-f(x)\big]^2\,dx = \|f-g\|_2^2\end{aligned}$$

which implies that

$$\lim_{g\to f}\frac{\big|\mathrm{I}(g) - \mathrm{I}(f) - (D\mathrm{I})(f)(g-f)\big|}{\|g-f\|_2} = 0\,.$$

Example 5.22. The function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2 + y^2$ is differentiable at every $(a,b) \in \mathbb{R}^2$. In fact, $(Df)(a,b)(h,k)$, the linear map $(Df)(a,b)$ acting on the vector (h,k) is $(Df)(a,b)(h,k) = 2ah + 2bk$, since

$$\begin{aligned}&\lim_{(x,y)\to(a,b)} \frac{\big|f(x,y) - f(a,b) - (Df)(a,b)(x-a,y-b)\big|}{\|(x,y)-(a,b)\|} \\ &\quad= \lim_{(h,k)\to(0,0)} \frac{\big|f(a+h,b+k) - f(a,b) - (Df)(a,b)(h,k)\big|}{\|(h,k)\|} \\ &\quad= \lim_{(h,k)\to(0,0)} \frac{\big|(a+h)^2 + (b+k)^2 - a^2 - b^2 - 2ah - 2bk\big|}{\sqrt{h^2+k^2}} \\ &\quad= \lim_{(h,k)\to(0,0)} \frac{h^2+k^2}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2} = 0\,.\end{aligned}$$

On the other hand, Example 5.5 shows that the function $g : \mathbb{C} \to \mathbb{R}$ given by $g(z) = |z|^2$ is differentiable only if $g'(z) \equiv (Dg)(z) = 0$. Therefore, if g is differentiable at z_0, then

$$0 = \lim_{z \to z_0} \frac{\big||z|^2 - |z_0|^2\big|}{|z - z_0|} = \lim_{h \to 0} \frac{\big|(z_0 + h) \cdot \overline{z_0 + h} - z_0\overline{z_0}\big|}{|h|} = \lim_{h \to 0} \Big|\bar{z}_0 - z_0\frac{\bar{h}}{h}\Big| ;$$

thus $z_0 = 0$ since $\displaystyle\lim_{h \to 0} \frac{\bar{h}}{h}$ does not exist.

By treating $\mathbb{R}$ as a subset of $\mathbb{C}$, we treat g as a function from $\mathbb{C}$ to $\mathbb{C}$ (note that there are more maps in $\mathscr{B}(\mathbb{C}, \mathbb{C})$ so in principle it is easier to have differentiable functions from $\mathbb{C}$ to $\mathbb{C}$). Assume the contrary that $g'(z_0) \equiv (Dg)(z_0) \in \mathscr{B}(\mathbb{C}, \mathbb{C})$ exists, then

$$\begin{aligned} 0 &= \lim_{z \to z_0} \frac{\big|g(z) - g(z_0) - g'(z_0)(z - z_0)\big|}{|z - z_0|} = \lim_{h \to 0} \frac{\big|g(z_0 + h) - g(z_0) - g'(z_0)h\big|}{|h|} \\ &= \lim_{h \to 0} \frac{\big|(z_0 + h)\overline{(z_0 + h)} - z_0\overline{z_0} - g'(z_0)h\big|}{|h|} = \lim_{h \to 0} \frac{\big|h\bar{z}_0 + z_0\bar{h} + |h|^2 - g'(z_0)h\big|}{|h|} \\ &= \lim_{h \to 0} \Big|\bar{z}_0 - g'(z_0) + z_0\frac{\bar{h}}{|h|}\Big| ; \end{aligned}$$

thus $\displaystyle\lim_{h \to 0} \Big(\bar{z}_0 - g'(z_0) + z_0\frac{\bar{h}}{|h|}\Big) = 0$ or equivalently, $\displaystyle z_0\Big(\lim_{h \to 0} \frac{\bar{h}}{|h|}\Big) = g'(z_0) - \bar{z}_0$. If $z_0 \neq 0$, then the fact that $\displaystyle\lim_{h \to 0} \frac{\bar{h}}{|h|}$ does not exist shows that the identity above cannot be true; thus g is not differentiable at $z_0 \neq 0$. On the other hand, if $z_0 = 0$, the choice of $g'(z_0) = 0$ makes the identity valid. Therefore, g is differentiable only at 0 and $g'(0) = 0$. This agrees with the observation by treating g as a function from $\mathbb{C}$ to $\mathbb{R}$.

Note that by writing $z = x+iy$, we indeed have $g(x+iy) = x^2+y^2$; thus $f(x, y) = g(x+iy)$. Even though f is differentiable at every point in $\mathbb{R}^2$, g is not. The reason behind this is that there are "much more" bounded linear maps in $\mathscr{B}(\mathbb{R}^2, \mathbb{R})$ than bounded linear maps in $\mathscr{B}(\mathbb{C}, \mathbb{R})$ or $\mathscr{B}(\mathbb{C}, \mathbb{C})$ so that it is easier to make a function from $\mathbb{R}^2 \to \mathbb{R}$ differentiable.

Theorem 5.23. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed vector spaces, A be a subset of X, and $f : A \to Y$ be differentiable at a. If $a \in \mathring{A}$, then $(Df)(a)$ is uniquely determined by f.*

Proof. Suppose $L_1, L_2 \in \mathscr{B}(X, Y)$ are derivatives of f at a. Let $\varepsilon > 0$ be given and $\mathrm{e} \in X$ be a unit vector; that is, $\|\mathrm{e}\|_X = 1$. Since a is an interior point of A, there exists $r > 0$ such that $B(a, r) \subseteq A$. By Definition 5.12, there exists $0 < \delta < r$ such that

$$\frac{\|f(x) - f(a) - L_1(x - a)\|_Y}{\|x - a\|_X} < \frac{\varepsilon}{2} \qquad \text{and} \qquad \frac{\|f(x) - f(a) - L_2(x - a)\|_Y}{\|x - a\|_X} < \frac{\varepsilon}{2}$$

if $0 < \|x-a\|_X < \delta$. Letting $x = a + \lambda \mathrm{e}$ with $0 < |\lambda| < \delta$, we have

$$\begin{aligned}\|L_1\mathrm{e} - L_2\mathrm{e}\|_Y &= \frac{1}{|\lambda|}\|L_1(x-a) - L_2(x-a)\|_Y \\ &\leqslant \frac{1}{|\lambda|}\big(\|f(x)-f(a)-L_1(x-a)\|_Y + \|f(x)-f(a)-L_2(x-x_2)\|_Y\big) \\ &= \frac{\|f(x)-f(a)-L_1(x-a)\|_Y}{\|x-a\|_X} + \frac{\|f(x)-f(a)-L_2(x-a)\|_Y}{\|x-a\|_X} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $L_1\mathrm{e} = L_2\mathrm{e}$ for all unit vectors $\mathrm{e} \in X$ which guarantees that $L_1 = L_2$ (since if $x \neq 0$, $L_1 x = \|x\|_X L_1\Big(\frac{x}{\|x\|_X}\Big) = \|x\|_X L_2\Big(\frac{x}{\|x\|_X}\Big) = L_2 x$). ▫

Example 5.24. $(Df)(a)$ may not be unique if the domain of f is not open. For example, let $A = \big\{(x,y)\,\big|\,0 \leqslant x \leqslant 1, y = 0\big\}$ be a subset of $\mathbb{R}^2$, and $f : A \to \mathbb{R}$ be given by $f(x,y) = 0$. Fix $a = (h,0) \in A$, then both of the linear map

$$L_1(x,y) = 0 \quad \text{and} \quad L_2(x,y) = hy \qquad \forall\,(x,y) \in \mathbb{R}^2$$

satisfy Definition 5.12 since

$$\lim_{(x,0)\to(h,0)} \frac{\big|f(x,0)-f(h,0)-L_1(x-h,0)\big|}{\big\|(x,0)-(h,0)\big\|_{\mathbb{R}^2}} = \lim_{(x,0)\to(h,0)} \frac{\big|f(x,0)-f(h,0)-L_2(x-h,0)\big|}{\big\|(x,0)-(h,0)\big\|_{\mathbb{R}^2}} = 0\,.$$

Remark 5.25. Let $U \subseteq \mathbb{R}^n$ be an open set and suppose that $f : U \to \mathbb{R}^m$ is differentiable on U. Then $Df : U \to \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$. Treating Df as a map from U to the normed space $\big(\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m), \|\cdot\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\big)$, and suppose that Df is also differentiable on U. Then the derivative of Df, denoted by $D^2 f$, is a map from U to $\mathscr{B}(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n,\mathbb{R}^m))$. In other words, for each $a \in U$, $(D^2 f)(a) \in \mathscr{B}(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n,\mathbb{R}^m))$ satisfying

$$\lim_{x\to a} \frac{\big\|(Df)(x) - (Df)(a) - (D^2f)(a)(x-a)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}}{\|x-a\|_{\mathbb{R}^n}} = 0\,,$$

here $(D^2 f)(a)$ is bounded linear map from $\mathbb{R}^n$ to $\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)$; thus $(D^2f)(a)(x-a) \in \mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)$.

Definition 5.26. Let $\{\mathrm{e}_k\}_{k=1}^n$ be the standard basis of $\mathbb{R}^n$, $U \subseteq \mathbb{R}^n$ be an open set, $a \in U$ and $f : U \to \mathbb{R}$ be a function. The partial derivative of f at a in the direction e_j, denoted by $\dfrac{\partial f}{\partial x_j}(a)$, is the limit

$$\lim_{h\to 0}\frac{f(a+h\mathrm{e}_j)-f(a)}{h}$$

if it exists. In other words, if $a=(a_1,\cdots,a_n)$, then

$$\frac{\partial f}{\partial x_j}(a)=\lim_{h\to 0}\frac{f(a_1,\cdots,a_{j-1},a_j+h,a_{j+1},\cdots,a_n)-f(a_1,\cdots,a_n)}{h}.$$

Theorem 5.27. *Suppose $U\subseteq\mathbb{R}^n$ is an open set and $f:U\to\mathbb{R}^m$ is differentiable at $a\in U$. Then the partial derivatives $\frac{\partial f_i}{\partial x_j}(a)$ exists for all $i=1,\cdots m$ and $j=1,\cdots n$, and the matrix representation of the linear map $Df(a)$ with respect to the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ is given by*

$$\big[(Df)(a)\big]=\begin{bmatrix}\frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a)\\ \vdots & \ddots & \vdots\\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a)\end{bmatrix}\qquad \textit{or}\qquad \big[(Df)(a)\big]_{ij}=\frac{\partial f_i}{\partial x_j}(a).$$

Proof. Since U is open and $a\in U$, there exists $r>0$ such that $B(a,r)\subseteq U$. By the differentiability of f at a, there is $L\in\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)$ such that for any given $\varepsilon>0$, there exists $0<\delta<r$ such that

$$\|f(x)-f(a)-L(x-a)\|_{\mathbb{R}^m}\leqslant\varepsilon\|x-a\|_{\mathbb{R}^n}\quad\text{whenever}\quad x\in B(a,\delta).$$

In particular, for each $i=1,\cdots,m$,

$$\Big|\frac{f_i(a+h\mathrm{e}_j)-f_i(a)}{h}-(L\mathrm{e}_j)_i\Big|\leqslant\Big\|\frac{f(a+h\mathrm{e}_j)-f(a)}{h}-L\mathrm{e}_j\Big\|_{\mathbb{R}^m}\leqslant\varepsilon\quad\forall\,0<|h|<\delta,h\in\mathbb{R},$$

where $(L\mathrm{e}_j)_i$ denotes the i-th component of $L\mathrm{e}_j$ in the standard basis. As a consequence, for each $i=1,\cdots,m$,

$$\lim_{h\to 0}\frac{f_i(a+h\mathrm{e}_j)-f_i(a)}{h}=(L\mathrm{e}_j)_i\text{ exists}$$

and by definition, we must have $(L\mathrm{e}_j)_i=\frac{\partial f_i}{\partial x_j}(a)$. Therefore, $L_{ij}=\frac{\partial f_i}{\partial x_j}(a)$. □

Definition 5.28. Let $U\subseteq\mathbb{R}^n$ be an open set, and $f:U\to\mathbb{R}^m$. The matrix

$$(Jf)(x)\equiv\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}\end{bmatrix}(x)\equiv\begin{bmatrix}\frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x)\\ \vdots & \ddots & \vdots\\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x)\end{bmatrix}$$

is called the ***Jacobian matrix*** of f at x (if each entry exists). If $n=m$, the determinant of $(Jf)(x)$ is called the ***Jacobian*** of f at x.

Remark 5.29. A function f might not be differential even if the Jacobian matrix Jf exists; however, if f is differentiable at x_0, then $(Df)(x)$ can be represented by $(Jf)(x)$; that is, $[(Df)(x)] = (Jf)(x)$.

Example 5.30. Let $f:\mathbb{R}^2\to\mathbb{R}^3$ be given by $f(x_1,x_2)=(x_1^2, x_1^3x_2, x_1^4x_2^2)$. Suppose that f is differentiable at $x=(x_1,x_2)$, then

$$\big[(Df)(x)\big] = \left[\begin{array}{cc} 2x_1 & 0 \\ 3x_1^2x_2 & x_1^3 \\ 4x_1^3x_2^2 & 2x_1^4x_2 \end{array}\right].$$

Example 5.31. Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by

$$f(x,y)=\left\{\begin{array}{cl} \dfrac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\,, \\ 0 & \text{if } (x,y)=(0,0)\,. \end{array}\right.$$

Then $\dfrac{\partial f}{\partial x}(0,0)=\dfrac{\partial f}{\partial y}(0,0)=0$; thus if f is differentiable at $(0,0)$, then $[(Df)(0,0)]=\begin{bmatrix}0 & 0\end{bmatrix}$. However,

$$\Big|f(x,y)-f(0,0)-\begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\Big| = \frac{|xy|}{x^2+y^2} = \frac{|xy|}{(x^2+y^2)^{\frac{3}{2}}}\sqrt{x^2+y^2}\,;$$

thus f is not differentiable at $(0,0)$ since $\dfrac{|xy|}{(x^2+y^2)^{\frac{3}{2}}}$ cannot be arbitrarily small even if x^2+y^2 is small.

Example 5.32. Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by

$$f(x,y)=\left\{\begin{array}{cl} x & \text{if } y=0\,, \\ y & \text{if } x=0\,, \\ 1 & \text{otherwise}\,. \end{array}\right.$$

Then $\dfrac{\partial f}{\partial x}(0,0)=\lim\limits_{h\to0}\dfrac{f(h,0)-f(0,0)}{h}=\lim\limits_{h\to0}\dfrac{h}{h}=1$. Similarly, $\dfrac{\partial f}{\partial y}(0,0)=1$; thus if f is differentiable at $(0,0)$, then $[(Df)(0,0)]=\begin{bmatrix}1 & 1\end{bmatrix}$. However,

$$\Big|f(x,y)-f(0,0)-\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\Big| = \big|f(x,y)-(x+y)\big|\,;$$

thus if $xy \neq 0$,

$$\big|f(x,y) - (x+y)\big| = |1-x-y| \nrightarrow 0 \quad \text{as} \quad (x,y) \to (0,0), xy \neq 0\,.$$

Therefore, f is not differentiable at $(0,0)$.

Definition 5.33. Let $U \subseteq \mathbb{R}^n$ be an open set. The derivative of a scalar function $f : U \to \mathbb{R}$ is called the ***gradient*** of f and is denoted by grad$f$ or ∇f.

5.3 Continuity of Differentiable Maps

Theorem 5.34. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, $U \subseteq X$ be open, and $f : U \to Y$ be differentiable at $a \in U$. Then f is continuous at a.*

Proof. Since f is differentiable at a, there exists $L \in \mathscr{B}(X,Y)$ such that

$$\exists\, \delta > 0 \ni \big\|f(x) - f(a) - L(x-a)\big\|_Y \leqslant \|x-a\|_X \quad \forall\, x \in B(a,\delta)\,.$$

As a consequence,

$$\big\|f(x) - f(a)\big\|_Y \leqslant \big(\|L\| + 1\big)\|x-a\|_X \quad \forall\, x \in B(a,\delta)\,; \tag{5.3.1}$$

thus $\lim\limits_{x\to a} \|f(x) - f(a)\|_Y = 0$. ▫

Remark 5.35. In fact, if f is differentiable at x_0, then f satisfies the "local Lipschitz property"; that is,

$$\exists\, M = M(x_0) > 0 \text{ and } \delta = \delta(x_0) > 0 \ni \|f(x) - f(x_0)\|_Y \leqslant M\|x - x_0\|_X \text{ if } \|x - x_0\|_X < \delta$$

since we can choose $M = \|L\| + 1$ and $\delta = \delta_1$ (see (5.3.1)).

Example 5.36. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given in Example 5.31. We have shown that f is not differentiable at $(0,0)$. In fact, f is not even continuous at $(0,0)$ since when approaching the origin along the straight line $x_2 = mx_1$,

$$\lim_{(x_1,mx_1)\to(0,0)} f(x_1, mx_1) = \lim_{x_1\to 0} \frac{mx_1^2}{(m^2+1)x_1^2} = \frac{m^2}{m^2+1} \neq f(0,0) \text{ if } m \neq 0\,.$$

Example 5.37. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given in Example 5.32. Then f is not continuous at $(0,0)$; thus not differentiable at $(0,0)$.

Example 5.38. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then $f_x(0,0) = 1$ and $f_y(0,0) = 0$. However,

$$\frac{\left| f(x,y) - f(0,0) - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right|}{\sqrt{x^2+y^2}} = \frac{|x|y^2}{(x^2+y^2)^{\frac{3}{2}}} \nrightarrow 0 \quad \text{as} \quad (x,y) \to (0,0)\,.$$

Therefore, f is not differentiable at $(0,0)$. On the other hand, f is continuous at $(0,0)$ since

$$\big|f(x,y) - f(0,0)\big| = \big|f(x,y)\big| \leqslant |x| \to 0 \quad \text{as} \;\; (x,y) \to (0,0).$$

5.4 Conditions for Differentiability

Proposition 5.39. *Let $U \subseteq \mathbb{R}^n$ be open, $a \in U$, and $f = (f_1, \cdots, f_m) : U \to \mathbb{R}^m$. Then f is differentiable at a if and only if f_i is differentiable at a for all $i = 1, \cdots, m$. In other words, for vector-valued functions defined on an open subset of $\mathbb{R}^n$,*

$$\textit{Componentwise differentiable} \;\Leftrightarrow\; \textit{Differentiable.}$$

Proof. By the definition of differentiability and Proposition 2.49,

$$\begin{aligned}
&f \text{ is differentiable at } a \\
&\quad \Leftrightarrow (\exists\, L \in \mathcal{M}_{m\times n})\Big(\lim_{x\to a} \frac{\|f(x) - f(a) - L(x-a)\|_{\mathbb{R}^m}}{\|x-a\|_{\mathbb{R}^n}} = 0\Big) \\
&\quad \Leftrightarrow (\exists\, L \in \mathcal{M}_{m\times n})\Big(\lim_{x\to a} \Big\| \frac{f(x) - f(a)}{\|x-a\|_{\mathbb{R}^n}} - L\Big(\frac{[x-a]}{\|x-a\|_{\mathbb{R}^n}}\Big)\Big\|_{\mathbb{R}^m} = 0\Big) \\
&\quad \Leftrightarrow (\exists\, L \in \mathcal{M}_{m\times n})(\forall\, 1 \leqslant i \leqslant n)\Big(\lim_{x\to a} \Big| \frac{f_i(x) - f_i(a)}{\|x-a\|_{\mathbb{R}^n}} - (\mathrm{e}_i^{\mathrm{T}} L)\Big(\frac{[x-a]}{\|x-a\|_{\mathbb{R}^n}}\Big)\Big| = 0\Big) \\
&\quad \Leftrightarrow (\exists\, L \in \mathcal{M}_{m\times n})(\forall\, 1 \leqslant i \leqslant n)\Big(\lim_{x\to a} \frac{\big|f_i(x) - f_i(a) - (\mathrm{e}_i^{\mathrm{T}} L)(x-a)\big|}{\|x-a\|_{\mathbb{R}^n}} = 0\Big) \\
&\quad \Leftrightarrow (\exists\, L_i \in \mathcal{M}_{m\times 1})\Big(\lim_{x\to a} \frac{\big|f_i(x) - f_i(a) - L_i(x-a)\big|}{\|x-a\|_{\mathbb{R}^n}} = 0\Big) \\
&\quad \Leftrightarrow f_i \text{ is differentiable at } a \text{ for each } 1 \leqslant i \leqslant n.
\end{aligned}$$

□

Theorem 5.40. *Let $U \subseteq \mathbb{R}^n$ be open, $a \in U$, and $f : U \to \mathbb{R}$. If each entry of the Jacobian matrix $\Big[\dfrac{\partial f}{\partial x_1} \;\cdots\; \dfrac{\partial f}{\partial x_n}\Big]$ of f*

1. *exists in a neighborhood of a, and*
2. *is continuous at a except perhaps one entry.*

Then f is differentiable at a.

Proof. W.L.O.G. we can assume that $\dfrac{\partial f}{\partial x_i}$ is continuous at a for $1 \leqslant i < n$. Let $\{\mathrm{e}_j\}_{j=1}^n$ be the standard basis of $\mathbb{R}^n$, and $\varepsilon > 0$ be given. Since $\dfrac{\partial f}{\partial x_i}$ is continuous at a for $i = 1, \cdots, n-1$,

$$\exists\, \delta_i > 0 \ni \Big|\frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_i}(a)\Big| < \frac{\varepsilon}{\sqrt{n}} \quad \text{whenever} \quad \|x-a\|_{\mathbb{R}^n} < \delta_i\,.$$

On the other hand, by the definition of the partial derivatives,

$$\exists\, \delta_n > 0 \ni \Big|\frac{f(a+h\mathrm{e}_n)-f(a)}{h} - \frac{\partial f}{\partial x_n}(a)\Big| \leqslant \frac{\varepsilon}{\sqrt{n}} \quad \text{whenever} \quad |h| < \delta_n\,.$$

Let $k = x - a$ and $\delta = \min\big\{\delta_1, \cdots, \delta_n\big\}$. Then

$$\begin{aligned}
&\Big|f(x) - f(a) - \Big[\frac{\partial f}{\partial x_1}(a)(x_1-a_1) + \cdots + \frac{\partial f}{\partial x_n}(a)(x_n-a_n)\Big]\Big| \\
&\quad = \Big|f(a+k) - f(a) - \frac{\partial f}{\partial x_1}(a)k_1 - \cdots - \frac{\partial f}{\partial x_n}(a)k_n\Big| \\
&\quad = \Big|f(a_1+k_1, \cdots, a_n+k_n) - f(a_1, \cdots, a_n) - \frac{\partial f}{\partial x_1}(a)k_1 - \cdots - \frac{\partial f}{\partial x_n}(a)k_n\Big| \\
&\quad \leqslant \Big|f(a_1+k_1, \cdots, a_n+k_n) - f(a_1, a_2+k_2, \cdots, a_n+k_n) - \frac{\partial f}{\partial x_1}(a)k_1\Big| \\
&\qquad + \Big|f(a_1, a_2+k_2, \cdots, a_n+k_n) - f(a_1, a_2, a_3+k_3, \cdots, a_n+k_n) - \frac{\partial f}{\partial x_2}(a)k_2\Big| \\
&\qquad + \cdots + \Big|f(a_1, \cdots, a_{n-1}, a_n+k_n) - f(a_1, \cdots, a_n) - \frac{\partial f}{\partial x_n}(a)k_n\Big|\,.
\end{aligned}$$

By the mean value theorem,

$$\begin{aligned}
f(a_1, \cdots, a_{j-1}, a_j+k_j, \cdots, a_n+k_n) - f(a_1, \cdots, a_j, a_{j+1}+k_{j+1}, \cdots, a_n+k_n) \\
= k_j \frac{\partial f}{\partial x_j}(a_1, \cdots, a_{j-1}, a_j+\theta_j k_j, a_{j+1}+k_{j+1}, \cdots, a_n+k_n)
\end{aligned}$$

for some $0 < \theta_j < 1$; thus for $j = 1, \cdots, n-1$, if $\|x-a\|_{\mathbb{R}^n} = \|k\|_{\mathbb{R}^n} < \delta$,

$$\begin{aligned}
&\Big|f(a_1, \cdots, a_{j-1}, a_j+k_j, \cdots, a_n+k_n) - f(a_1, \cdots, a_j, a_{j+1}+k_{j+1}, \cdots, a_n+k_n) - \frac{\partial f}{\partial x_j}(a)k_j\Big| \\
&\quad = \Big|\frac{\partial f}{\partial x_j}(a_1, \cdots, a_{j-1}, a_j+\theta_j k_j, a_{j+1}+k_{j+1}, \cdots, a_n+k_n) - \frac{\partial f}{\partial x_j}(a)\Big||k_j| \leqslant \frac{\varepsilon}{\sqrt{n}}|k_j|\,.
\end{aligned}$$

Moreover, if $\|x-a\|_{\mathbb{R}^n} < \delta$, then $|k_n| \leqslant \|k\|_{\mathbb{R}^n} = \|x-a\|_{\mathbb{R}^n} < \delta \leqslant \delta_n$; thus

$$\Big| f(a_1, \cdots, a_{n-1}, a_n + k_n) - f(a_1, \cdots, a_n) - \frac{\partial f}{\partial x_n}(a) k_n \Big| \leqslant \frac{\varepsilon}{\sqrt{n}} |k_n| \,.$$

As a consequence, if $\|x-a\|_{\mathbb{R}^n} < \delta$, by Cauchy's inequality,

$$\begin{aligned} &\Big| f(x) - f(a) - \Big[\frac{\partial f}{\partial x_1}(a)(x_1 - a_1) + \cdots + \frac{\partial f}{\partial x_n}(a)(x_n - a_n) \Big] \Big| \\ &\qquad \leqslant \frac{\varepsilon}{\sqrt{n}} \sum_{j=1}^{n} |k_j| \leqslant \varepsilon \|k\|_{\mathbb{R}^n} = \varepsilon \|x-a\|_{\mathbb{R}^n} \end{aligned}$$

which implies that f is differentiable at a. □

Remark 5.41. When two or more components of the Jacobian matrix $\Big[\frac{\partial f}{\partial x_1} \ \cdots \ \frac{\partial f}{\partial x_n} \Big]$ of a scalar function f are discontinuous at a point $a \in U$, in general f is not differentiable at a. For example, both components of the Jacobian matrix of the functions given in Example 5.31, 5.32, 5.38 are discontinuous at $(0,0)$, and these functions are not differentiable at $(0,0)$.

Example 5.42. Let $U = \mathbb{R}^2 \backslash \big\{ (x,0) \in \mathbb{R}^2 \,\big|\, x \geqslant 0 \big\}$, and $f : U \to \mathbb{R}$ be given by

$$f(x,y) = \arg(x+iy) = \begin{cases} \arccos \dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y > 0\,, \\ \pi & \text{if } y = 0\,, \\ 2\pi - \arccos \dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y < 0\,. \end{cases}$$

Then

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} -\dfrac{y}{x^2+y^2} & \text{if } y \neq 0\,, \\ 0 & \text{if } y = 0\,, \end{cases} \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x}{x^2+y^2} & \text{if } y \neq 0\,, \\ \dfrac{1}{x} & \text{if } y = 0\,. \end{cases}$$

Since $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are both continuous on U, f is differentiable on U.

Definition 5.43. Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^m$ be differentiable on U. f is said to be ***continuously differentiable*** on U if $Df : U \to \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$ is continuous on U. The collection of all continuously differentiable mappings from U to $\mathbb{R}^m$ is denoted by $\mathscr{C}^1(U;\mathbb{R}^m)$. The collection of all bounded differentiable functions from U to $\mathbb{R}^m$ whose derivative is continuous and bounded is denoted by $\mathscr{C}^1_b(U;\mathbb{R}^m)$. In other words,

$$\mathscr{C}^1(U;\mathbb{R}^m) = \big\{ f : U \to \mathbb{R}^m \text{ is differentiable on } U \,\big|\, Df : U \to \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m) \text{ is continuous} \big\}$$

and

$$\mathscr{C}_b^1(U;\mathbb{R}^m) = \Big\{ f \in \mathscr{C}^1(U;\mathbb{R}^m) \,\Big|\, \sup_{x\in U} |f(x)| + \sup_{x\in U} \|Df(x)\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} < \infty \Big\}.$$

Theorem 5.44. *Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^m$. Then $f \in \mathscr{C}^1(U;\mathbb{R}^m)$ if and only if the partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ exist and are continuous on U for $i = 1, \cdots, m$ and $j = 1, \cdots, n$.*

Proof. Note that $\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$ is finite dimensional. By Example 4.29, there exist c and $C > 0$ such that

$$c\sum_{i=1}^m \sum_{j=1}^n |a_{ij}| \leqslant \|L\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} \leqslant C\sum_{i=1}^m \sum_{j=1}^n |a_{ij}| \quad \forall\, L \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m) \text{ with representation } [L] = [a_{ij}]\,.$$

Therefore, for every $a \in U$,

$$c\sum_{i=1}^m \sum_{j=1}^n \Big|\frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(a)\Big| \leqslant \big\|(Df)(x) - (Df)(a)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} \leqslant C\sum_{i=1}^m \sum_{j=1}^n \Big|\frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(a)\Big|;$$

thus $\lim\limits_{x\to a} \|(Df)(x) - (Df)(a)\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} = 0$ if and only if $\lim\limits_{x\to a} \Big|\dfrac{\partial f_i}{\partial x_j}(x) - \dfrac{\partial f_i}{\partial x_j}(a)\Big| = 0$ for all $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. □

Example 5.45. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a, must f' be continuous at a? In other words, is it always true that $\lim\limits_{x\to a} f'(x) = f'(a)$?
Answer: No! For example, take

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is differentiable at $x = 0$ since the limit

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h\to 0} h \sin \frac{1}{h} = 0$$

exists. Therefore,

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

However, $\lim\limits_{x\to 0} f'(x)$ does not exist.

5.5 The Product Rule and the Chain Rule

Theorem 5.46. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed vector spaces over field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $U \subseteq X$ be open, and $f : U \to Y$ and $g : U \to \mathbb{F}$ be differentiable at $a \in U$. Then $gf : U \to Y$ is differentiable at a, and*

$$D(gf)(a)(v) = g(a)(Df)(a)(v) + (Dg)(a)(v)f(a)\,. \tag{5.5.1}$$

Moreover, if $g(a) \neq 0$, then $\dfrac{f}{g} : U \to Y$ is also differentiable at a, and $D(\dfrac{f}{g})(a) : X \to Y$ is given by

$$D\big(\frac{f}{g}\big)(a)(v) = \frac{g(a)\big((Df)(a)(v)\big) - (Dg)(a)(v)f(a)}{g^2(a)}\,. \tag{5.5.2}$$

Proof. We only prove (5.5.1), and (5.5.2) is left as an exercise.

Define $A : X \to Y$ by $A(v) = g(a)(Df)(a)(v) + (Dg)(a)(v)f(a)$. Then clearly $A \in \mathscr{L}(X, Y)$. Moreover,

$$\begin{aligned}
\|Av\|_Y &\leqslant \big\|g(a)(Df)(a)(v)\big\|_Y + \big\|(Dg)(a)(v)f(a)\big\|_Y \\
&\leqslant \big|g(a)\big|\big\|(Df)(a)(v)\big\|_Y + \big|(Dg)(a)(v)\big|\big\|f(a)\big\|_Y \\
&\leqslant \big|g(a)\big|\big\|(Df)(a)\big\|_{\mathscr{B}(X,Y)}\|v\|_X + \big\|(Dg)(a)\big\|_{\mathscr{B}(X,\mathbb{F})}\|v\|_X\|f(a)\|_Y \\
&= \Big[\big|g(a)\big|\big\|(Df)(a)\big\|_{\mathscr{B}(X,Y)} + \big\|(Dg)(a)\big\|_{\mathscr{B}(X,\mathbb{F})}\|f(a)\|_Y\Big]\|v\|_X
\end{aligned}$$

so that A is bounded. Note that

$$\begin{aligned}
(gf)(x) - (gf)(a) - A(x-a) = g(x)&\big(f(x) - f(a) - (Df)(a)(x-a)\big) \\
&+ \big(g(x) - g(a) - (Dg)(a)(x-a)\big)f(a) \\
&+ \big(g(x) - g(a)\big((Df)(a)(x-a)\big)\big)\,.
\end{aligned}$$

As a consequence, for $x \neq a$,

$$\begin{aligned}
\frac{\big\|(gf)(x) - (gf)(a) - A(x-a)\big\|_Y}{\|x-a\|_X} &\leqslant \big|g(x)\big|\frac{\big\|f(x) - f(a) - (Df)(a)(x-a)\big\|_Y}{\|x-a\|_X} \\
&+ \frac{\big|g(x) - g(a) - (Dg)(a)(x-a)\big|}{\|x-a\|_X}\|f(a)\|_Y + \big\|(Df)(a)\big\|_{\mathscr{B}(X,Y)}\big|g(x) - g(a)\big|
\end{aligned}$$

and the right-hand side approaches zero as $x \to a$ since f and g are differentiable at a (so that g is continuous at a). Therefore, gf is differentiable at a with derivative $D(gf)(a)$ given by (5.5.1). □

Theorem 5.47. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be normed vector spaces, $U \subseteq X$ and $V \subseteq Y$ be open sets. Suppose that $f : U \to Y$ is differentiable at $a \in U$, $f(U) \subseteq V$, and $g : V \to Z$ is differentiable at $f(a)$. Then the map $F = g \circ f : U \to Z$ defined by*

$$F(x) = g\big(f(x)\big) \qquad \forall\, x \in U$$

is differentiable at a, and

$$(DF)(a)(h) = (Dg)\big(f(a)\big)\big((Df)(a)(h)\big) \qquad \forall\, h \in X\,.$$

In particular, if $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ and $Z = \mathbb{R}^\ell$, then

$$\big((DF)(a)\big)_{ij} = \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}\big(f(a)\big)\frac{\partial f_k}{\partial x_j}(a)\,.$$

Proof. To simplify the notation, we write $b = f(a)$, $A = (Df)(a) \in \mathscr{B}(X,Y)$, and $B = (Dg)(b) \in \mathscr{B}(Y,Z)$. Since U and V are open, there exists $r_1, r_2 > 0$ such that $B_X(a, r_1) \subseteq U$ and $B_Y(b, r_2) \subseteq V$, where B_X and B_Y denote balls in X and Y, respectively.

Let $\varepsilon > 0$ be given. Define $u : B_X(0, r_1) \to Y$ and $v : B_Y(0, r_2) \to Z$ by

$$u(h) = f(a+h) - f(a) - Ah \qquad \text{and} \qquad v(k) = g(b+k) - g(b) - Bk\,.$$

By the differentiability of f and g at a and b, there exist $0 < \delta_1 < r_1$ and $0 < \delta_2 < r_2$ such that

$$\|u(h)\|_Y \leqslant \min\Big\{1, \frac{\varepsilon}{2(\|B\|+1)}\Big\}\|h\|_X \quad \text{whenever} \quad \|h\|_X < \delta_1\,,$$
$$\|v(k)\|_Z \leqslant \frac{\varepsilon}{2(\|A\|+1)}\|k\|_Y \quad \text{whenever} \quad \|k\|_Y < \delta_2\,.$$

Let $k = f(a+h) - f(a) = Ah + u(h)$. Then $\lim\limits_{h \to 0} k = 0$; thus there exists $\delta_3 > 0$ such that

$$\|k\|_Y < \delta_2 \quad \text{whenever} \quad \|h\|_X < \delta_3\,.$$

Define $\delta = \min\{\delta_1, \delta_3\}$. Then $\delta > 0$; thus by the fact that

$$\begin{aligned} F(a+h) - F(a) &= g(b+k) - g(b) = Bk + v(k) = B\big(Ah + u(h)\big) + v(k) \\ &= BAh + Bu(h) + v(k)\,, \end{aligned}$$

we find that if $\|h\|_X < \delta$,

$$\begin{aligned} \|F(a+h) - F(a) - BAh\|_Z &\leqslant \|Bu(h)\|_Z + \|v(k)\|_Z \leqslant \|B\|\|u(h)\|_Y + \frac{\varepsilon}{2(\|A\|+1)}\|k\|_Y \\ &\leqslant \frac{\varepsilon}{2}\|h\|_X + \frac{\varepsilon}{2(\|A\|+1)}\big(\|A\|\|h\|_X + \|u(h)\|_Y\big) \leqslant \frac{\varepsilon}{2}\|h\|_X + \frac{\varepsilon}{2}\|h\|_X = \varepsilon\|h\|_X\,. \end{aligned}$$

Since $BA = B \circ A \in \mathscr{B}(X,Z)$ by Proposition 5.9, we conclude that F is differentiable at a and $(DF)(a) = BA$. □

Example 5.48. Let $X = \mathscr{C}([a,b];\mathbb{R})$ and $\|\cdot\|_X$ be the maximum norm; that is, $\|f\|_X = \max\limits_{x\in[a,b]} \big|f(x)\big|$. Let $I : X \to X$ be defined by $I(f) = f^2$ and $J : X \to \mathbb{R}$ be defined by $J(f) = \int_a^b f(x)^2\,dx$. Then I is differentiable on X (with $(DI)(f)(h) = 2fh$) and J is differentiable on X $\big($with $(DJ)(f)(h) = \int_a^b 2f(x)h(x)\,dx\big)$. Therefore, the chain rule implies that $J \circ I$ is differentiable on X and

$$D(J\circ I)(f)(h) = (DJ)(f^2)\big((DI)(f)(h)\big) = (DJ)(f^2)(2fh) = \int_a^b 4f^3(x)h(x)\,dx\,.$$

Example 5.49. Let $f : \mathrm{GL}(n) \to \mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)$ and $g : \mathscr{B}(\mathbb{R}^n,\mathbb{R}^n) \to \mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)$ be defined by $f(L) = L^{-1}$ and $g(L) = L^2$. Then for $H \in \mathrm{GL}(n)$, Example 5.19 and 5.20 imply that

$$(Df)(L)(H) = -L^{-1}HL^{-1} \qquad \text{and} \qquad (Dg)(L)(H) = LH + HL\,.$$

Therefore, the chain rule shows that

$$\begin{aligned} D(g\circ f)(L)(H) &= (Dg)(L^{-1})\big((Df)(L)(H)\big) = L^{-1}\big((Df)(L)(H)\big) + \big((Df)(L)(H)\big)L^{-1} \\ &= -L^{-2}HL^{-1} - L^{-1}HL^{-2}\,. \end{aligned}$$

5.6 Higher Derivatives of Functions

Let $U \subseteq X$ be open, and $f : U \to Y$ is differentiable. By Proposition 5.8, the space $\big(\mathscr{B}(X,Y), \|\cdot\|_{\mathscr{B}(X,Y)}\big)$ is a normed space, so it is legitimate to ask if $Df : U \to \mathscr{B}(X,Y)$ is differentiable or not. If Df is differentiable at a, we call f twice differentiable at a, and denote the twice derivative of f at a as $(D^2f)(a)$. If Df is differentiable on U, then $D^2f : U \to \mathscr{B}\big(X, \mathscr{B}(X,Y)\big)$. Similar, we can talk about three times differentiability of a function if it is twice differentiable. In general, we have the following

Definition 5.50. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and $U \subseteq X$ be open. A function $f : U \to Y$ is said to be ***twice differentiable*** at $a \in U$ if

1. f is (once) differentiable in a neighborhood of a;

2. there exists $L_2 \in \mathscr{B}\big(X, \mathscr{B}(X,Y)\big)$, usually denoted by $(D^2 f)(a)$ and called the ***second derivative*** of f at a, such that

$$\lim_{x \to a} \frac{\big\|(Df)(x) - (Df)(a) - L_2(x-a)\big\|_{\mathscr{B}(X,Y)}}{\|x-a\|_X} = 0\,.$$

For two vectors $u, v \in X$, $(D^2 f)(a)(v) \in \mathscr{B}(X,Y)$ and $(D^2 f)(a)(v)(u) \in Y$. The vector $(D^2 f)(a)(v)(u)$ is usually denoted by $(D^2 f)(a)(u,v)$.

In general, a function f is said to be ***k-times differentiable*** at $a \in U$ if

1. f is $(k-1)$-times differentiable in a neighborhood of a;

2. there exists $L_k \in \mathscr{B}(\underbrace{X, \mathscr{B}(X, \cdots, \mathscr{B}(X}_{k \text{ copies of "}X\text{"}}, Y\underbrace{)\cdots))}_{k \text{ copies of ")"}}$, usually denoted by $(D^k f)(a)$ and called the ***k-th derivative*** of f at a, such that

$$\lim_{x \to a} \frac{\big\|(D^{k-1} f)(x) - (D^{k-1} f)(a) - L_k(x-a)\big\|_{\mathscr{B}(X, \mathscr{B}(X, \cdots, \mathscr{B}(X,Y)\cdots))}}{\|x-a\|_X} = 0\,.$$

For k vectors $u^{(1)}, \cdots, u^{(k)} \in X$, the vector $(D^k f)(a)(u^{(1)}, \cdots, u^{(k)})$ is defined as the vector

$$(D^k f)(a)(u^{(k)})(u^{(k-1)}) \cdots (u^{(1)})\,,$$

where $(D^k f)(a)(u^{(k)}) \in \mathscr{B}(\underbrace{X, \mathscr{B}(X, \cdots, \mathscr{B}(X}_{(k-1) \text{ copies of "}X\text{"}}, Y \underbrace{)\cdots))}_{(k-1) \text{ copies of ")"}}$ so that $(D^k f)(a)(u^{(k)})(u^{(k-1)}) \in \mathscr{B}(\underbrace{X, \mathscr{B}(X, \cdots, \mathscr{B}(X}_{(k-2) \text{ copies of "}X\text{"}}, Y \underbrace{)\cdots))}_{(k-2) \text{ copies of ")"}}$, and etc.

Example 5.51. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces, and $f(x) = Lx$ for some $L \in \mathscr{B}(X,Y)$. From Example 5.18, $(Df)(a) = L$ for all $a \in X$; thus $(D^2 f)(a) = 0$ since $Df : U \in \mathscr{B}(X,Y)$ is a "constant" map. In fact, one can also conclude from

$$\lim_{x \to a} \frac{\big\|(Df)(x) - (Df)(a) - 0(x-a)\big\|_{\mathscr{B}(X,Y)}}{\|x-a\|_X} = 0$$

that $(D^2 f)(a) = 0$ for all $a \in X$.

Remark 5.52. We focus on what $(D^k f)(a)(u_k)(\cdots)(u_1)$ means in this remark. We first look at the case that f is twice differentiable at a. With $x = a + tv$ for $v \in X$ with $\|v\|_X = 1$ in the definition, we find that

$$\lim_{t \to 0} \frac{\big\|(Df)(a+tv) - (Df)(a) - t(D^2 f)(a)(v)\big\|_{\mathscr{B}(X,Y)}}{|t|} = 0\,.$$

Since $(Df)(a+tv)-(Df)(a)-t(D^2f)(a)(v) \in \mathscr{B}(X,Y)$, for all $u \in X$ with $\|u\|_X = 1$ we have

$$\begin{aligned}
&\lim_{t\to 0}\left\|\frac{(Df)(a+tv)(u)-(Df)(a)(u)}{t} - (D^2f)(a)(v)(u)\right\|_Y \\
&\quad= \lim_{t\to 0}\frac{\|(Df)(a+tv)(u)-(Df)(a)(u)-t(D^2f)(a)(v)(u)\|_Y}{|t|} \\
&\quad= \lim_{t\to 0}\frac{\big\|\big[(Df)(a+tv)-(Df)(a)-t(D^2f)(a)(v)\big](u)\big\|_Y}{|t|} \\
&\quad\leqslant \lim_{t\to 0}\frac{\|(Df)(a+tv)-(Df)(a)-t(D^2f)(a)(v)\|_{\mathscr{B}(X,Y)}}{\|(a+tv)-a\|_X} = 0\,.
\end{aligned}$$

On the other hand, by the definition of the direction derivative (see Remark 5.15),

$$(Df)(a+tv)(u)-(Df)(a)(u) = \lim_{s\to 0}\Big[\frac{f(a+tv+su)-f(a+tv)}{s} - \frac{f(a+su)-f(a)}{s}\Big]\,;$$

thus the limit above implies that

$$\begin{aligned}
(D^2f)(a)(v)(u) &= \lim_{t\to 0}\lim_{s\to 0}\frac{f(a+tv+su)-f(a+tv)-f(a+su)+f(a)}{st} \\
&= \lim_{t\to 0}\frac{\displaystyle\lim_{s\to 0}\frac{f(a+tv+su)-f(a+tv)}{s} - \lim_{s\to 0}\frac{f(a+su)-f(a)}{s}}{t} \\
&= D_v(D_uf)(a)\,.
\end{aligned}$$

Therefore, $(D^2f)(a)(v)(u)$ is obtained by first differentiating f around a in the u-direction, then differentiating (Df) at a in the v-direction.

In general, $(D^kf)(a)(u_k)\cdots(u_1)$ is obtained by first differentiating f around a in the u_1-direction, then differentiating (Df) near a in the u_2-direction, and so on, and finally differentiating $(D^{k-1}f)$ at a in the u_k-direction.

Remark 5.53. Since $(D^2f)(a) \in \mathscr{B}(X,\mathscr{B}(X,Y))$, if $v_1, v_2 \in X$ and $c \in \mathbb{R}$, we have $(D^2f)(a)(cv_1+v_2) = c(D^2f)(a)(v_1)+(D^2f)(a)(v_2)$ (treated as "vectors" in $\mathscr{B}(X,Y)$); thus

$$(D^2f)(a)(cv_1+v_2)(u) = c(D^2f)(a)(v_1)(u)+(D^2f)(a)(v_2)(u) \qquad \forall\, u, v_1, v_2 \in X\,.$$

On the other hand, since $(D^2f)(a)(v) \in \mathscr{B}(X,Y)$,

$$(D^2f)(a)(v)(cu_1+u_2) = c(D^2f)(a)(v)(u_1)+(D^2f)(a)(v)(u_2) \qquad \forall\, u_1, u_2, v \in X\,.$$

Therefore, $(D^2f)(a)(v)(u)$ is linear in both u and v variables. A map with such kind of property is called a ***bilinear*** map (meaning 2-linear). In particular, $(D^2f)(a) : X \times X \to Y$ is a bilinear map.

In general, the vector $(D^k f)(a)(u^{(1)}, \cdots, u^{(k)})$ is linear in $u^{(1)}, \cdots, u^{(k)}$; that is,

$$\begin{aligned}(D^k f)(a)&(u^{(1)}, \cdots, u^{(i-1)}, \alpha v + \beta w, u^{(i+1)}, \cdots, u^{(k)})\\ &= \alpha(D^k f)(a)(u^{(1)}, \cdots, u^{(i-1)}, v, u^{(i+1)}, \cdots, u^{(k)})\\ &\quad + \beta(D^k f)(a)(u^{(1)}, \cdots, u^{(i-1)}, w, u^{(i+1)}, \cdots, u^{(k)})\end{aligned}$$

for all $v, w \in X$, $\alpha, \beta \in \mathbb{R}$, and $i = 1, \cdots, n$. Such kind of map which is linear in each component when the other $k-1$ components are fixed is called ***k-linear***.

Consider the case that X is finite dimensional with $\dim(X) = n$, $\{e_1, e_2, \ldots, e_n\}$ is a basis of X, and $Y = \mathbb{R}$. Then $(D^2 f)(a) : X \times X \to Y$ is a bilinear form (here the term "form" means that $Y = \mathbb{R}$). A bilinear form $B : X \times X \to \mathbb{R}$ can be represented as follows: Let $a_{ij} = B(e_i, e_j) \in \mathbb{R}$ for $i, j = 1, 2, \cdots, n$. Given $x, y \in \mathbb{R}^n$, write $u = \sum_{i=1}^{n} u_i e_i$ and $v = \sum_{j=1}^{n} v_j e_j$. Then by the bilinearity of B,

$$B(u, v) = B\Big(\sum_{i=1}^{n} u_i e_i, \sum_{j=1}^{n} v_j e_j\Big) = \sum_{i,j=1}^{n} u_i v_j a_{ij} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

Therefore, if $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is twice differentiable at a, then the bilinear form $(D^2 f)(a)$ can be represented as

$$(D^2 f)(a)(u, v) = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} (D^2 f)(e_1, e_1) & \cdots & (D^2 f)(a)(e_1, e_n) \\ \vdots & \ddots & \vdots \\ (D^2 f)(e_n, e_1) & \cdots & (D^2 f)(a)(e_n, e_n) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

The following proposition is an analogy of Proposition 5.39. The proof is similar to the one of Proposition 5.39, and is left as an exercise.

Proposition 5.54. *Let $U \subseteq \mathbb{R}^n$ be open, $a \in U$, and $f = (f_1, \cdots, f_m) : U \to \mathbb{R}^m$. Then f is k-times differentiable at a if and only if f_i is k-times differentiable at a for all $i = 1, \cdots, m$.*

Due to the proposition above, when talking about the higher-order differentiability of $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and a point $a \in U$, from now on we only focus on the case $m = 1$.

Example 5.55. In this example, we focus on what the second derivative $(D^2 f)(a)$ of a function f is, or in particular, what $(D^2 f)(a)(e_i, e_j)$ (which appears in the Remark 5.53) is, if $X = \mathbb{R}^2$.

Let $f:\mathbb{R}^2\to\mathbb{R}$ be differentiable, then

$$\big[(Df)(x,y)\big]=\big[f_x(x,y)\quad f_y(x,y)\big]=\left[\frac{\partial f}{\partial x}(x,y)\quad \frac{\partial f}{\partial y}(x,y)\right].$$

Suppose that f is twice differentiable at (a,b), and let $L_2=(D^2f)(a,b)$. Then

$$\lim_{(x,y)\to(a,b)}\frac{\big\|(Df)(x,y)-(Df)(a,b)-L_2\big((x-a,y-b)\big)\big\|_{\mathscr{B}(\mathbb{R}^2,\mathbb{R})}}{\sqrt{(x-a)^2+(y-b)^2}}=0$$

or equivalently,

$$\lim_{(x,y)\to(a,b)}\frac{\big\|\,\big[f_x(x,y)\quad f_y(x,y)\big]-\big[f_x(a,b)\quad f_y(a,b)\big]-\big[L_2\big((x-a,y-b)\big)\big]\big\|_{\mathscr{B}(\mathbb{R}^2,\mathbb{R})}}{\sqrt{(x-a)^2+(y-b)^2}}=0\,,$$

where $\big[L_2\big((x-a,y-b)\big)\big]$ denotes the matrix representation of the linear map $L_2\big((x-a,y-b)\big)\in\mathscr{B}(\mathbb{R}^2,\mathbb{R})$. In particular, we must have

$$\lim_{x\to a}\left\|\left[\frac{f_x(x,b)-f_x(a,b)}{x-a}\quad \frac{f_y(x,b)-f_y(a,b)}{x-a}\right]-\big[L_2\mathrm{e}_1\big]\right\|_{\mathscr{B}(\mathbb{R}^2,\mathbb{R})}=0$$

and

$$\lim_{y\to b}\left\|\left[\frac{f_x(a,y)-f_x(a,b)}{y-b}\quad \frac{f_y(a,y)-f_y(a,b)}{y-b}\right]-\big[L_2\mathrm{e}_2\big]\right\|_{\mathscr{B}(\mathbb{R}^2,\mathbb{R})}=0\,.$$

Using the notation of second partial derivatives, we find that

$$\big[L_2\mathrm{e}_1\big]=\big[f_{xx}(a,b)\quad f_{yx}(a,b)\big]\quad\text{and}\quad\big[L_2\mathrm{e}_2\big]=\big[f_{xy}(a,b)\quad f_{yy}(a,b)\big]\,,$$

where $f_{xy}=(f_x)_y=\dfrac{\partial}{\partial y}\Big(\dfrac{\partial f}{\partial x}\Big)$ and $f_{yx}=(f_y)_x=\dfrac{\partial}{\partial x}\Big(\dfrac{\partial f}{\partial y}\Big)$. Therefore, if $v=v_1\mathrm{e}_1+v_2\mathrm{e}_2$,

$$[L_2v]=\big[L_2(v_1\mathrm{e}_1+v_2\mathrm{e}_2)\big]=\big[v_1f_{xx}(a,b)+v_2f_{xy}(a,b)\quad v_1f_{yx}(a,b)+v_2f_{yy}(a,b)\big]\,.\qquad(5.6.1)$$

Symbolically, we can write

$$\big[L_2\big]=\Big[\,\big[f_{xx}(a,b)\quad f_{yx}(a,b)\big]\quad\big[f_{xy}(a,b)\quad f_{yy}(a,b)\big]\,\Big]$$

so that

$$\begin{aligned}\big[L_2(v_1\mathrm{e}_1+v_2\mathrm{e}_2)\big]&=\big[L_2\big]\begin{bmatrix}v_1\\v_2\end{bmatrix}=\Big[\,\big[f_{xx}(a,b)\quad f_{yx}(a,b)\big]\quad\big[f_{xy}(a,b)\quad f_{yy}(a,b)\big]\,\Big]\begin{bmatrix}v_1\\v_2\end{bmatrix}\\&=v_1\,\big[f_{xx}(a,b)\quad f_{yx}(a,b)\big]+v_2\,\big[f_{xy}(a,b)\quad f_{yy}(a,b)\big]\,.\end{aligned}$$

For two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, what does $(D^2f)(a,b)(\boldsymbol{v})(\boldsymbol{u})$ or $(D^2f)(a,b)(\boldsymbol{u},\boldsymbol{v})$ mean? To see this, let $\boldsymbol{u} = u_1\mathrm{e}_1 + u_2\mathrm{e}_2$ and $\boldsymbol{v} = v_1\mathrm{e}_1 + v_2\mathrm{e}_2$. Then

$$\begin{aligned}\big[(D^2f)(a,b)(\boldsymbol{v})(\boldsymbol{u})\big] &= \big[(D^2f)(a,b)(\boldsymbol{v})\big]\big[\boldsymbol{u}\big] = \big[L_2(v_1\mathrm{e}_1+v_2\mathrm{e}_2)\big]\begin{bmatrix}u_1\\u_2\end{bmatrix}\\ &= v_1\begin{bmatrix}f_{xx}(a,b) & f_{yx}(a,b)\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} + v_2\begin{bmatrix}f_{xy}(a,b) & f_{yy}(a,b)\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}\\ &= \begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}f_{xx}(a,b) & f_{yx}(a,b)\\ f_{xy}(a,b) & f_{yy}(a,b)\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}.\end{aligned}$$

Therefore, $(D^2f)(a,b)(\mathrm{e}_1,\mathrm{e}_1) = f_{xx}(a,b)$, $(D^2f)(a,b)(\mathrm{e}_1,\mathrm{e}_2) = f_{xy}(a,b)$, $(D^2f)(a,b)(\mathrm{e}_2,\mathrm{e}_1) = f_{yx}(a,b)$ and $(D^2f)(a,b)(\mathrm{e}_2,\mathrm{e}_2) = f_{yy}(a,b)$.

On the other hand, we can identify $\mathscr{B}(\mathbb{R}^2;\mathbb{R})$ as $\mathbb{R}^2$ (every 1×2 matrix is a "row" vector), and treat $g \equiv \big[Df\big]^{\mathrm{T}} : \mathbb{R}^2 \to \mathbb{R}^2$ as a vector-valued function. By Theorem 5.27 $(Dg)(a,b)$ can be represented as a 2×2 matrix given by

$$\big[(Dg)(a,b)\big] = \begin{bmatrix}f_{xx}(a,b) & f_{xy}(a,b)\\ f_{yx}(a,b) & f_{yy}(a,b)\end{bmatrix}.$$

We note that the representation above means

$$\lim_{(x,y)\to(a,b)} \frac{\left\|\begin{bmatrix}f_x(x,y)\\f_y(x,y)\end{bmatrix} - \begin{bmatrix}f_x(a,b)\\f_y(a,b)\end{bmatrix} - \begin{bmatrix}f_{xx}(a,b) & f_{xy}(a,b)\\ f_{yx}(a,b) & f_{yy}(a,b)\end{bmatrix}\begin{bmatrix}x-a\\y-b\end{bmatrix}\right\|_{\mathbb{R}^2}}{\sqrt{(x-a)^2+(y-b)^2}} = 0\,.$$

The equality above is equivalent to that

$$\lim_{(x,y)\to(a,b)} \frac{\left\|\big[(Df)(x,y)\big] - \big[(Df)(a,b)\big] - \begin{bmatrix}x-a & y-b\end{bmatrix}\begin{bmatrix}f_{xx}(a,b) & f_{yx}(a,b)\\ f_{xy}(a,b) & f_{yy}(a,b)\end{bmatrix}\right\|_{\mathbb{R}^2}}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

According to the equality above, $L_2 = (D^2f)(a,b)$ should be defined by

$$\big[L_2(v_1\mathrm{e}_1+v_2\mathrm{e}_2)\big] = \begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}f_{xx}(a,b) & f_{yx}(a,b)\\ f_{xy}(a,b) & f_{yy}(a,b)\end{bmatrix} = \left(\begin{bmatrix}f_{xx}(a,b) & f_{xy}(a,b)\\ f_{yx}(a,b) & f_{yy}(a,b)\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)^{\mathrm{T}}$$

which agrees with what (5.6.1) provides.

Proposition 5.56. *Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$. Suppose that f is k-times differentiable at a. Then for k vectors $u^{(1)},\cdots,u^{(k)} \in \mathbb{R}^n$,*

$$(D^kf)(a)(u^{(1)},\cdots,u^{(k)}) = \sum_{j_1,\cdots,j_k=1}^{n} \frac{\partial^k f}{\partial x_{j_k}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a)u^{(1)}_{j_1}u^{(2)}_{j_2}\cdots u^{(k)}_{j_k}\,,$$

where $u^{(i)} = (u^{(i)}_1, u^{(i)}_2, \cdots, u^{(i)}_n)$ *for all* $i = 1, \cdots, k$（上標括號中的數字指所給定的 k 個向量中的第幾個向量，下標指每一個固定向量的第幾個分量）*and*

$$\frac{\partial^k f}{\partial x_{j_k} \partial x_{j_{k-1}} \cdots \partial x_{j_1}}(a) = \frac{\partial}{\partial x_{j_k}}\Big|_{x=a}\Big(\frac{\partial}{\partial x_{j_{k-1}}}\Big(\cdots \frac{\partial}{\partial x_{j_2}}\Big(\frac{\partial f}{\partial x_{j_1}}\Big)\cdots\Big)\Big).$$

Proof. We prove the proposition by induction. Let $\{\mathrm{e}_j\}_{j=1}^n$ be the standard basis of $\mathbb{R}^n$. By Remark 5.53 (on multi-linearity), it suffices to show that

$$(D^k f)(a)(\mathrm{e}_{j_k})(\mathrm{e}_{j_{k-1}})\cdots(\mathrm{e}_{j_2})(\mathrm{e}_{j_1}) = (D^k f)(a)(\mathrm{e}_{j_1}, \cdots, \mathrm{e}_{j_k}) = \frac{\partial^k f}{\partial x_{j_k} \partial x_{j_{k-1}} \cdots \partial x_{j_1}}(a) \tag{5.6.2}$$

provided that f is k-times differentiable at a since if so, we must have

$$\begin{aligned}
(D^k f)(a)(u^{(1)}, \cdots, u^{(k)}) &= (D^k f)(a)\Big(\sum_{j_1=1}^n u^{(1)}_{j_1}\mathrm{e}_{j_1}, \cdots, \sum_{j_k=1}^n u^{(k)}_{j_k}\mathrm{e}_{j_k}\Big) \\
&= \sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_k=1}^n (D^k f)(a)(\mathrm{e}_{j_1}, \cdots, \mathrm{e}_{j_k}) u^{(1)}_{j_1} u^{(2)}_{j_2} \cdots u^{(k)}_{j_k} \\
&= \sum_{j_1, \cdots, j_k=1}^n \frac{\partial^k f}{\partial x_{j_k} \partial x_{j_{k-1}} \cdots \partial x_{j_1}}(a) u^{(1)}_{j_1} u^{(2)}_{j_2} \cdots u^{(k)}_{j_k}.
\end{aligned}$$

Note that the case $k = 1$ is true because of Theorem 5.27. Next we assume that (5.6.2) holds true for $k = \ell$ if f is $(\ell - 1)$-times differentiable in a neighborhood of a and f is ℓ-times differentiable at a. Now we show that (5.6.2) also holds true for $k = \ell + 1$ if f is ℓ-times differentiable in a neighborhood of a, and f is $(\ell + 1)$-times differentiable at a. By the definition of $(\ell + 1)$-times differentiability at a,

$$\lim_{x \to a} \frac{\big\|(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1} f)(a)(x - a)\big\|_{\mathscr{B}(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n, \cdots, \mathscr{B}(\mathbb{R}^n, \mathbb{R})\cdots))}}{\|x - a\|_{\mathbb{R}^n}} = 0.$$

Since

$$\begin{aligned}
&\Big|\big[(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1} f)(a)(x - a)\big](\mathrm{e}_{j_\ell})\cdots(\mathrm{e}_{j_2})(\mathrm{e}_{j_1})\Big| \\
&\quad \leqslant \Big\|\big[(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1} f)(a)(x - a)\big](\mathrm{e}_{j_\ell})\cdots(\mathrm{e}_{j_2})\Big\|_{\mathscr{B}(\mathbb{R}^n, \mathbb{R})} \|\mathrm{e}_{j_1}\|_{\mathbb{R}^n} \\
&\quad \leqslant \big\|(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1} f)(a)(x - a)\big\|_{\mathscr{B}(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n, \cdots, \mathscr{B}(\mathbb{R}^n, \mathbb{R})\cdots))} \|\mathrm{e}_{j_1}\|_{\mathbb{R}^n} \cdots \|\mathrm{e}_{j_\ell}\|_{\mathbb{R}^n} \\
&\quad = \big\|(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1} f)(a)(x - a)\big\|_{\mathscr{B}(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n, \cdots, \mathscr{B}(\mathbb{R}^n, \mathbb{R})\cdots))},
\end{aligned}$$

using (5.6.2) (for the case $k = \ell$) we conclude that for $x \neq a$,

$$\begin{aligned}
&\frac{\Big|\dfrac{\partial^\ell f}{\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(x) - \dfrac{\partial^\ell f}{\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a) - (D^{\ell+1}f)(a)(\mathrm{e}_{j_1},\cdots,\mathrm{e}_{j_\ell},x-a)\Big|}{\|x-a\|_{\mathbb{R}^n}} \\
&\quad = \frac{\big|(D^\ell f)(x)(\mathrm{e}_{j_1},\cdots,\mathrm{e}_{j_\ell}) - (D^\ell f)(a)(\mathrm{e}_{j_1},\cdots,\mathrm{e}_{j_\ell}) - (D^{\ell+1}f)(a)(x-a)(\mathrm{e}_{j_1},\cdots,\mathrm{e}_{j_\ell})\big|}{\|x-a\|_{\mathbb{R}^n}} \\
&\quad \leqslant \frac{\big\|(D^\ell f)(x) - (D^\ell f)(a) - (D^{\ell+1}f)(a)(x-a)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathscr{B}(\mathbb{R}^n,\cdots,\mathscr{B}(\mathbb{R}^n,\mathbb{R})\cdots))}}{\|x-a\|_{\mathbb{R}^n}}
\end{aligned}$$

and the right-hand side approaches zero as $x \to a$ so that

$$\lim_{x\to a}\frac{\Big|\dfrac{\partial^\ell f}{\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(x) - \dfrac{\partial^\ell f}{\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a) - (D^{\ell+1}f)(a)(\mathrm{e}_{j_1},\cdots,\mathrm{e}_{j_\ell},x-a)\Big|}{\|x-a\|_{\mathbb{R}^n}} = 0\,.$$

In particular, we pass to the limit as $x \to a$ in the way $x = a + t\mathrm{e}_{j_{\ell+1}}$ as $t \to 0$ for some $j_{\ell+1} = 1,\cdots,n$ and conclude from the definition of partial derivatives that

$$\begin{aligned}
(D^{\ell+1}f)(a)(\mathrm{e}_{j_1},\cdots,\mathrm{e}_{j_\ell},\mathrm{e}_{j_{\ell+1}}) &= \lim_{t\to 0}\frac{\dfrac{\partial^\ell f}{\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a+t\mathrm{e}_{j_{\ell+1}}) - \dfrac{\partial^\ell f}{\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a)}{t} \\
&= \frac{\partial^{\ell+1} f}{\partial x_{j_{\ell+1}}\partial x_{j_\ell}\partial x_{j_{k-1}}\cdots\partial x_{j_1}}(a)
\end{aligned}$$

which is (5.6.2) for the case $k = \ell + 1$. □

Example 5.57. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1, x_2) = x_1^2\cos x_2$, and $u^{(1)} = (2,0)$, $u^{(2)} = (1,1)$, $u^{(3)} = (0,-1)$. Suppose that f is three-times differentiable at $a = (0,0)$ (in fact it is, but we have not talked about this yet). Then

$$\begin{aligned}
(D^3 f)(a)(u^{(1)},u^{(2)},u^{(3)}) &= \sum_{i,j,k=1}^{2}\frac{\partial^3 f}{\partial x_k\partial x_j\partial x_i}(a)u_i^{(1)}u_j^{(2)}u_k^{(3)} = \sum_{j=1}^{2}\frac{\partial^3 f}{\partial x_2\partial x_j\partial x_1}(a)\cdot 2\cdot u_j^{(2)}\cdot(-1) \\
&= \frac{\partial^3 f}{\partial x_2\partial x_1^2}(0,0)\cdot 2\cdot 1\cdot(-1) + \frac{\partial^3 f}{\partial x_2^2\partial x_1}(0,0)\cdot 2\cdot 1\cdot(-1) = 0\,.
\end{aligned}$$

Corollary 5.58. *Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$ be $(k+1)$-times differentiable at a. Then for $u^{(1)},\cdots,u^{(k)},u^{(k+1)} \in \mathbb{R}^n$,*

$$(D^{k+1}f)(a)(u^{(1)},\cdots,u^{(k)},u^{(k+1)}) = \sum_{j=1}^{n}u_j^{(k+1)}\frac{\partial}{\partial x_j}\Big|_{x=a}(D^k f)(x)(u^{(1)},\cdots,u^{(k)})\,.$$

In other words, (using the terminology in Remark 5.15) $(D^{k+1}f)(a)(u^{(1)},\cdots,u^{(k)},u^{(k+1)})$ is the "directional derivative" of the function $(D^k f)(\cdot)(u^{(1)},\cdots,u^{(k)})$ at a in the "direction" $u^{(k+1)}$.

Proof. By Proposition 5.56,

$$
\begin{aligned}
(D^{k+1}f)(a)(u^{(1)},\cdots,u^{(k)},u^{(k+1)}) &= \sum_{j_1,\cdots,j_k,j_{k+1}=1}^{n} \frac{\partial^{k+1} f}{\partial x_{j_{k+1}}\partial x_{j_k}\cdots\partial x_{j_1}}(a)u^{(1)}_{j_1}\cdots u^{(k)}_{j_k}u^{(k+1)}_{j_{k+1}} \\
&= \sum_{j_{k+1}=1}^{n} u^{(k+1)}_{j_{k+1}} \sum_{j_1,\cdots,j_k=1}^{n} \frac{\partial^{k+1} f}{\partial x_{j_{k+1}}\partial x_{j_k}\cdots\partial x_{j_1}}(a)u^{(1)}_{j_1}\cdots u^{(k)}_{j_k} \\
&= \sum_{j_{k+1}=1}^{n} u^{(k+1)}_{j_{k+1}} \frac{\partial}{\partial x_{j_{k+1}}}\Big|_{x=a} \sum_{j_1,\cdots,j_k=1}^{n} \frac{\partial^{k} f}{\partial x_{j_k}\cdots\partial x_{j_1}}(x)u^{(1)}_{j_1}\cdots u^{(k)}_{j_k} \\
&= \sum_{j_{k+1}=1}^{n} u^{(k+1)}_{j_{k+1}} \frac{\partial}{\partial x_{j_{k+1}}}\Big|_{x=a} (D^k f)(x)(u^{(1)},\cdots,u^{(k)})\,. \qquad \square
\end{aligned}
$$

Example 5.59. Let $f:\mathbb{R}^2\to\mathbb{R}$ be twice differentiable at $a=(a_1,a_2)\in\mathbb{R}^2$. Then the proposition above shows that for $u=(u_1,u_2), v=(v_1,v_2)\in\mathbb{R}^2$,

$$
\begin{aligned}
(D^2f)(a)(v)(u) = (D^2f)(a)(u,v) &= \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial x_j\partial x_i}(a)u_iv_j \\
&= \frac{\partial^2 f}{\partial x_1^2}(a)u_1v_1 + \frac{\partial^2 f}{\partial x_2\partial x_1}(a)u_1v_2 + \frac{\partial^2 f}{\partial x_1\partial x_2}(a)u_2v_1 + \frac{\partial^2 f}{\partial x_2^2}(a)u_2v_2 \\
&= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(a) & \dfrac{\partial^2 f}{\partial x_2\partial x_1}(a) \\ \dfrac{\partial^2 f}{\partial x_1\partial x_2}(a) & \dfrac{\partial^2 f}{\partial x_2^2}(a) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.
\end{aligned}
$$

In general, if $f:\mathbb{R}^n\to\mathbb{R}$ be twice differentiable at $a=(a_1,\cdots,a_n)\in\mathbb{R}^n$. Then for $u=(u_1,\cdots,u_n), v=(v_1,\cdots,v_n)\in\mathbb{R}^n$,

$$
(D^2f)(a)(v)(u) = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(a) & \cdots & \dfrac{\partial^2 f}{\partial x_n\partial x_1}(a) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_1\partial x_n}(a) & \cdots & \dfrac{\partial^2 f}{\partial x_n^2}(a) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.
$$

The bilinear form $B:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ given by

$$
B(u,v) = (D^2f)(a)(v)(u) \qquad \forall\, u,v\in\mathbb{R}^n
$$

is called the ***Hessian*** of f, and is represented (in the matrix form) as an $n \times n$ matrix by

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(a) & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_1}(a) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(a) & \cdots & \dfrac{\partial^2 f}{\partial x_n^2}(a) \end{bmatrix}.$$

If the second partial derivatives $\dfrac{\partial^2 f}{\partial x_j \partial x_i}(a)$ of f at a exists for all $i, j = 1, \cdots, n$ (here the twice differentiability of f at a is ignored), the matrix (on the right-hand side of equality) above is also called the ***Hessian matrix*** of f at a.

Even though there is no reason to believe that $(D^2 f)(a)(u, v) = (D^2 f)(a)(v, u)$ (since the left-hand side means first differentiating f in u-direction and then differentiating Df in v-direction, while the right-hand side means first differentiating f in v-direction then differentiating Df in u-direction), it is still reasonable to ask whether $(D^2 f)(a)$ is symmetric or not; that is, could it be true that $(D^2 f)(a)(u, v) = (D^2 f)(a)(v, u)$ for all $u, v \in \mathbb{R}^n$? When f is twice differentiable at a, this is equivalent of asking (by plugging in $u = \mathrm{e}_i$ and $v = \mathrm{e}_j$) that whether or not

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a) . \tag{5.6.3}$$

The following example provides a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that (5.6.3) does not hold at $a = (0, 0)$. We remark that the function in the following example is not twice differentiable at a even though the Hessian matrix of f at a can still be computed.

Example 5.60. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) , \\ 0 & \text{if } (x, y) = (0, 0) . \end{cases}$$

Then

$$f_x(x, y) = \begin{cases} \dfrac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) , \\ 0 & \text{if } (x, y) = (0, 0) , \end{cases}$$

and

$$f_y(x, y) = \begin{cases} \dfrac{x^5 - 4x^3 y^2 - xy^4}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) , \\ 0 & \text{if } (x, y) = (0, 0) , \end{cases}$$

It is clear that f_x and f_y are continuous on $\mathbb{R}^2$; thus f is differentiable on $\mathbb{R}^2$. However,

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = -1\,,$$

while

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = 1\,;$$

thus the Hessian matrix of f at the origin is not symmetric.

Theorem 5.61 (Clairaut's Theorem)**.** *Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$. Suppose that the mixed partial derivatives $\dfrac{\partial f}{\partial x_i}$, $\dfrac{\partial f}{\partial x_j}$, $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$, $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ exist in a neighborhood of a, and $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ is continuous at a. Then*

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a)\,. \tag{5.6.4}$$

Proof. Let $a \in U$ be given. For real numbers $h, k \neq 0$ such that $a + h\mathrm{e}_i + k\mathrm{e}_j \in U$, define

$$Q(h,k) \equiv \frac{f(a + h\mathrm{e}_i + k\mathrm{e}_j) - f(a + h\mathrm{e}_i) - f(a + k\mathrm{e}_j) + f(a)}{hk}\,.$$

Then

$$\begin{aligned}\lim_{k\to 0} Q(h,k) &= \frac{1}{h} \lim_{k\to 0} \Big(\frac{f(a + h\mathrm{e}_i + k\mathrm{e}_j) - f(a + h\mathrm{e}_i)}{k} - \frac{f(a + k\mathrm{e}_j) + f(a)}{k} \Big) \\ &= \frac{1}{h} \Big(\frac{\partial f}{\partial x_j}(a + h\mathrm{e}_j) - \frac{\partial f}{\partial x_j}(a) \Big)\,;\end{aligned}$$

thus

$$\lim_{h\to 0} \lim_{k\to 0} Q(h,k) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a)\,. \tag{5.6.5}$$

Define $\varphi(x) = f(x + k\mathrm{e}_j) - f(x)$. Then the mean value theorem implies that

$$\begin{aligned}Q(h,k) = \frac{\varphi(a + h\mathrm{e}_i) - \varphi(a)}{hk} &= \frac{1}{k} \frac{\partial \varphi}{\partial x_i}(a + \theta_1 h\mathrm{e}_i) \\ &= \frac{1}{k} \Big(\frac{\partial f}{\partial x_i}(a + \theta_1 h\mathrm{e}_i + k\mathrm{e}_j) - \frac{\partial f}{\partial x_i}(a + \theta_1 h\mathrm{e}_i) \Big) \\ &= \frac{\partial^2 f}{\partial x_j \partial x_i}(a + \theta_1 h\mathrm{e}_i + \theta_2 k\mathrm{e}_j)\end{aligned}$$

for some functions $\theta_1 = \theta_1(h,k)$ and $\theta_2 = \theta_2(h,k)$ satisfying $0 < \theta_1, \theta_2 < 1$. Therefore, we establish that there exist functions $\theta_1 = \theta_1(h,k)$ and $\theta_2 = \theta_2(h,k)$ such that $\theta_1, \theta_2 \in (0,1)$ and

$$Q(h,k) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a + \theta_1 h\mathrm{e}_i + \theta_2 k\mathrm{e}_j)\,.$$

Passing to the limit as $k \to 0$ first then $h \to 0$, using (5.6.5) and the continuity of $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ we conclude that

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \lim_{h\to 0}\lim_{k\to 0} Q(h,k) = \lim_{h\to 0}\lim_{k\to 0} \frac{\partial^2 f}{\partial x_j \partial x_i}(a + \theta_1 h\mathrm{e}_i + \theta_2 k\mathrm{e}_j) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a)\,. \qquad \square$$

Remark 5.62. In view of Remark 5.52, (5.6.4) is the same as the following identity

$$\begin{aligned}&\lim_{h\to 0}\lim_{k\to 0} \frac{f(a+h\mathrm{e}_i+k\mathrm{e}_j) - f(a+h\mathrm{e}_i) - f(a+k\mathrm{e}_j) + f(a)}{hk} \\ &\qquad = \lim_{k\to 0}\lim_{h\to 0} \frac{f(a+h\mathrm{e}_i+k\mathrm{e}_j) - f(a+h\mathrm{e}_i) - f(a+k\mathrm{e}_j) + f(a)}{hk}\end{aligned}$$

which implies that the order of the two limits $\lim\limits_{h\to 0}$ and $\lim\limits_{k\to 0}$ can be interchanged without changing the value of the limit (under certain conditions).

Example 5.63. Let $f(x,y) = yx^2 \cos y^2$. Then

$$\begin{aligned} f_{xy}(x,y) &= (2xy\cos y^2)_y = 2x\cos y^2 - 2xy(2y)\sin y^2 = 2x\cos y^2 - 4xy^2\sin y^2\,,\\ f_{yx}(x,y) &= (x^2\cos y^2 - yx^2(2y)\sin y^2)_x = (x^2\cos y^2 - 2x^2y^2\sin y^2)_x \\ &= 2x\cos y^2 - 4xy^2\sin y^2 = f_{xy}(x,y)\,. \end{aligned}$$

Definition 5.64. A function is said to be ***of class*** $\mathscr{C}^k$ if the first k derivatives exist and are continuous. A function is said to be ***smooth*** or ***of class*** $\mathscr{C}^\infty$ if it is of class $\mathscr{C}^k$ for all positive integer k.

Now we would like to answer the question of what kind of functions are k-times differentiable. Suppose that $U \subseteq \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$. Note that by the definition of differentiability, f is k-times differentiable in U if and only if $D^{k-1}f$ is differentiable in U. This would further imply that f is k-times differentiable in U if and only if $D^{k-2}f$ is twice differentiable in U. Therefore, Proposition 5.39 and Theorem 5.44 imply that

$$
\begin{aligned}
& f \text{ is } k\text{-times (continuously) differentiable in } U \\
& \quad \Leftrightarrow Df \text{ is } (k-1)\text{-times (continuously) differentiable in } U \\
& \quad \Leftrightarrow \Big[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\Big] \text{ is } (k-1)\text{-times (continuously) differentiable in } U \\
& \quad \Leftrightarrow \frac{\partial f}{\partial x_{j_1}} \text{ is } (k-1)\text{-times (continuously) differentiable in } U \text{ for all } 1 \leqslant j_1 \leqslant n \\
& \quad \Leftrightarrow D\frac{\partial f}{\partial x_{j_1}} \text{ is } (k-2)\text{-times (continuously) differentiable in } U \text{ for all } 1 \leqslant j_1 \leqslant n \\
& \quad \Leftrightarrow \Big[\frac{\partial^2 f}{\partial x_1 \partial x_{j_1}}, \cdots, \frac{\partial^2 f}{\partial x_n \partial x_{j_1}}\Big] \text{ is } (k-2)\text{-times (continuously) differentiable in } U \\
& \qquad \text{for all } 1 \leqslant j_1 \leqslant n \\
& \quad \Leftrightarrow \frac{\partial^2 f}{\partial x_{j_2} \partial x_{j_1}} \text{ is } (k-2)\text{-times (continuously) differentiable in } U \text{ for all } 1 \leqslant j_1, j_2 \leqslant n\,.
\end{aligned}
$$

Applying similar argument several times, we obtain the following theorem which is an analogy of Theorem 5.44.

Theorem 5.65. *Let $U \to \mathbb{R}^n$ and $f : U \to \mathbb{R}$. Suppose that the partial derivative $\dfrac{\partial^k f}{\partial x_{j_k} \partial x_{j_{k-1}} \cdots \partial x_{j_1}}$ exists in a neighborhood of $a \in U$ and is continuous at a for all $j_1, \cdots, j_k = 1, \cdots, n$. Then f is k-times differentiable at a. Moreover, $\dfrac{\partial^k f}{\partial x_{j_k} \partial x_{j_{k-1}} \cdots \partial x_{j_1}}$ is continuous on U if and only if f is of class $\mathscr{C}^k$.*

Corollary 5.66. *Let $U \subseteq \mathbb{R}^n$ be open, and f is of class $\mathscr{C}^2$. Then*

$$(D^2 f)(a)(u,v) = (D^2 f)(a)(v,u) \qquad \forall\, a \in U \text{ and } u, v \in \mathbb{R}^n\,.$$

5.7 Taylor's Theorem

Recall Taylor's Theorem for functions of one variable:

Theorem 5.67 (Taylor)**.** *Suppose that for some $k \in \mathbb{N}$, $f : (a,b) \to \mathbb{R}$ be $(k+1)$-times differentiable and $c \in (a,b)$. Then for all $x \in (a,b)$, there exists d in between c and x such that*

$$f(x) = \sum_{\ell=0}^{k} \frac{f^{(\ell)}(c)}{\ell!}(x-c)^\ell + \frac{f^{(k+1)}(d)}{(k+1)!}(x-c)^{(k+1)},$$

where $f^{(\ell)}$ denotes the ℓ-th derivative of f.

Theorem 5.68 (Taylor)**.** *Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$ be $(k+1)$-times differentiable. Suppose that $x, a \in U$ and the line segment joining x and a lies in U. Then there exists a point c on the line segment joining x and a such that*

$$\begin{aligned} f(x) - f(a) = & \sum_{\ell=1}^{k} \frac{1}{\ell!}(D^\ell f)(a)(\overbrace{x-a, \cdots, x-a}^{\ell \text{ copies of } x-a}) \\ & + \frac{1}{(k+1)!}(D^{k+1} f)(c)(\underbrace{x-a, \cdots, x-a}_{(k+1) \text{ copies of } x-a}) . \end{aligned} \tag{5.7.1}$$

Proof. Let $g(t) = f\big((1-t)a + tx\big)$. Since $\overline{xa} \subseteq U$ and U is open, there exists $\delta > 0$ such that $(1-t)a + tx \in U$ for all $t \in (-\delta, 1+\delta)$. By the chain rule, for $t \in (-\delta, 1+\delta)$,

$$g'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big((1-t)a + tx\big)(x_i - a_i) = (Df)\big((1-t)a + tx\big)(x-a) ;$$

thus for $t \in (-\delta, 1+\delta)$, Proposition 5.56 shows that

$$g''(t) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}\big((1-t)a + tx\big)(x_i - a_i)(x_j - a_j) = (D^2 f)\big((1-t)a + tx\big)(x-a, x-a).$$

By induction, we conclude that

$$g^{(\ell)}(t) = (D^\ell f)\big((1-t)a + tx\big)(\underbrace{x-a, \cdots, x-a}_{\ell \text{ copies of } x-a}) .$$

By the fact that f is $(k+1)$-times differentiable, $g : (-\delta, 1+\delta) \to \mathbb{R}$ is $(k+1)$-times differentiable as well. Theorem 5.67 then implies that for some $t_0 \in (0,1)$,

$$g(1) - g(0) = \sum_{\ell=1}^{k} \frac{g^{(\ell)}(0)}{\ell!} + \frac{g^{(k+1)}(t_0)}{(k+1)!} . \tag{5.7.2}$$

Letting $c = (1-t_0)a + t_0 x$, (5.7.2) implies (5.7.1). □

Definition 5.69. Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$ be k-times differentiable. The k-th (order) Taylor polynomial for f at a is the polynomial

$$\sum_{\ell=0}^{k} \frac{1}{\ell!}(D^\ell f)(a)(\underbrace{x-a, \cdots, x-a}_{\ell \text{ copies } x-a}).$$

Corollary 5.70. *Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be $(k+1)$-times differentiable, and define the remainder*

$$R_k(a,h) = f(a+h) - \sum_{\ell=0}^{k} \frac{1}{\ell!}(D^\ell f)(a)(h,\cdots,h).$$

Then $\lim\limits_{h\to 0} \dfrac{R_k(a,h)}{\|h\|^k_{\mathbb{R}^n}} = 0$*, or in notation,* $R_k(a,h) = \mathscr{o}(\|h\|^k_{\mathbb{R}^n})$ *as* $h \to 0$.

Remark 5.71. An n-dimensional multi-index is a vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ of non-negative integers (that is, $\alpha_j \in \mathbb{N} \cup \{0\}$ for all $1 \leqslant j \leqslant n$). Given an n-dimensional multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha|$ and $\alpha!$ are numbers defined by

$$|\alpha| = \sum_{k=1}^{n} \alpha_k \quad \text{and} \quad \alpha! = \prod_{k=1}^{n} \alpha_k!\,,$$

and the differential operator D_x^α is defined by

$$D_x^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\,.$$

We also use D^α to denote D_x^α when the variable of differentiation is clear. For a vector $h = (h_1, \cdots, h_n) \in \mathbb{R}^n$ and an n-dimensional multi-index α, we use h^α to denote the number $h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_n^{\alpha_n}$.

Suppose that $f : U \to \mathbb{R}$ is $(k+1)$-times differentiable. Then $D^\ell f$ is continuous on U for $1 \leqslant \ell \leqslant k$; that is, f is of class $\mathscr{C}^k$; thus Theorem 5.65 implies that all the mixed partial derivatives $\dfrac{\partial^\ell f}{\partial x_{j_\ell} \partial x_{j_{\ell+1}} \cdots \partial x_{j_1}}$ are continuous on U. Therefore, the Clairaut Theorem shows that

$$(D^\ell f)(x)(h,\cdots,h) = \sum_{|\alpha|=\ell} \frac{|\alpha|!}{\alpha!}(D^\alpha f)(x) h^\alpha \qquad \forall\, x \in U\,, h \in \mathbb{R}^n\,,$$

and the Taylor Theorem further implies that

$$f(x) = \sum_{\ell=0}^{k} \sum_{|\alpha|=k} \frac{1}{\alpha!}(D^\alpha f)(a)(x-a)^\alpha + \sum_{|\alpha|=m+1} \frac{1}{\alpha!}(D^\alpha f)(c)(x-a)^\alpha\,.$$

Example 5.72. Let $f(x,y) = e^x \cos y$. Compute the fourth degree Taylor polynomial for f at $(0,0)$.

Solution: We compute the zeroth, the first, the second, the third and the fourth mixed derivatives of f at $(0,0)$ as follows:

$$\begin{aligned}
f(0,0) &= 1\,, \qquad f_x(0,0) = 1\,, \qquad f_y(0,0) = 0\,,\\
f_{xx}(0,0) &= 1\,, \qquad f_{xy}(0,0) = f_{yx}(0,0) = 0\,, \qquad f_{yy}(0,0) = -1\,,\\
f_{xxx}(0,0) &= 1\,, \qquad f_{xxy}(0,0) = f_{xyx}(0,0) = f_{yxx}(0,0) = 0\,,\\
f_{yyy}(0,0) &= 0\,, \qquad f_{yyx}(0,0) = f_{yxy}(0,0) = f_{xyy}(0,0) = -1\,,
\end{aligned}$$

and

$$\begin{aligned}
&f_{xxxx}(0,0) = 1\,, \qquad f_{yyyy}(0,0) = 1\,,\\
&f_{xxxy}(0,0) = f_{xxyx}(0,0) = f_{xyxx}(0,0) = f_{yxxx}(0,0) = 0\,,\\
&f_{xyyy}(0,0) = f_{yxyy}(0,0) = f_{yyxy}(0,0) = f_{yyyx}(0,0) = 0\,,\\
&f_{xxyy}(0,0) = f_{xyxy}(0,0) = f_{xyyx}(0,0) = f_{yxxy}(0,0) = f_{yxyx}(0,0) = f_{yyxx}(0,0) = -1\,.
\end{aligned}$$

Then the fourth order Taylor polynomial for f at $(0,0)$ is

$$\begin{aligned}
f(0,0) &+ f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\Big[f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\Big]\\
&+ \frac{1}{6}\Big[f_{xxx}(0,0)x^3 + 3f_{xxy}(0,0)x^2y + 3f_{xyy}(0,0)xy^2 + f_{yyy}(0,0)y^3\Big]\\
&+ \frac{1}{24}\Big[f_{xxxx}(0,0)x^4 + 4f_{xxxy}(0,0)x^3 + 6f_{xxyy}(0,0)x^2y^2\\
&\qquad + 4f_{xyyy}(0,0)xy^3 + f_{yyyy}(0,0)y^4\Big]\\
&= 1 + x + \frac{1}{2}\big(x^2 - y^2\big) + \frac{1}{6}\big(x^3 - 3xy^2\big) + \frac{1}{24}\big(x^4 - 6x^2y^2 + y^4\big)\,.
\end{aligned}$$

Observing that using the Taylor expansions

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots \quad \text{and} \quad \cos y = 1 - \frac{1}{2}y^2 + \frac{1}{24}y^4 + \cdots\,,$$

we can "formally" compute $e^x \cos y$ by multiplying the two "polynomials" above and obtain that

$$e^x \cos y \text{ ``=''}\ 1 + x + \frac{1}{2}\big(x^2 - y^2\big) + \big(\frac{1}{6}x^3 - \frac{1}{2}xy^2\big) + \big(\frac{1}{24}x^4 - \frac{1}{4}x^2y^2 + \frac{1}{24}y^2\big) + \text{h.o.t.}\,;$$

where h.o.t. stands for the higher order terms which are terms with fifth or higher degree.

Theorem 5.73. *Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$ be of class $\mathscr{C}^k$ and $(D^\ell f)(a) = 0$ for $\ell = 1, \cdots, k-1$. If $(D^k f)(a)(u, u, \cdots, u) > 0$ for all non-zero vectors $u \in \mathbb{R}^n$, then f has a local minimum at a; that is, there exists $\delta > 0$ such that*

$$f(x) \geqslant f(a) \qquad \forall\, x \in B(a, \delta)\,.$$

Proof. Let $a \in U$. Since U is open, there exists $r > 0$ such that $B(a,r) \subseteq U$. Note that $g : B(a,r) \times \mathbb{R}^n \to \mathbb{R}$ defined by $g(x,u) = (D^k f)(x)(u,\cdots,u)$ is continuous since

$$\begin{aligned}
\big|g(x,u) - g(y,v)\big| &= \big|(D^k f)(x)(u,\cdots,u) - (D^k f)(y)(v,\cdots,v)\big| \\
&\leqslant \big|(D^k f)(x)(u,\cdots,u) - (D^k f)(x)(v,\cdots,v)\big| + \Big|\big[(D^k f)(x) - (D^k f)(y)\big](v,\cdots,v)\Big| \\
&\leqslant \big|(D^k f)(x)(u,\cdots,u) - (D^k f)(x)(v,\cdots,v)\big| + \big|(D^k f)(x) - (D^k f)(y)\big\|\|v\|_{\mathbb{R}^n}^k \\
&\leqslant \big|(D^k f)(x)(u-v,u,\cdots,u)\big| + \big|(D^k f)(x)(v,u,\cdots,u) - (D^k f)(x)(v,\cdots,v)\big| \\
&\quad + \big|(D^k f)(x) - (D^k f)(y)\big\|\|v\|_{\mathbb{R}^n}^k \\
&\leqslant \big\|(D^k f)(x)\big\|\|u-v\|_{\mathbb{R}^n}\|u\|_{\mathbb{R}^n}^{k-1} + \big|(D^k f)(x)(v,u-v,u\cdots,u) - (D^k f)(x)(v,\cdots,v)\big| \\
&\quad + \big|(D^k f)(x)(v,v,u\cdots,u) - (D^k f)(x)(v,\cdots,v)\big| + \big|(D^k f)(x) - (D^k f)(y)\big\|\|v\|_{\mathbb{R}^n}^k \\
&\leqslant \cdots\cdots \\
&\leqslant \big\|(D^k f)(x)\big\|\|u-v\|_{\mathbb{R}^n}\big(\|u\|_{\mathbb{R}^n}^{k-1} + \|u\|_{\mathbb{R}^n}^{k-2}\|v\|_{\mathbb{R}^n} + \cdots + \|u\|_{\mathbb{R}^n}\|v\|_{\mathbb{R}^n}^{k-2} + \|v\|_{\mathbb{R}^n}^{k-1}\big) \\
&\quad + \big\|(D^k f)(x) - (D^k f)(y)\big\|\|v\|_{\mathbb{R}^n}^k
\end{aligned}$$

so that

$$\begin{aligned}
&\big|g(x,u) - g(y,v)\big| \\
&\qquad \leqslant \big\|(D^k f)(x)\big\|\big(\|u\|_{\mathbb{R}^n} + \|v\|_{\mathbb{R}^n}\big)^{k-1}\|u-v\|_{\mathbb{R}^n} + \big\|(D^k f)(x) - (D^k f)(y)\big\|\|v\|_{\mathbb{R}^n}^k
\end{aligned} \tag{5.7.3}$$

and the right-hand side approaches zero as $x \to y$ and $u \to v$. In particular, by the compactness of $\mathbb{S}^{n-1} \equiv \big\{x \in \mathbb{R}^n \,\big|\, \|x\| = 1\big\}(= B[0,1]\backslash B(0,1)$ which is closed and bounded), $g(a,\cdot)$ attains its minimum at some point $w \in \mathbb{S}^{n-1}$; that is,

$$g(a,u) \geqslant g(a,w) \qquad \forall\, u \in \mathbb{S}^{n-1}\,.$$

Let $\lambda = g(a,w) = (D^k f)(a)(w,\cdots,w) > 0$. Since f is of class $\mathscr{C}^k$, there exists $0 < \delta < r$ such that

$$\|(D^k f)(x) - (D^k f)(a)\| < \frac{\lambda}{2} \quad \text{whenever} \quad x \in B(a,\delta)\,.$$

Let $x \in B(a,\delta)\backslash\{a\}$ be given. By Taylor's Theorem there exists $c \in \overline{xa}$ (so that $c \in B(a,\delta)$) such that

$$f(x) = f(a) + \sum_{\ell=1}^{k-1} \frac{1}{\ell!}(D^\ell f)(a)(\overbrace{x-a,\cdots,x-a}^{\ell \text{ copies of } x-a}) + \frac{1}{k!}(D^k f)(c)(\overbrace{x-a,\cdots,x-a}^{k \text{ copies of } x-a})\,.$$

Since $(D^\ell f)(a)(u,u,\cdots,u) = 0$ for $1 \leqslant j \leqslant k-1$, we conclude that

$$f(x) = f(a) + \frac{1}{k!}(D^k f)(c)(x-a,x-a,\cdots,x-a) = f(a) + \frac{1}{k!}g(c,x-a)\,.$$

Note that (5.7.3) implies that

$$\Big|g\big(c,\frac{x-a}{\|x-a\|}\big)-g\big(a,\frac{x-a}{\|x-a\|}\big)\Big| \leqslant \big\|(D^kf)(c)-(D^kf)(a)\big\| < \frac{\lambda}{2}\,;$$

thus

$$g\big(c,\frac{x-a}{\|x-a\|}\big) > g\big(a,\frac{x-a}{\|x-a\|}\big)-\frac{\lambda}{2} \geqslant \frac{\lambda}{2}\,.$$

By the fact that $g(c,x-a)=g\big(c,\frac{x-a}{\|x-a\|}\big)\|x-a\|^k$, we conclude that

$$f(x) > f(a)+\frac{\lambda}{2k!}\|x-a\|^k \qquad \forall\, x\in B(a,\delta)\backslash\{a\}\,;$$

thus $f(x)\geqslant f(a)$ for all $x\in B(a,\delta)$. □

Corollary 5.74. *Let $U\subseteq\mathbb{R}^n$ be open, $a\in U$, and $f:U\to\mathbb{R}$ be of class $\mathscr{C}^2$. If $\dfrac{\partial f}{\partial x_\ell}(a)=0$ for all $1\leqslant\ell\leqslant n$ and the Hessian matrix of f at a is positive (cf. negative) definitive, then f has a local minimum (cf. maximum) at a.*

Definition 5.75. Let $U\subseteq\mathbb{R}^n$ be open. A function $f:U\to\mathbb{R}$ is said to be ***real analytic*** at $a\in U$ if $f(x)=\sum\limits_{k=0}^{\infty}\frac{1}{k!}(D^kf)(a)(x-a,\cdots,x-a)$ in a neighborhood of a.

Example 5.76. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}\exp\big(-\dfrac{1}{|x|^2}\big) & \text{if } x>0\,,\\ 0 & \text{if } x<0\,.\end{cases}$$

Then f is of class $\mathscr{C}^\infty$, and $f^{(k)}(0)=0$ for all $k\in\mathbb{N}$. Therefore, f is not real analytic at 0.

Chapter 6

Integration of Functions

6.1 Integrable Functions

In this chapter, we discuss the integration of (bounded) real-valued functions defined on bounded sets. We first recall the integral of functions of one variable that we learned from Calculus.

Definition 6.1. A finite set $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is called a partition of the closed interval $[a, b]$ if $a = x_0 < x_1 < \cdots < x_n = b$. Such a partition $\mathcal{P}$ is usually denoted by $\{a = x_0 < x_1 < \cdots < x_n = b\}$. The norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is the number $\max\{x_i - x_{i-1} \,\big|\, 1 \leqslant i \leqslant n\}$.

Let $f : [a, b] \to \mathbb{R}$ be a function. A Riemann sum of f for the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ is a sum which takes the form

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}),$$

where $\xi_k \in [x_{k-1}, x_k]$ for each $1 \leqslant k \leqslant n$. f is said to be Riemann integrable on $[a, b]$ if there exists a real number A such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums for the partition $\mathcal{P}$ belongs to the interval $(A - \varepsilon, A + \varepsilon)$. Such a number A (is unique and) is called the Riemann integral of f on $[a, b]$ and is denoted by $\displaystyle\int_{[a,b]} f(x)\, dx$.

To define the partition of a bounded set A in $\mathbb{R}^n$, we start with the simplest case $n = 1$.

Definition 6.2. Let $A \subseteq \mathbb{R}$ be a bounded set. A collection of point $\mathcal{P} = \{x_0, x_1, \cdots, x_N\}$ is called a partition of A if $\mathcal{P}$ is a partition of the closed interval $[\inf A, \sup A]$. Such a partition $\mathcal{P}$ is usually denoted by $\{[x_k, x_{k+1}] \,|\, 0 \leqslant k \leqslant N-1\}$, and the norm of $\mathcal{P}$ is the maximum of the length of intervals in $\mathcal{P}$; that is,

$$\|\mathcal{P}\| \equiv \max\left\{x_k - x_{k-1} \,\middle|\, 1 \leqslant k \leqslant N\right\}.$$

Next we look at how a partition of a bounded set in the plane is defined.

Definition 6.3. Let $A \subseteq \mathbb{R}^2$ be a bounded set. Define

$$\begin{aligned}
a_1 &= \inf\left\{x \in \mathbb{R} \,\middle|\, (x,y) \in A \text{ for some } y \in \mathbb{R}\right\},\\
b_1 &= \sup\left\{x \in \mathbb{R} \,\middle|\, (x,y) \in A \text{ for some } y \in \mathbb{R}\right\},\\
a_2 &= \inf\left\{y \in \mathbb{R} \,\middle|\, (x,y) \in A \text{ for some } x \in \mathbb{R}\right\},\\
b_2 &= \sup\left\{y \in \mathbb{R} \,\middle|\, (x,y) \in A \text{ for some } x \in \mathbb{R}\right\}.
\end{aligned}$$

A collection of rectangles $\mathcal{P}$ is called a ***partition*** of A if there exists a partition $\mathcal{P}_x$ of $[a_1, b_1]$ and a partition P_y of $[a_2, b_2]$, where

$$\mathcal{P}_x = \left\{a_1 = x_0 < x_1 < \cdots < x_n = b_1\right\} \quad \text{and} \quad \mathcal{P}_y = \left\{a_2 = y_0 < y_1 < \cdots < y_m = b_2\right\},$$

such that

$$\mathcal{P} = \left\{\Delta_{ij} \,\middle|\, \Delta_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \text{ for } i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m\right\}.$$

The norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$ and also called the ***mesh size*** of the partition $\mathcal{P}$, is a real number defined by

$$\|\mathcal{P}\| = \max\left\{\sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2} \,\middle|\, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m\right\}.$$

The number $\sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}$ is often denoted by $\operatorname{diam}(\Delta_{ij})$, and is called the ***diameter*** of Δ_{ij} (thus the norm of $\mathcal{P}$ is the maximum of the diameter of rectangles in $\mathcal{P}$).

In general, the partition of a bounded set $A \subseteq \mathbb{R}^n$ is defined as follows.

Definition 6.4. Let $A \subseteq \mathbb{R}^n$ be a bounded set. Define the numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ by

$$\begin{aligned}
a_k &= \inf\left\{x_k \in \mathbb{R} \,\middle|\, x = (x_1, \cdots, x_n) \in A \text{ for some } x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n \in \mathbb{R}\right\},\\
b_k &= \sup\left\{x_k \in \mathbb{R} \,\middle|\, x = (x_1, \cdots, x_n) \in A \text{ for some } x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n \in \mathbb{R}\right\}.
\end{aligned}$$

A collection of rectangles $\mathcal{P}$ is called a ***partition*** of A if there exists partitions $\mathcal{P}^{(k)}$ of $[a_k, b_k]$, $k = 1, \cdots, n$, $\mathcal{P}^{(k)} = \{a_k = x_0^{(k)} < x_1^{(k)} < \cdots < x_{N_k}^{(k)} = b_k\}$, such that

$$\mathcal{P} = \Big\{\Delta_{i_1 i_2 \cdots i_n} \,\Big|\, \Delta_{i_1 i_2 \cdots i_n} = [x_{i_1-1}^{(1)}, x_{i_1}^{(1)}] \times [x_{i_2-1}^{(2)}, x_{i_2}^{(2)}] \times \cdots \times [x_{i_n-1}^{(n)}, x_{i_n}^{(n+1)}], \\ i_k = 1, 2, \cdots, N_k, k = 1, \cdots, n\Big\}.$$

The norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$ and also called the ***mesh size*** of the partition $\mathcal{P}$, is a real number defined by

$$\|\mathcal{P}\| = \max\left\{\sqrt{\sum_{k=1}^{n} (x_{i_k}^{(k)} - x_{i_k-1}^{(k)})^2} \,\middle|\, i_k = 1, 2, \cdots, N_k, k = 1, \cdots, n\right\}.$$

The number $\sqrt{\sum\limits_{k=1}^{n} (x_{i_k}^{(k)} - x_{i_{k-1}}^{(k)})^2}$ is often denoted by $\text{diam}(\Delta_{i_1 i_2 \cdots i_n})$, and is called the ***diameter*** of the rectangle $\Delta_{i_1 i_2 \cdots i_n}$. The volume of $\Delta_{i_1 i_2 \cdots i_n}$, denoted by $\nu_n(\Delta_{i_1 i_2 \cdots i_n})$ (or simply $\nu(\Delta_{i_1 i_2 \cdots i_n})$ is n is clear to us), is defined by

$$\nu(\Delta_{i_1 i_2 \cdots i_n}) = \prod_{k=1}^{n} \big(x_{i_k}^{(k)} - x_{i_k-1}^{(k)}\big) = \big(x_{i_1}^{(1)} - x_{i_1-1}^{(1)}\big)\big(x_{i_2}^{(2)} - x_{i_2-1}^{(2)}\big) \cdots \big(x_{i_n}^{(n)} - x_{i_n-1}^{(n)}\big).$$

Next we define the Riemann sum of a function $f : A \to \mathbb{R}$ for a partition $\mathcal{P}$ of A.

Definition 6.5. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a (bounded) function. A ***Riemann sum*** of f for the partition $\mathcal{P} = \{\Delta_1, \Delta_2, \cdots, \Delta_N\}$ of A is a sum which takes the form

$$\sum_{k=1}^{N} \overline{f}^A(\xi_k)\nu(\Delta_k),$$

where $\overline{f}^A$ is a function given by

$$\overline{f}^A(x) = \begin{cases} f(x) & x \in A, \\ 0 & x \notin A. \end{cases} \tag{6.1.1}$$

and the set $\Xi = \{\xi_1, \xi_2, \cdots, \xi_N\}$ satisfies that $\xi_k \in \Delta_k$ for all $1 \leqslant k \leqslant N$.

Definition 6.6. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a (bounded) function. The function f is Riemann integrable on A if and only if there exists (a unique) $\mathrm{I} \in \mathbb{R}$ such that for every given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of A satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums of f for the partition $\mathcal{P}$ belongs to the interval $(\mathrm{I}-\varepsilon, \mathrm{I}+\varepsilon)$.

In other words, f is Riemann integrable on A if and only if there exists $\mathrm{I} \in \mathbb{R}$ such that for every given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \sum_{k=1}^{N} \overline{f}^{A}(\xi_k)\nu(\Delta_k) - \mathrm{I} \right| < \varepsilon \tag{6.1.2}$$

whenever $\mathcal{P} = \{\Delta_1, \cdots, \Delta_N\}$ is a partition of A satisfying $\|\mathcal{P}\| < \delta$ and the set $\Xi = \{\xi_1, \xi_2, \cdots, \xi_N\}$ satisfies that $\xi_k \in \Delta_k$ for all $1 \leqslant k \leqslant N$. The number I is denoted by (R)$\displaystyle\int_A f(x)\,dx$.

The definition of the integrability of functions given above is due to Bernhard Riemann; however, the definition above somehow lacks of flexibility for developing the theory of integration of functions. In the following, we adopt another point of view due to Gaston Darboux to discuss the integration of (bounded) functions $f : A \to \mathbb{R}$ for general bounded set $A \subseteq \mathbb{R}^n$.

Definition 6.7. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a (bounded) function. For a partition $\mathcal{P} = \{\Delta_1, \Delta_2, \cdots, \Delta_N\}$, the ***upper sum*** and the ***lower sum*** of f for the partition $\mathcal{P}$, denoted by $U(f, \mathcal{P})$ and $L(f, \mathcal{P})$ respectively, are numbers defined by

$$U(f, \mathcal{P}) = \sum_{k=1}^{N} \sup_{x \in \Delta_k} \overline{f}^{A}(x)\nu(\Delta_k) \qquad \text{and} \qquad L(f, \mathcal{P}) = \sum_{k=1}^{N} \inf_{x \in \Delta_k} \overline{f}^{A}(x)\nu(\Delta_k)\,.$$

The two numbers

$$\overline{\int_A} f(x)\,dx \equiv \inf\big\{U(f, \mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition of } A\big\}\,,$$

and

$$\underline{\int_A} f(x)\,dx \equiv \sup\big\{L(f, \mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition of } A\big\}$$

are called the ***upper integral*** and ***lower integral*** of f on A, respectively. The function f is said to be ***Darboux integrable*** (on A) if $\displaystyle\overline{\int_A} f(x)\,dx$ and $\displaystyle\underline{\int_A} f(x)\,dx$ are identical real numbers, and in this case, we express the upper and lower integral as (D)$\displaystyle\int_A f(x)\,dx$, called the ***Darboux integral*** of f on A.

Using the property of supremum and infimum, we immediately obtain the following

Proposition 6.8. *Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f, g : A \to \mathbb{R}$ be functions. If $\mathcal{P}$ is a partition of A, then*

$$L(f,\mathcal{P}) + L(g,\mathcal{P}) \leqslant L(f+g,\mathcal{P}) \leqslant U(f+g,\mathcal{P}) \leqslant U(f,\mathcal{P}) + U(g,\mathcal{P}) . \tag{6.1.3}$$

Definition 6.9. A partition $\mathcal{P}'$ of a bounded set $A \subseteq \mathbb{R}^n$ is called a ***refinement*** of another partition $\mathcal{P}$ of A if for any $\Delta' \in \mathcal{P}'$, there is $\Delta \in \mathcal{P}$ such that $\Delta' \subseteq \Delta$. A partition $\mathcal{P}$ of a bounded set $A \subseteq \mathbb{R}^n$ is called the ***common refinement*** of another partitions $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_k$ of A if

1. $\mathcal{P}$ is a refinement of $\mathcal{P}_j$ for all $1 \leqslant j \leqslant k$.

2. If $\mathcal{P}'$ is a refinement of $\mathcal{P}_j$ for all $1 \leqslant j \leqslant k$, then $\mathcal{P}'$ is also a refinement of $\mathcal{P}$.

In other words, $\mathcal{P}$ is a common refinement of $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_k$ if it is the coarsest refinement.

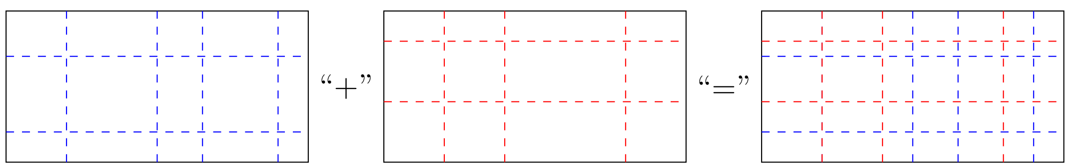


Figure 6.1: The common refinement of two partitions

Qualitatively speaking, $\mathcal{P}$ is a common refinement of $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_k$ if for each $j = 1, \cdots n$, the j-th component c_j of the vertex $(c_1, \cdots, c_n)$ of each rectangle $\Delta \in \mathcal{P}$ belongs to $\mathcal{P}_i^{(j)}$ for some $i = 1, \cdots, k$.

The following proposition should be clear to the readers, and the proof is left as an exercise.

Proposition 6.10. *Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a function. If $\mathcal{P}$ and $\mathcal{P}'$ are partitions of A and $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then*

$$L(f,\mathcal{P}) \leqslant L(f,\mathcal{P}') \leqslant U(f,\mathcal{P}') \leqslant U(f,\mathcal{P}) .$$

Corollary 6.11. *Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a function. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions of A, then $L(f,\mathcal{P}_1) \leqslant U(f,\mathcal{P}_2)$.*

Proof. Let $\mathcal{P}$ be the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then Proposition 6.10 implies that

$$L(f,\mathcal{P}_1) \leqslant L(f,\mathcal{P}) \leqslant U(f,\mathcal{P}) \leqslant U(f,\mathcal{P}_2) .$$

□

Corollary 6.12. *Let $A \subseteq \mathbb{R}^n$ be a bounded subset, and $f : A \to \mathbb{R}$ be a function. Then*

$$\underline{\int_A} f(x)dx \leqslant \overline{\int_A} f(x)dx\,.$$

Proof. Note that for each given partition $\mathcal{P}$ of A, the previous corollary implies that $L(f, \mathcal{P})$ is a lower bound for all possible upper sum. Therefore,

$$L(f, \mathcal{P}) \leqslant \overline{\int_A} f(x)dx \qquad \text{for all partitions } \mathcal{P} \text{ of } A$$

which further implies that $\overline{\int_A} f(x)dx$ is an upper bound for all possible lower sum; thus $\underline{\int_A} f(x)dx \leqslant \overline{\int_A} f(x)dx\,.$ □

In the following proposition, we state an equivalent condition for Darboux integrability of bounded functions (on bounded sets).

Proposition 6.13 (Riemann's condition)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a (bounded) function. Then f is Darboux integrable on A if and only if*

$$\forall\, \varepsilon > 0, \exists \text{ a partition } \mathcal{P} \text{ of } A \ni U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon\,.$$

Proof. "⇒" Let $\varepsilon > 0$ be given. By the definition of infimum and supremum, there exist partition $\mathcal{P}_1$ and $\mathcal{P}_2$ of A such that

$$\underline{\int_A} f(x)\,dx - \frac{\varepsilon}{2} < L(f, \mathcal{P}_2) \quad \text{and} \quad \overline{\int_A} f(x)\,dx + \frac{\varepsilon}{2} > U(f, \mathcal{P}_1)\,.$$

Let $\mathcal{P}$ be a common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. Since f is Darboux integrable on A, $\underline{\int_A} f(x)dx = \overline{\int_A} f(x)dx$; thus Proposition 6.10 implies that

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &\leqslant U(f, \mathcal{P}_1) - L(f, \mathcal{P}_2) \\ &< \overline{\int_A} f(x)\,dx + \frac{\varepsilon}{2} - \Big(\underline{\int_A} f(x)\,dx - \frac{\varepsilon}{2}\Big) = \varepsilon\,. \end{aligned}$$

"⇐" Let $\varepsilon > 0$ be given. By assumption there exists a partition $\mathcal{P}$ of A such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$. Then

$$0 \leqslant \overline{\int_A} f(x)\,dx - \underline{\int_A} f(x)\,dx \leqslant U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrary, we must have $\underline{\int}_A f(x)dx = \overline{\int}_A f(x)dx$; thus f is Darboux integrable on A. ▫

The following theorem establishes the equivalence between the Riemann integrals and the Darboux integrals.

Theorem 6.14 (Darboux)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a (bounded) function. Then f is Riemann integrable on A if and only if f is Darboux integrable on A. In either cases,*

$$(\text{R})\int_A f(x)\,dx = (\text{D})\int_A f(x)\,dx\,.$$

Proof. The boundedness of A guarantees that $A \subseteq \big[-\frac{r}{2}, \frac{r}{2}\big]^n$ for some $r > 0$. Let $R = \big[-\frac{r}{2}, \frac{r}{2}\big]^n$. Then $\nu(R) = r^n$.

"⇒" Suppose that f is Riemann integrable on A with $(\text{R})\int_A f(x)\,dx = \text{I}$. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of A satisfying $\|\mathcal{P}\| < \delta$, then any Riemann of f for $\mathcal{P}$ locates in $\big(\text{I} - \frac{\varepsilon}{4}, \text{I} + \frac{\varepsilon}{4}\big)$.

Let $\mathcal{P} = \{\Delta_1, \cdots, \Delta_N\}$ be a partition of A with $\|\mathcal{P}\| < \delta$. For each $1 \leqslant k \leqslant N$, choose $\xi_k, \eta_k \in \Delta_k$ such that

(a) $\sup\limits_{x\in\Delta_k} \overline{f}^A(x) - \dfrac{\varepsilon}{4\nu(R)} < \overline{f}^A(\xi_k) \leqslant \sup\limits_{x\in\Delta_k} \overline{f}^A(x)$;

(b) $\inf\limits_{x\in\Delta_k} \overline{f}^A(x) + \dfrac{\varepsilon}{4\nu(R)} > \overline{f}^A(\eta_k) \geqslant \inf\limits_{x\in\Delta_k} \overline{f}^A(x)$.

Then

$$\begin{aligned} U(f,\mathcal{P}) &= \sum_{k=1}^N \sup_{x\in\Delta_k} \overline{f}^A(x)\nu(\Delta_k) < \sum_{k=1}^N \big[\overline{f}^A(\xi_k) + \frac{\varepsilon}{4\nu(R)}\big]\nu(\Delta_k) \\ &= \sum_{k=1}^N \overline{f}^A(\xi_k)\nu(\Delta_k) + \frac{\varepsilon}{4\nu(R)}\sum_{k=1}^N \nu(\Delta_k) < \text{I} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \text{I} + \frac{\varepsilon}{2} \end{aligned}$$

and

$$\begin{aligned} L(f,\mathcal{P}) &= \sum_{k=1}^N \inf_{x\in\Delta_k} \overline{f}^A(x)\nu(\Delta_k) > \sum_{k=1}^N \big[\overline{f}^A(\eta_k) - \frac{\varepsilon}{4\nu(R)}\big]\nu(\Delta_k) \\ &= \sum_{k=1}^N \overline{f}^A(\eta_k)\nu(\Delta_k) - \frac{\varepsilon}{4\nu(R)}\sum_{k=1}^N \nu(\Delta_k) > \text{I} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \text{I} - \frac{\varepsilon}{2}\,. \end{aligned}$$

As a consequence, $\mathrm{I}-\frac{\varepsilon}{2} < L(f,\mathcal{P}) \leqslant U(f,\mathcal{P}) < \mathrm{I}+\frac{\varepsilon}{2}$; thus $U(f,\mathcal{P})-L(f,\mathcal{P}) < \varepsilon$ which shows that f is Darboux integrable on A. Moreover, since $\varepsilon > 0$ is given arbitrarily and $L(f,\mathcal{P}) \leqslant (\mathrm{D})\int_A f(x)\,dx \leqslant U(f,\mathcal{P})$, we must have $\mathrm{I} = (\mathrm{D})\int_A f(x)\,dx$.

"$\Leftarrow$" Let $\mathrm{I} = (\mathrm{D})\int_A f(x)dx$, and $\varepsilon > 0$ be given. Since f is Darboux integrable on A, there exists a partition $\mathcal{P}_1$ of A such that $U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \frac{\varepsilon}{2}$. Suppose that $\mathcal{P}_1^{(i)} = \{y_0^{(i)}, y_1^{(i)}, \cdots, y_{m_i}^{(i)}\}$ for $1 \leqslant i \leqslant n$. We define

$$\delta = \frac{\varepsilon}{4r^{n-1}(m_1+m_2+\cdots+m_n+n)\big(\sup \overline{f}^A(R) - \inf \overline{f}^A(R) + 1\big)}\,.$$

Then $\delta > 0$.

Assume that $\mathcal{P} = \{\Delta_1, \Delta_2, \cdots, \Delta_N\}$ is a given partition of A with $\|\mathcal{P}\| < \delta$. Let $\mathcal{P}'$ be the common refinement of $\mathcal{P}$ and $\mathcal{P}_1$. Write $\mathcal{P}' = \{\Delta'_1, \Delta'_2, \cdots, \Delta'_{N'}\}$ and $\Delta_k = \Delta_k^{(1)} \times \Delta_k^{(2)} \times \cdots \times \Delta_k^{(n)}$ as well as $\Delta'_k = \Delta_k'^{(1)} \times \Delta_k'^{(2)} \times \cdots \times \Delta_k'^{(n)}$. Define two classes of rectangles in $\mathcal{P}$ and $\mathcal{P}'$ by

$$\begin{aligned} C_1 &= \big\{\Delta \in \mathcal{P} \,\big|\, y_j^{(i)} \notin \Delta^{(i)} \text{ for all } i,j\big\}, \qquad & C_2 &= \big\{\Delta \in \mathcal{P} \,\big|\, y_j^{(i)} \in \Delta^{(i)} \text{ for some } i,j\big\}, \\ D_1 &= \big\{\Delta' \in \mathcal{P}' \,\big|\, y_j^{(i)} \notin \Delta'^{(i)} \text{ for all } i,j\big\}, \qquad & D_2 &= \big\{\Delta \in \mathcal{P}' \,\big|\, y_j^{(i)} \in \Delta'^{(i)} \text{ for some } i,j\big\}. \end{aligned}$$

By the definition of the upper sum,

$$U(f,\mathcal{P}) = \sum_{k=1}^{N} \sup_{x\in\Delta_k} \overline{f}^A(x)\nu(\Delta_k) = \sum_{\Delta\in C_1} \sup_{x\in\Delta} \overline{f}^A(x)\nu(\Delta) + \sum_{\Delta\in C_2} \sup_{x\in\Delta} \overline{f}^A(x)\nu(\Delta)$$

and similarly,

$$U(f,\mathcal{P}') = \sum_{k=1}^{N'} \sup_{x\in\Delta'_k} \overline{f}^A(x)\nu(\Delta'_k) = \sum_{\Delta'\in D_1} \sup_{x\in\Delta'} \overline{f}^A(x)\nu(\Delta') + \sum_{\Delta'\in D_2} \sup_{x\in\Delta'} \overline{f}^A(x)\nu(\Delta')\,.$$

By the fact that $C_1 = D_1$, we must have

$$\sum_{\Delta\in C_1} \sup_{x\in\Delta} \overline{f}^A(x)\nu(\Delta) = \sum_{\Delta'\in D_1} \sup_{x\in\Delta'} \overline{f}^A(x)\nu(\Delta')$$

and

$$\sum_{\Delta\in C_2} \nu(\Delta_k) = \sum_{\Delta'\in D_2} \nu(\Delta')\,.$$

The equalities above further imply that

$$\begin{aligned}
U(f,\mathcal{P}) - U(f,\mathcal{P}') &= \sum_{\Delta\in C_2} \sup_{x\in\Delta} \overline{f}^A(x)\nu(\Delta) - \sum_{\Delta'\in D_2} \sup_{x\in\Delta'} \overline{f}^A(x)\nu(\Delta') \\
&\leqslant \big(\sup \overline{f}^A(R) - \inf \overline{f}^A(R)\big) \sum_{\Delta\in C_2} \nu(\Delta) \\
&= \big(\sup \overline{f}^A(R) - \inf \overline{f}^A(R)\big) \sum_{1\leqslant k\leqslant N \text{ with } y_j^{(i)} \in \Delta_k^{(i)} \text{ for some } i,j} \nu(\Delta_k) \\
&= \big(\sup \overline{f}^A(R) - \inf \overline{f}^A(R)\big) \sum_{i=1}^{n}\sum_{j=0}^{m_i} \sum_{1\leqslant k\leqslant N \text{ with } y_j^{(i)}\in\Delta_k^{(i)}} \nu(\Delta_k)\,.
\end{aligned}$$

Moreover, for each fixed i, j,

$$\bigcup_{1\leqslant k\leqslant N \text{ with } y_j^{(i)}\in\Delta_k^{(i)}} \Delta_k \subseteq \Big[-\frac{r}{2},\frac{r}{2}\Big]^{i-1} \times \big[y_j^{(i)} - \delta, y_j^{(i)} + \delta\big] \times \Big[-\frac{r}{2},\frac{r}{2}\Big]^{n-i};$$

thus

$$\sum_{1\leqslant k\leqslant N \text{ with } y_j^{(i)}\in\Delta_k^{(i)}} \nu(\Delta_k) \leqslant 2\delta r^{n-1} \qquad \forall\, 1\leqslant i\leqslant n, 1\leqslant j\leqslant m_i\,.$$

Therefore,

$$\begin{aligned}
U(f,\mathcal{P}) - U(f,\mathcal{P}') &\leqslant \big(\sup \overline{f}^A(R) - \inf \overline{f}^A(R)\big) \sum_{i=1}^{n}\sum_{j=0}^{m_i} \sum_{1\leqslant k\leqslant N \text{ with } y_j^{(i)}\in\Delta_k^{(i)}} \nu(\Delta_k) \\
&\leqslant \big(\sup \overline{f}^A(R) - \inf \overline{f}^A(R)\big) \sum_{i=1}^{n}\sum_{j=0}^{m_i} 2\delta r^{n-1} \\
&\leqslant 2\delta r^{n-1}(m_1+m_2+\cdots+m_n+n)\big(\sup \overline{f}^A(R) - \inf \overline{f}^A(R)\big) < \frac{\varepsilon}{2}\,,
\end{aligned}$$

and the fact that $U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \dfrac{\varepsilon}{2}$ shows that

$$\begin{aligned}
U(f,\mathcal{P}) - \mathrm{I} &\leqslant U(f,\mathcal{P}) - \mathrm{I} + U(f,\mathcal{P}_1) - U(f,\mathcal{P}_1) \\
&\leqslant U(f,\mathcal{P}) - L(f,\mathcal{P}_1) + U(f,\mathcal{P}_1) - U(f,\mathcal{P}') < \varepsilon\,.
\end{aligned}$$

Similar argument can be used to show that $L(f,\mathcal{P}) - \mathrm{I} > \varepsilon$. Therefore,

$$\mathrm{I} - \varepsilon < L(f,\mathcal{P}) \leqslant U(f,\mathcal{P}) < \mathrm{I} + \varepsilon$$

which implies that any Riemann sum of f for $\mathcal{P}$ locates in $(\mathrm{I}-\varepsilon, \mathrm{I}+\varepsilon)$. □

Notation: If $f : A \to \mathbb{R}$ is Riemann/Darboux integrable on A,

$$(\text{R})\int_A f(x)\,dx = (\text{D})\int_A f(x)\,dx$$

and we use $\int_A f(x)\,dx$ to denote this common number.

From now on, we will simply use $\bar{f}$ to denote the zero extension of f when the domain outside which the zero extension is made is clear.

6.2 The Lebesgue Theorem

In this section, we talk about another equivalent condition of Riemann/Darboux integrability, named the Lebesgue theorem. The Lebesque theorem provides a more practical way to check the Riemann/Darboux integrability in the development of theory. To understand the Lebesgue theorem, we need to talk about a new concept, sets of measure zero.

6.2.1 Volume and sets of measure zero

Definition 6.15. A bounded set $A \subseteq \mathbb{R}^n$ is said to ***have volume*** if the characteristic function or the indicator function of A, denoted by $\mathbf{1}_A$ and given by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

is Riemann integrable on A, and the number $\int_A \mathbf{1}_A(x)\,dx$ is called the ***volume*** of A and is denoted by $\nu(A)$. If $\nu(A) = 0$, then A is said to have volume zero or be a set of volume zero.

Remark 6.16. Not all bounded set has volume.

Proposition 6.17. *Let $A \subseteq \mathbb{R}^n$ be bounded. Then the following three statements are equivalent.*

(a) *A has volume zero;*

(b) *for every $\varepsilon > 0$, there exists finite open rectangles $S_1, \cdots, S_N$ whose sides are parallel to the coordinate axes such that*

$$A \subseteq \bigcup_{k=1}^{N} S_k \qquad \text{and} \qquad \sum_{k=1}^{N} \nu(S_k) < \varepsilon \tag{6.2.1}$$

(c) *for every* $\varepsilon > 0$*, there exist finite rectangles* $S_1, \cdots, S_N$ *such that* (6.2.1) *holds.*

Proof. It suffices to show (a)⇒(b) and (c)⇒(a) since it is clear that (b)⇒(c).

"(a)⇒(b)" Let $\varepsilon > 0$ be given. Since A has volume zero, $\overline{\int_A} \mathbf{1}_A(x)\,dx = 0$; thus there exists a partition $\mathcal{P}$ of A such that

$$U(\mathbf{1}_A, \mathcal{P}) < \overline{\int_A} \mathbf{1}_A(x)\,dx + \frac{\varepsilon}{2} = \frac{\varepsilon}{2}\,.$$

Since $\sup\limits_{x\in\Delta} \mathbf{1}_A(x) = \begin{cases} 1 & \text{if } \Delta \cap A \neq \varnothing\,, \\ 0 & \text{otherwise}\,, \end{cases}$ we must have $\sum\limits_{\substack{\Delta\in\mathcal{P}\\ \Delta\cap A\neq\varnothing}} \nu(\Delta) < \frac{\varepsilon}{2}$. Now if $\Delta \in \mathcal{P}$ and $\Delta \cap A \neq \varnothing$, we can find an open rectangle $\square$ whose sides are parallel to the coordinate axes such that $\Delta \subseteq \square$ and $\nu(\square) < 2\nu(\Delta)$. Let $S_1, \cdots, S_N$ be those open rectangles $\square$. Then $A \subseteq \bigcup\limits_{k=1}^{N} S_k$ and $\sum\limits_{k=1}^{N} \nu(S_k) < \varepsilon$.

"(c)⇒(a)" Let $\varepsilon > 0$ be given. By assumption there exist rectangles $S_1, S_2, \cdots, S_N$ such that (6.2.1) holds. W.L.O.G. we can assume that the ratio of the maximum length and minimum length of sides of S_k is less than 2 for all $k = 1, \cdots, N$ (otherwise we can divide S_k into smaller rectangles so that each smaller rectangle satisfies this requirement). Then each S_k can be covered by a closed rectangle $\square_k$ whose sides are parallel to the coordinate axes with the property that $\nu(\square_k) \leqslant 2^{n-1}\sqrt{n}^{\,n}\nu(S_k)$. Let $\mathcal{P}$ be a partition of A such that for each $\Delta \in \mathcal{P}$ with $\Delta \cap A \neq \varnothing$, $\Delta \subseteq \square_k$ for some $k = 1, \cdots, N$. Then

$$U(\mathbf{1}_A, \mathcal{P}) = \sum_{\substack{\Delta\in\mathcal{P}\\ \Delta\cap A\neq\varnothing}} \nu(\Delta) \leqslant \sum_{k=1}^{N} \nu(\square_k) \leqslant 2^{n-1}\sqrt{n}^{\,n} \sum_{k=1}^{N} \nu(S_k) < 2^{n-1}\sqrt{n}^{\,n}\varepsilon\,;$$

thus the upper integral $\overline{\int_A} \mathbf{1}_A(x)\,dx = 0$. Since the lower integral cannot be negative, we must have $\overline{\int_A} \mathbf{1}_A(x)\,dx = \underline{\int_A} \mathbf{1}_A(x)\,dx = 0$ which shows that A has volume zero. □

Example 6.18. Each point in $\mathbb{R}^n$ has volume zero.

Definition 6.19. A set $A \subseteq \mathbb{R}^n$ (not necessarily bounded) is said to ***have measure zero*** (測度為零) or be ***a set of measure zero*** (零測度集) if for every $\varepsilon > 0$, there exist countable many rectangles $S_1, S_2, \cdots$ such that $\{S_k\}_{k=1}^{\infty}$ is a cover of A $\big($that is, $A \subseteq \bigcup\limits_{k=1}^{\infty} S_k\big)$ and $\sum\limits_{k=1}^{\infty} \nu(S_k) < \varepsilon$.

Example 6.20. The real line $\mathbb{R} \times \{0\}$ on $\mathbb{R}^2$ has measure zero: for any given $\varepsilon > 0$, let $S_k = [-k, k] \times \big[\frac{-\varepsilon}{2^{k+3}k}, \frac{\varepsilon}{2^{k+3}k}\big]$. Then

$$\mathbb{R} \times \{0\} \subseteq \bigcup_{k=1}^{\infty} S_k \quad \text{and} \quad \sum_{k=1}^{\infty} \nu(S_k) = \sum_{k=1}^{\infty} 2k \cdot \frac{2\varepsilon}{2^{k+3}k} = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2} < \varepsilon\,.$$

Similarly, any hyperplane in $\mathbb{R}^n$ also has measure zero.

Proposition 6.21. *Let $A \subseteq \mathbb{R}^n$ be a set of measure zero. If $B \subseteq A$, then B also has measure zero.*

Modifying the proof of Proposition 6.17, we can also conclude the following

Proposition 6.22. *A set $A \subseteq \mathbb{R}^n$ has measure zero if and only if for every $\varepsilon > 0$, there exist countable many open rectangles S_1, S_2, $\cdots$ whose sides are parallel to the coordinate axes such that $A \subseteq \bigcup_{k=1}^{\infty} S_k$ and $\sum_{k=1}^{\infty} \nu(S_k) < \varepsilon$.*

Remark 6.23. If a set A has volume zero, then it has measure zero.

Proposition 6.24. *Let $K \subseteq \mathbb{R}^n$ be a compact set of measure zero. Then K has volume zero.*

Proof. Let $\varepsilon > 0$ be given. Then there are countable open rectangles $S_1, S_2, \cdots$ such that

$$K \subseteq \bigcup_{k=1}^{\infty} S_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} \nu(S_k) < \varepsilon\,.$$

Since $\{S_k\}_{k=1}^{\infty}$ is an open cover of K, by the compactness of K there exists $N > 0$ such that $K \subseteq \bigcup_{k=1}^{N} S_k$, while $\sum_{k=1}^{N} \nu(S_k) \leqslant \sum_{k=1}^{\infty} \nu(S_k) < \varepsilon$. As a consequence, K has volume zero. □

Since the boundary of a rectangle has measure zero, we also have the following

Corollary 6.25. *Let $S \subseteq \mathbb{R}^n$ be a bounded rectangle with positive volume. Then S is not a set of measure zero.*

Theorem 6.26. *If A_1, A_2, $\cdots$ are sets of measure zero in $\mathbb{R}^n$, then $\bigcup_{k=1}^{\infty} A_k$ has measure zero.*

Proof. Let $\varepsilon > 0$ be given. Since $A_k's$ are sets of measure zero, there exist countable rectangles $\{S_j^{(k)}\}_{j=1}^{\infty}$, such that

$$A_k \subseteq \bigcup_{j=1}^{\infty} S_j^{(k)} \quad \text{and} \quad \sum_{j=1}^{\infty} \nu(S_j^{(k)}) < \frac{\varepsilon}{2^{k+1}} \qquad \forall\, k \in \mathbb{N}\,.$$

Consider the collection consisting of all $S_j^{(k)}$'s. Since there are countable many rectangles in this collection, we can label them as S_1, S_2, $\cdots$, and we have

$$\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty} S_j^{(k)} = \bigcup_{\ell=1}^{\infty} S_\ell$$

and

$$\sum_{k=1}^{\infty} \nu(S_\ell) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \nu(S_j^{(k)}) \leqslant \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2} < \varepsilon\,.$$

Therefore, $\bigcup_{k=1}^{\infty} A_k$ has measure zero. □

Corollary 6.27. *The set of rational numbers in* $\mathbb{R}$ *has measure zero.*

Theorem 6.28. *Let* $A \subseteq \mathbb{R}^n$ *be bounded and* $B \subseteq \mathbb{R}^m$ *be a set of measure zero. Then* $A \times B$ *has measure zero in* $\mathbb{R}^{n+m}$*.*

Proof. Let $\varepsilon > 0$ be given. Since A is bounded, there exist a bounded rectangle R such that $A \subseteq R$. Since B has measure zero, there exist countable rectangles $\{S_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^m$ such that

$$B \subseteq \bigcup_{k=1}^{\infty} S_k \quad \text{and} \quad \sum_{k=1}^{\infty} \nu_m(S_k) < \frac{\varepsilon}{\nu(R)}\,.$$

Then $A \times B \subseteq \bigcup_{k=1}^{\infty} (R \times S_k)$, and

$$\sum_{k=1}^{\infty} \nu_{n+m}(R \times S_k) = \sum_{k=1}^{\infty} \nu_n(R)\nu_m(S_k) = \nu_n(R)\sum_{k=1}^{\infty} \nu_m(S_k) < \varepsilon\,.$$

Since $R \times S_k$ is a rectangle for all $k \in \mathbb{N}$, we conclude that $A \times B$ has measure zero. □

6.2.2 The Lebesgue theorem

在之前我們提到了函數 Riemann 可積的兩個等價條件：Riemann's condition 和 Darboux 定理。在這一節中，我們將引進函數是 Riemman 可積的另一個等價條件。這個等價條件

說的是一個函數 f 在 A 上是 Riemann 可積的若且唯若 f 的延拓 $\overline{f}^A$（在函數可積分的定義中有定義）的不連續點所構成的集合其測度為零。為了證明這個敘述，我們先對一個函數的連續點做一個量化的刻劃。這個刻劃的方式，可以很容易用來檢驗一個函數在一個點是否連續。

Definition 6.29. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. For any $x \in \mathbb{R}^n$, the ***oscillation*** of f at x is the quantity

$$\operatorname{osc}(f, x) \equiv \inf_{\delta>0} \sup_{x_1,x_2\in B(x,\delta)} \big|f(x_1) - f(x_2)\big| .$$

我們注意到在上述定義中被取 infimum 的這個量 $h(\delta; x) \equiv \sup\limits_{x_1,x_2\in B(x,\delta)} \big|f(x_1) - f(x_2)\big|$ 是個 δ 的遞減函數（x 固定），而 $\operatorname{osc}(f, x)$ 則是 $h(\delta; x)$ 當 $\delta \to 0$ 時的極限。另外，我們也注意到說 $h(\delta; x)$ 也可以寫成 $\sup\limits_{y\in B(x,\delta)} f(y) - \inf\limits_{y\in B(x,\delta)} f(y)$.

以下的 Lemma 是關於如何檢驗一個函數在一個點是連續的。

Lemma 6.30. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, and $x_0 \in \mathbb{R}^n$. Then f is continuous at x_0 if and only if* $\operatorname{osc}(f, x_0) = 0$.

Proof. "⇒" Let $\varepsilon > 0$ be given. Since f is continuous at x_0,

$$\exists\, \delta > 0 \ni \big|f(x) - f(x_0)\big| < \frac{\varepsilon}{3} \qquad \text{whenever } x \in B(x_0, \delta).$$

In particular, for any $x_1, x_2 \in B(x_0, \delta)$,

$$\big|f(x_1) - f(x_2)\big| \leqslant \big|f(x_1) - f(x_0)\big| + \big|f(x_0) - f(x_2)\big| < \frac{2\varepsilon}{3} ;$$

thus $\sup\limits_{x_1,x_2\in B(x_0,\delta)} \big|f(x_1) - f(x_2)\big| \leqslant \dfrac{2\varepsilon}{3}$ which further implies that

$$0 \leqslant \operatorname{osc}(f, x_0) \leqslant \frac{2\varepsilon}{3} < \varepsilon.$$

Since ε is given arbitrarily, $\operatorname{osc}(f, x_0) = 0$.

"⇐" Let $\varepsilon > 0$ be given. By the definition of infimum, there exists $\delta > 0$ such that

$$\sup_{x_1,x_2\in B(x_0,\delta)} \big|f(x_1) - f(x_2)\big| < \varepsilon .$$

In particular, $\big|f(x) - f(x_0)\big| \leqslant \sup\limits_{x_1,x_2\in B(x_0,\delta)} \big|f(x_1) - f(x_2)\big| < \varepsilon$ for all $x \in B(x_0, \delta)$. □

Lemma 6.31. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then for all $\varepsilon > 0$, the set $D_\varepsilon = \big\{x \in \mathbb{R}^n \,\big|\, \text{osc}(f,x) \geqslant \varepsilon\big\}$ is closed.*

Proof. Suppose that $\{y_k\}_{k=1}^\infty \subseteq D_\varepsilon$ and $y_k \to y$. Then for any $\delta > 0$, there exists $N > 0$ such that $y_k \in B(y,\delta)$ for all $k \geqslant N$. Since $B(y,\delta)$ is open, for each $k \geqslant N$ there exists $\delta_k > 0$ such that $B(y_k,\delta_k) \subseteq B(y,\delta)$; thus we find that

$$\sup_{x_1,x_2 \in B(y_k,\delta_k)} \big|f(x_1) - f(x_2)\big| \leqslant \sup_{x_1,x_2 \in B(y,\delta)} \big|f(x_1) - f(x_2)\big| \qquad \forall\, k \geqslant N\,.$$

The inequality above implies that $\text{osc}(f,y) \geqslant \varepsilon$; thus $y \in D_\varepsilon$ and D_ε is closed. □

Theorem 6.32 (Lebesgue)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded set, $f : A \to \mathbb{R}$ be a bounded function, and $\overline{f}^A$ be the extension of f by zero outside A; that is,*

$$\overline{f}^A(x) = \left\{ \begin{array}{cl} f(x) & \text{if } x \in A\,, \\ 0 & \text{otherwise}\,. \end{array} \right.$$

Then f is Riemann integrable on A if and only if the collection of discontinuity of $\overline{f}^A$ is a set of measure zero.

Proof. Let $D = \big\{x \in \mathbb{R}^n \,\big|\, \text{osc}(\overline{f}^A, x) > 0\big\}$ and $D_\varepsilon = \big\{x \in \mathbb{R}^n \,\big|\, \text{osc}(\overline{f}^A, x) \geqslant \varepsilon\big\}$. We remark here that $D = \bigcup\limits_{k=1}^\infty D_{\frac{1}{k}}$.

"⇒" We show that $D_{\frac{1}{k}}$ has measure zero for all $k \in \mathbb{N}$ (if so, then Theorem 6.26 implies that D has measure zero).

Let $k \in \mathbb{N}$ be fixed, and $\varepsilon > 0$ be given. By Riemann's condition there exists a partition $\mathcal{P}$ of A such that

$$\sum_{\Delta \in \mathcal{P}} \Big[\sup_{x \in \Delta} \overline{f}^A(x) - \inf_{x \in \Delta} \overline{f}^A(x)\Big] \nu(\Delta) < \frac{\varepsilon}{k}\,.$$

Define

$$\begin{aligned} D^{(1)}_{\frac{1}{k}} &\equiv \big\{x \in D_{\frac{1}{k}} \,\big|\, x \in \partial\Delta \text{ for some } \Delta \in \mathcal{P}\big\}\,, \\ D^{(2)}_{\frac{1}{k}} &\equiv \big\{x \in D_{\frac{1}{k}} \,\big|\, x \in \text{int}(\Delta) \text{ for some } \Delta \in \mathcal{P}\big\}\,. \end{aligned}$$

Then $D_{\frac{1}{k}} = D^{(1)}_{\frac{1}{k}} \cup D^{(2)}_{\frac{1}{k}}$. We note that $D^{(1)}_{\frac{1}{k}}$ has measure zero since it is contained in $\bigcup\limits_{\Delta \in \mathcal{P}} \partial\Delta$ while each $\partial\Delta$ has measure zero. Now we show that $D^{(2)}_{\frac{1}{k}}$ also has measure zero. Let $C = \big\{\Delta \in \mathcal{P} \,\big|\, \text{int}(\Delta) \cap D_{\frac{1}{k}} \neq \varnothing\big\}$. Then $D^{(2)}_{\frac{1}{k}} \subseteq \bigcup\limits_{\Delta \in C} \Delta$. Moreover, we also note that if $\Delta \in C$, $\sup\limits_{x\in\Delta} \overline{f}^A(x) - \inf\limits_{x\in\Delta} \overline{f}^A(x) \geqslant \dfrac{1}{k}$. In fact, if $\Delta \in C$, there exists $y \in \text{int}(\Delta) \cap D_{\frac{1}{k}}$; thus choosing $\delta > 0$ such that $B(y,\delta) \subseteq \text{int}(\Delta)$,

$$\begin{aligned}\sup_{x\in\Delta} \overline{f}^A(x) - \inf_{x\in\Delta} \overline{f}^A(x) &= \sup_{x_1,x_2\in\Delta} \big|\overline{f}^A(x_1) - \overline{f}^A(x_2)\big| \geqslant \sup_{x_1,x_2\in B(y,\delta)} \big|\overline{f}^A(x_1) - \overline{f}^A(x_2)\big| \\ &\geqslant \inf_{\delta>0} \sup_{x_1,x_2\in B(y,\delta)} \big|\overline{f}^A(x_1) - \overline{f}^A(x_2)\big| = \text{osc}(\overline{f}^A, y) \geqslant \frac{1}{k}\,.\end{aligned}$$

As a consequence,

$$\frac{1}{k}\sum_{\Delta\in C} \nu(\Delta) \leqslant \sum_{\Delta\in\mathcal{P}} \Big[\sup_{x\in\Delta} \overline{f}^A(x) - \inf_{x\in\Delta} \overline{f}^A(x)\Big]\nu(\Delta) = U(f,\mathcal{P}) - L(f,\mathcal{P}) < \frac{\varepsilon}{k}$$

which implies that $\sum\limits_{\Delta\in C} \nu(\Delta) < \varepsilon$. In other words, we establish that $D^{(2)}_{\frac{1}{k}}$ has measure zero. Therefore, $D_{\frac{1}{k}}$ has measure zero for all $k \in \mathbb{N}$; thus D has measure zero.

"$\Leftarrow$" Let R be a bounded closed rectangle with sides parallel to the coordinate axes and $\bar{A} \subseteq \text{int}(R)$, and $\varepsilon > 0$ be given. Define $\varepsilon' = \dfrac{\varepsilon}{2\|f\|_\infty + \nu(R) + 1}$, where $\|f\|_\infty = \sup\limits_{x\in A} |f(x)|$.

1. Since $D_{\varepsilon'}$ is a subset of D, Proposition 6.21 implies that $D_{\varepsilon'}$ has measure zero; thus Proposition 6.22 provides open rectangles $S_1, S_2, \cdots$ whose sides are parallel to the coordinate axes such that $D_{\varepsilon'} \subseteq \bigcup\limits_{k=1}^{\infty} S_k$, and $\sum\limits_{k=1}^{\infty} \nu(S_k) < \varepsilon'$. In addition, we can assume that $S_k \subseteq R$ for all $k \in \mathbb{N}$ since $D_{\varepsilon'} \subseteq R$.
2. Since $D_{\varepsilon'} \subseteq R$ is bounded, Lemma 6.31 implies that $D_{\varepsilon'}$ is compact; thus $D_{\varepsilon'} \subseteq \bigcup\limits_{k=1}^{N} S_k$ for some $N \in \mathbb{N}$.

Let $\square_k = \overline{S_k}$, and $\mathcal{P}_1$ be a partition of R satisfying

(a) For each $\Delta \in \mathcal{P}_1$ with $\Delta \cap D_{\varepsilon'} \neq \varnothing$, $\Delta \subseteq \square_k$ for some $k = 1, \cdots, N$.

(b) For each $k = 1, \cdots, N$, $\square_k$ is the union of rectangles in $\mathcal{P}_1$.

(c) Some collection of $\Delta \in \mathcal{P}_1$ forms a partition $\mathcal{P}_2$ of A.

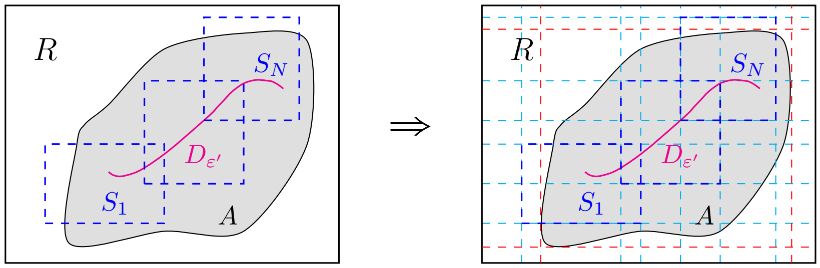


Figure 6.2: Constructing partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ from finite rectangles $S_1, \cdots, S_N$

Rectangles in $\mathcal{P}_1$ fall into two families: $C_1 = \{\Delta \in \mathcal{P}_1 \,|\, \Delta \subseteq \Box_k \text{ for some } k = 1, \cdots, N\}$, and $C_2 = \{\Delta \in \mathcal{P}_1 \,|\, \Delta \nsubseteq \Box_k \text{ for all } k = 1, \cdots, N\}$. By the definition of the oscillation function, for $x \notin D_{\varepsilon'}$ we let $\delta_x > 0$ be such that

$$\sup_{x \in B(y,\delta_y)} \overline{f}^A(y) - \inf_{x \in B(y,\delta_y)} \overline{f}^A(y) = \sup_{x_1,x_2 \in B(x,\delta_x)} \big|\overline{f}^A(x_1) - \overline{f}^A(x_2)\big| < \varepsilon' .$$

Since $K = \bigcup_{\Delta \in C_2} \Delta$ is compact, there exists $r > 0$ (the Lebesgue number associated with K and open cover $\bigcup_{x \in K} B(x, \delta_x)$) such that for each $a \in K$, $B(a, r) \subseteq B(y, \delta_y)$ for some $y \in K$. Let $\mathcal{P}'$ be a refinement of $\mathcal{P}_1$ such that $\|\mathcal{P}'\| < r$. Then if $\Delta' \in \mathcal{P}'$ satisfies that $\Delta' \subseteq \Delta$ for some $\Delta \in C_2$, we must have $\Delta' \subseteq B(y, \delta_y)$ for some $y \in K$; thus

$$\begin{aligned}\sup_{x \in \Delta'} \overline{f}^A(x) - \inf_{x \in \Delta'} \overline{f}^A(x) &\leqslant \sup_{x \in B(y,\delta_y)} \overline{f}^A(y) - \inf_{x \in B(y,\delta_y)} \overline{f}^A(y) \\ &= \sup_{x_1,x_2 \in B(y,\delta_y)} \big|\overline{f}^A(x_1) - \overline{f}^A(x_2)\big| < \varepsilon'\end{aligned}$$

if $\Delta' \subseteq \Delta$ for some $\Delta \in C_2$. Let $\mathcal{P} = \{\Delta' \in \mathcal{P}' \,|\, \Delta' \subseteq \Delta \text{ for some } \Delta \in \mathcal{P}_2\}$. Then $\mathcal{P}$ is a partition of A and

$$\begin{aligned}U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \Big(\sum_{\substack{\Delta' \in \mathcal{P}' \\ \Delta' \subseteq \Delta \in C_1}} + \sum_{\substack{\Delta' \in \mathcal{P}' \\ \Delta' \subseteq \Delta \in C_2}}\Big)\big(\sup_{x \in \Delta'} \overline{f}^A(x) - \inf_{x \in \Delta'} \overline{f}^A(x)\big)\nu(\Delta') \\ &\leqslant 2\|f\|_\infty \sum_{\substack{\Delta' \in \mathcal{P}' \\ \Delta' \subseteq \Delta \in C_1}} \nu(\Delta') + \varepsilon' \sum_{\substack{\Delta' \in \mathcal{P}' \\ \Delta' \subseteq \Delta \in C_2}} \nu(\Delta') \\ &\leqslant 2\|f\|_\infty \sum_{\Delta \in \mathcal{P} \cap C_1} \nu(\Delta) + \varepsilon' \nu(R) \\ &\leqslant 2\|f\|_\infty \sum_{k=1}^{N} \nu(S_k) + \varepsilon' \nu(R) < \big(2\|f\|_\infty + \nu(R)\big)\varepsilon' \leqslant \varepsilon ;\end{aligned}$$

thus f is Riemann integrable on A by Riemann's condition. □

Example 6.33. Let $A = \mathbb{Q} \cap [0,1]$, and $f : A \to \mathbb{R}$ be the constant function $f \equiv 1$. Then

$$\bar{f}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The collection of points of discontinuity of $\bar{f}$ is $[0,1]$ which, by Corollary 6.25, cannot be a set of measure zero; thus f is not Riemann integrable.

Another way to see that f is not Riemann integrable is $U(f, \mathcal{P}) = 1$ and $L(f, \mathcal{P}) = 0$ for all partitions $\mathcal{P}$ of A.

Corollary 6.34. *A bounded set $A \subseteq \mathbb{R}^n$ has volume if and only if the boundary of A has measure zero.*

Proof. It suffices to show that the collection of discontinuities of the function $\mathbf{1}_A$ (which is the same as $\overline{\mathbf{1}_A}^A$) is indeed ∂A.

1. If $x_0 \notin \partial A$, then there exists $\delta > 0$ such that either $B(x_0, \delta) \subseteq A$ or $B(x_0, \delta) \subseteq A^\complement$; thus $\mathbf{1}_A$ is continuous at $x_0 \notin \partial A$ since $\mathbf{1}_A(x)$ is constant for all $x \in B(x_0, \delta)$.

2. On the other hand, if $x_0 \in \partial A$, then there exists $x_k \in A$, $y_k \in A^\complement$ such that $x_k \to x_0$ and $y_k \to x_0$ as $k \to \infty$. This implies that $\mathbf{1}_A$ cannot be continuous at x_0 since $\mathbf{1}_A(x_k) = 1$ while $\mathbf{1}_A(y_k) = 0$ for all $k \in \mathbb{N}$.

As a consequence, the collection of discontinuity of $\mathbf{1}_A$ is exactly ∂A, and the corollary follows from Lebesgue's theorem. □

Corollary 6.35. *Let $A \subseteq \mathbb{R}^n$ be a bounded set with volume. A bounded function $f : A \to \mathbb{R}$ is Riemann integrable on A if and only if the collection of discontinuities of f has measure zero. In particular, a bounded function $f : A \to \mathbb{R}$ with a finite or countable number of points of discontinuity is Riemann integrable on A.*

Proof. Note that

$$\left\{x \in \mathbb{R}^n \,\middle|\, \operatorname{osc}(\bar{f}, x) > 0\right\} \subseteq \partial A \cup \left\{x \in A \,\middle|\, f \text{ is discontinuous at } x\right\}$$

and

$$\left\{x \in A \,\middle|\, f \text{ is discontinuous at } x\right\} \subseteq \left\{x \in \mathbb{R}^n \,\middle|\, \operatorname{osc}(\bar{f}, x) > 0\right\};$$

thus by the fact that ∂A has measure zero (Corollary 6.34) and Theorem 6.26 we conclude that the collection of discontinuities of $\bar{f}$ has measure zero if and only if the collection of discontinuities of f has measure zero. □

Corollary 6.36. *A bounded function is integrable on a compact set of measure zero.*

Proof. If $f : K \to \mathbb{R}$ is bounded, and K is a compact set of measure zero, then the collection of discontinuities of $\bar{f}$ is a subset of K. ◻

Corollary 6.37. *Suppose that $A, B \subseteq \mathbb{R}^n$ are bounded sets with volume, and $f : A \to \mathbb{R}$ is Riemann integrable on A. Then f is Riemann integrable on $A \cap B$.*

Proof. By the inclusion

$$\big\{x \in \text{int}(A \cap B) \,\big|\, \text{osc}(\overline{f}^{A\cap B}, x) > 0\big\} \subseteq \big\{x \in \mathbb{R}^n \,\big|\, \text{osc}(\overline{f}^{A}, x) > 0\big\},$$

we find that

$$\begin{aligned}\big\{x \in \mathbb{R}^n \,\big|\, \text{osc}(\overline{f}^{A\cap B}, x) > 0\big\} &\subseteq \partial(A \cap B) \cup \big\{x \in \text{int}(A \cap B) \,\big|\, \text{osc}(\overline{f}^{A\cap B}, x) > 0\big\} \\ &\subseteq \partial A \cup \partial B \cup \big\{x \in \mathbb{R}^n \,\big|\, \text{osc}(\overline{f}^{A}, x) > 0\big\}.\end{aligned}$$

Since ∂A and ∂B both have measure zero, the integrability of f on $A \cap B$ then follows from the integrability of f on A and the Lebesgue Theorem. ◻

Remark 6.38. Suppose that $A \subseteq \mathbb{R}^n$ is a bounded set of measure zero. Even if $f : A \to \mathbb{R}$ is continuous, f might not be Riemann integrable. For example, the function f given in Example 6.33 is not Riemann integrable even though f is continuous on A.

Remark 6.39. When $f : A \to \mathbb{R}$ is Riemann integrable on A, it is not necessary that A has volume. For example, the zero function is Riemann integrable on $A = \mathbb{Q} \cap [0, 1]$ even though A does not has volume.

Corollary 6.40 (Lebesgue's Differentiation Theorem, a simple version)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded set with volume, and $f : A \to \mathbb{R}$ be bounded and Riemann integrable on A. Then*

$$\lim_{r\to 0} \frac{1}{\nu(B(x_0, r) \cap A)} \int_{B(x_0,r)\cap A} f(x)\, dx = f(x_0) \tag{6.2.2}$$

for almost every $x_0 \in A$; that is, the equality above does not hold only for x_0 from a set of measure zero.

Proof. Let $\varepsilon > 0$ be given, and suppose that $\bar{f}$, the zero extension of f outside A, is continuous at x_0. Then there exists $\delta > 0$ such that

$$\big|\bar{f}(x) - \bar{f}(x_0)\big| < \frac{\varepsilon}{2} \qquad \forall\, x \in B(x_0, \delta) \cap A\,.$$

Since ∂A has measure zero, by the fact that $\partial(B(x_0,r)\cap A)\subseteq \partial B(x_0,r)\cup\partial A$ we find that $\partial(B(x_0,r)\cap A)$ also has measure zero for all $r>0$. In other words, $B(x_0,r)\cap A$ has volume. Then if $0<r<\delta$,

$$\begin{aligned}
&\Big|\frac{1}{\nu(B(x_0,r)\cap A)}\int_{B(x_0,r)\cap A} f(x)\,dx - f(x_0)\Big| \\
&\quad = \Big|\frac{1}{\nu(B(x_0,r)\cap A)}\int_{B(x_0,r)\cap A} \big(\bar f(x)-\bar f(x_0)\big)\,dx\Big| \\
&\quad \leqslant \frac{1}{\nu(B(x_0,r)\cap A)}\int_{B(x_0,r)\cap A} \big|\bar f(x)-\bar f(x_0)\big|\,dx \\
&\quad \leqslant \frac{\varepsilon}{2}\frac{1}{\nu(B(x_0,r)\cap A)}\int_{B(x_0,r)\cap A} 1\,dx = \frac{\varepsilon}{2} < \varepsilon\,.
\end{aligned}$$

This implies that (6.2.2) holds for all x_0 at which $\bar f$ is continuous. The theorem then follows from the Lebesgue theorem. ▫

6.3 Properties of the Integrals

Proposition 6.41. *Let $A\subseteq\mathbb{R}^n$ be a bounded set, and $f,g:A\to\mathbb{R}$ be bounded functions. Then*

(a) *If $B\subseteq A$, then $\underline{\int}_A (f\mathbf{1}_B)(x)\,dx = \underline{\int}_B f(x)\,dx$ and $\overline{\int}_A (f\mathbf{1}_B)(x)\,dx = \overline{\int}_B f(x)\,dx$.*

(b) $\underline{\int}_A f(x)\,dx + \underline{\int}_A g(x)\,dx \leqslant \underline{\int}_A (f+g)(x)\,dx \leqslant \overline{\int}_A (f+g)(x)\,dx \leqslant \overline{\int}_A f(x)\,dx + \overline{\int}_A g(x)\,dx.$

(c) *If $c\geqslant 0$, then $\underline{\int}_A (cf)(x)\,dx = c\underline{\int}_A f(x)\,dx$ and $\overline{\int}_A (cf)(x)\,dx = c\overline{\int}_A f(x)\,dx$. If $c<0$, then $\underline{\int}_A (cf)(x)\,dx = c\overline{\int}_A f(x)\,dx$ and $\overline{\int}_A (cf)(x)\,dx = c\underline{\int}_A f(x)\,dx$.*

(d) *If $f\leqslant g$ on A, then $\underline{\int}_A f(x)\,dx \leqslant \underline{\int}_A g(x)\,dx$ and $\overline{\int}_A f(x)\,dx \leqslant \overline{\int}_A g(x)\,dx$.*

(e) *If A has volume zero, then f is Riemann integrable on A, and $\int_A f(x)\,dx = 0$.*

Proof. We only prove (a), (b), (c) and (e) since (d) is trivial.

(a) Let $\varepsilon > 0$ be given. By the definition of the lower integral, there exist partition $\mathcal{P}_A$ of A and $\mathcal{P}_B$ of B such that

$$\underline{\int_A} (f\mathbf{1}_B)(x)\,dx - \varepsilon < L(f\mathbf{1}_B, \mathcal{P}_A) = \sum_{\Delta \in \mathcal{P}_A} \inf_{x\in\Delta} \overline{f\mathbf{1}_B}^A(x)\nu(\Delta)$$

and

$$\underline{\int_B} f(x)\,dx - \frac{\varepsilon}{2} < L(f, \mathcal{P}_B) = \sum_{\Delta \in \mathcal{P}_B} \inf_{x\in\Delta} \overline{f}^B(x)\nu(\Delta)\,.$$

Let $\mathcal{P}'_A$ be a refinement of $\mathcal{P}_A$ such that some collection of rectangles in $\mathcal{P}'_A$ forms a partition of B. Denote this partition of B by $\mathcal{P}'_B$. Since $\inf\limits_{x\in\Delta} \overline{f}^B(x) \leqslant 0$ if $\Delta \in \mathcal{P}'_A \backslash \mathcal{P}'_B$, Proposition 6.10 implies that

$$\begin{aligned}
\underline{\int_A} (f\mathbf{1}_B)(x)\,dx - \varepsilon < L(f\mathbf{1}_B, \mathcal{P}_A) \leqslant L(f\mathbf{1}_B, \mathcal{P}'_A) &= \sum_{\Delta \in \mathcal{P}'_A} \inf_{x\in\Delta} \overline{f\mathbf{1}_B}^A(x)\nu(\Delta) \\
&= \Big(\sum_{\Delta\in\mathcal{P}'_A\backslash\mathcal{P}'_B} + \sum_{\Delta\in\mathcal{P}'_B}\Big) \inf_{x\in\Delta} \overline{f}^B(x)\nu(\Delta) \\
&\leqslant \sum_{\Delta\in\mathcal{P}'_B} \inf_{x\in\Delta} \overline{f}^B(x)\nu(\Delta) = L(f, \mathcal{P}'_B) \leqslant \underline{\int_B} f(x)\,dx\,.
\end{aligned}$$

On the other hand, let $\widetilde{\mathcal{P}}_A$ be a partition of A such that $\mathcal{P}_B \subseteq \widetilde{\mathcal{P}}_A$ and

$$\sum_{\Delta\in\widetilde{\mathcal{P}}_A\backslash\mathcal{P}_B,\,\Delta\cap B\neq\varnothing} \nu(\Delta) \leqslant \frac{\varepsilon}{2(M+1)}\,,$$

where $M > 0$ is an upper bound of $|f|$. Then

$$\sum_{\Delta\in\widetilde{\mathcal{P}}_A\backslash\mathcal{P}_B,\,\Delta\cap B\neq\varnothing} \inf_{x\in\Delta} \overline{f}^B(x)\nu(\Delta) \geqslant -M \sum_{\Delta\in\widetilde{\mathcal{P}}_A\backslash\mathcal{P}_B,\,\Delta\cap B\neq\varnothing} \nu(\Delta) \geqslant -\frac{\varepsilon}{2}$$

which further implies that

$$\begin{aligned}
\underline{\int_A} (f\mathbf{1}_B)(x)\,dx \geqslant L(f\mathbf{1}_B, \widetilde{\mathcal{P}}_A) &= \sum_{\Delta\in\widetilde{\mathcal{P}}_A} \inf_{x\in\Delta} \overline{f\mathbf{1}_B}^A(x)\nu(\Delta) \\
&= \Big(\sum_{\Delta\in\mathcal{P}_B} + \sum_{\Delta\in\widetilde{\mathcal{P}}_A\backslash\mathcal{P}_B,\,\Delta\cap B\neq\varnothing} + \sum_{\Delta\in\widetilde{\mathcal{P}}_A\backslash\mathcal{P}_B,\,\Delta\cap B=\varnothing}\Big) \inf_{x\in\Delta} \overline{f}^B(x)\nu(\Delta) \\
&= L(f, \mathcal{P}_B) + \sum_{\Delta\in\widetilde{\mathcal{P}}_A\backslash\mathcal{P}_B,\,\Delta\cap B\neq\varnothing} \inf_{x\in\Delta} \overline{f}^B(x)\nu(\Delta) > \underline{\int_B} f(x)\,dx - \varepsilon\,.
\end{aligned}$$

Therefore, we establish that

$$\underline{\int_B} f(x)\,dx - \varepsilon < \underline{\int_A} (f\mathbf{1}_B)(x)\,dx < \underline{\int_B} f(x)\,dx + \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrarily, we conclude that $\underline{\int_A} (f\mathbf{1}_B)(x)\,dx = \underline{\int_B} f(x)\,dx$. Similar argument can be applied to conclude that $\overline{\int_A} (f\mathbf{1}_B)(x)\,dx = \overline{\int_B} f(x)\,dx$.

(b) Let $\varepsilon > 0$ be given. By the definition of the lower integral, there exist partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of A such that

$$\underline{\int_A} f(x)\,dx - \frac{\varepsilon}{2} < L(f,\mathcal{P}_1) \quad \text{and} \quad \underline{\int_A} g(x)\,dx - \frac{\varepsilon}{2} < L(g,\mathcal{P}_2)\,.$$

Let $\mathcal{P}$ be a common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then

$$\begin{aligned}\underline{\int_A} f(x)\,dx + \underline{\int_A} g(x)\,dx - \varepsilon &< L(f,\mathcal{P}_1) + L(f,\mathcal{P}_2) \leqslant L(f,\mathcal{P}) + L(g,\mathcal{P}) \\ &= \sum_{\Delta\in\mathcal{P}} \inf_{x\in\Delta} \bar{f}(x)\nu(\Delta) + \sum_{\Delta\in\mathcal{P}} \inf_{x\in\Delta} \bar{g}(x)\nu(\Delta) \\ &\leqslant \sum_{\Delta\in\mathcal{P}} \inf_{x\in\Delta} (\bar{f}+\bar{g})(x)\nu(\Delta) = L(f+g,\mathcal{P}) \leqslant \underline{\int_A} (f+g)(x)\,dx\,.\end{aligned}$$

Since $\varepsilon > 0$ is given arbitrarily, we conclude that

$$\underline{\int_A} f(x)\,dx + \underline{\int_A} g(x)\,dx \leqslant \underline{\int_A} (f+g)(x)\,dx\,.$$

Similarly, we have $\overline{\int_A} (f+g)(x)\,dx \leqslant \overline{\int_A} f(x)\,dx + \overline{\int_A} g(x)\,dx$; thus (b) is established.

(c) It suffices to show the case $c = -1$. Let $\varepsilon > 0$ be given. Then there exist partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of A such that

$$\underline{\int_A} -f(x)\,dx - \varepsilon < L(-f,\mathcal{P}_1) \qquad \text{and} \qquad U(f,\mathcal{P}_2) < \overline{\int_A} f(x)\,dx + \varepsilon\,.$$

Let $\mathcal{P}$ be the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then

$$\underline{\int_A} -f(x)\,dx - \varepsilon < L(-f,\mathcal{P}_1) \leqslant L(-f,\mathcal{P}) \leqslant \underline{\int_A} -f(x)\,dx$$

and

$$\overline{\int_A} f(x)\,dx \leqslant U(f,\mathcal{P}) \leqslant U(f,\mathcal{P}_2) < \overline{\int_A} f(x)\,dx + \varepsilon\,.$$

By the fact that

$$L(-f,\mathcal{P}) = \sum_{\Delta\in\mathcal{P}} \inf_{x\in\Delta} \overline{(-f)}^A(x)\nu(\Delta) = -\sum_{\Delta\in\mathcal{P}} \sup_{x\in\Delta} \overline{f}^A(x)\nu(\Delta) = -U(f,\mathcal{P})\,,$$

we find that

$$\underline{\int_A} -f(x)\,dx - \varepsilon < L(-f,\mathcal{P}) = -U(f,\mathcal{P}) \leqslant -\overline{\int_A} f(x)\,dx$$

and

$$\underline{\int_A} -f(x)\,dx \geqslant L(-f,\mathcal{P}) = -U(f,\mathcal{P}) > -\overline{\int_A} f(x)\,dx - \varepsilon\,.$$

Therefore,

$$\underline{\int_A} -f(x)\,dx - \varepsilon < -\overline{\int_A} f(x)\,dx < \underline{\int_A} -f(x)\,dx + \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrarily, we conclude (c).

(e) Since f is bounded on A, there exist $M > 0$ such that $-M \leqslant f(x) \leqslant M$ for all $x \in A$. Therefore, $-1_A \leqslant \dfrac{f}{M} \leqslant 1_A$ on A; thus (c) and (d) imply that

$$0 = \int_A \mathbf{1}_A(x)\,dx = \overline{\int_A} \mathbf{1}_A(x)\,dx \geqslant \overline{\int_A} \frac{f(x)}{M}\,dx = \frac{1}{M}\overline{\int_A} f(x)\,dx$$

which implies that $\overline{\int_A} f(x)\,dx \leqslant 0$. Similarly, $\overline{\int_A} -f(x)\,dx \leqslant 0$ which further implies that $\underline{\int_A} f(x)\,dx \geqslant 0$. Therefore, by Corollary 6.12 we conclude that

$$0 \leqslant \underline{\int_A} f(x)\,dx \leqslant \overline{\int_A} f(x)\,dx \leqslant 0$$

which implies that f is Riemann integrable on A and $\int_A f(x)\,dx = 0$. □

Remark 6.42. Let $A \subseteq \mathbb{R}^n$ be a bounded set.

1. If $f : A \to \mathbb{R}$ is a bounded function, then (a) of Proposition 6.41 shows that if $B \subseteq A$, then f is Riemann integrable on B if and only if $f\mathbf{1}_B$ is Riemann integrable on A.

2. If $f, g : A \to \mathbb{R}$ are bounded functions, then (b) of Proposition 6.41 also implies that

$$\underline{\int_A}(f-g)(x)\,dx \leqslant \underline{\int_A} f(x)\,dx - \underline{\int_A} g(x)\,dx \text{ and } \overline{\int_A} f(x)\,dx - \overline{\int_A} g(x)\,dx \leqslant \overline{\int_A}(f-g)(x)\,dx\,.$$

Corollary 6.43. *Let $A, B \subseteq \mathbb{R}^n$ be bounded sets such that $A \cap B$ has volume zero, and $f : A \cup B \to \mathbb{R}$ be a bounded function. Then*

$$\underline{\int_A} f(x)\,dx + \underline{\int_B} f(x)\,dx \leqslant \underline{\int_{A\cup B}} f(x)\,dx \leqslant \overline{\int_{A\cup B}} f(x)\,dx \leqslant \overline{\int_A} f(x)\,dx + \overline{\int_B} f(x)\,dx\,.$$

Proof. Note that $f\mathbf{1}_A + f\mathbf{1}_B = f\mathbf{1}_{A\cup B} + f\mathbf{1}_{A\cap B}$ on $A \cup B$. Therefore, (a), (b) of Proposition 6.41 and Remark 6.42 implies that

$$\begin{aligned}
\underline{\int_A} f(x)\,dx + \underline{\int_B} f(x)\,dx &= \underline{\int_{A\cup B}} (f\mathbf{1}_A)(x)\,dx + \underline{\int_{A\cup B}} (f\mathbf{1}_B)(x)\,dx \leqslant \underline{\int_{A\cup B}} (f\mathbf{1}_A + f\mathbf{1}_B)(x)\,dx \\
&= \underline{\int_{A\cup B}} \big(f\mathbf{1}_{A\cup B} - (-f\mathbf{1}_{A\cap B})\big)(x)\,dx \\
&\leqslant \underline{\int_{A\cup B}} f\mathbf{1}_{A\cup B}(x)\,dx - \underline{\int_{A\cup B}} (-f\mathbf{1}_{A\cap B})(x)\,dx \\
&= \underline{\int_{A\cup B}} f(x)\,dx - \underline{\int_{A\cap B}} (-f)(x)\,dx
\end{aligned}$$

which, with the help of Proposition 6.41 (e), further implies that

$$\underline{\int_A} f(x)\,dx + \underline{\int_B} f(x)\,dx \leqslant \underline{\int_{A\cup B}} f(x)\,dx\,.$$

The case of the upper integral can be proved in a similar fashion. □

Having established Proposition 6.41, it is easy to see the following theorem (except (c)). The proof is left as an exercise.

Theorem 6.44. *Let $A \subseteq \mathbb{R}^n$ be a bounded set, $c \in \mathbb{R}$, and $f, g : A \to \mathbb{R}$ be Riemann integrable functions. Then*

(a) *$f \pm g$ is Riemann integrable, and* $\displaystyle\int_A (f \pm g)(x)\,dx = \int_A f(x)\,dx \pm \int_A g(x)\,dx$.

(b) *cf is Riemann integrable, and* $\displaystyle\int_A (cf)(x)\,dx = c\int_A f(x)\,dx$.

(c) *$|f|$ is Riemann integrable, and* $\displaystyle\left|\int_A f(x)\,dx\right| \leqslant \int_A |f(x)|dx$.

(d) *If $f \leqslant g$, then* $\displaystyle\int_A f(x)\,dx \leqslant \int_A g(x)\,dx$.

(e) *If A has volume and $|f| \leqslant M$, then* $\displaystyle\left|\int_A f(x)\,dx\right| \leqslant M\nu(A)$.

Theorem 6.45. *Let $A \subseteq \mathbb{R}^n$ be bounded, and $f : A \to \mathbb{R}$ be a Riemann integrable function.*

1. *If A has measure zero, then $\int_A f(x)dx = 0$.*
2. *If $f(x) \geqslant 0$ for all $x \in A$, and $\int_A f(x)dx = 0$, then the set $\{x \in A \,|\, f(x) \neq 0\}$ has measure zero.*

Proof. 1. We show that $L(f,\mathcal{P}) \leqslant 0 \leqslant U(f,\mathcal{P})$ for all partitions $\mathcal{P}$ of A. Let $\mathcal{P} = \{\Delta_1, \cdots, \Delta_N\}$ be a partition of A. By Corollary 6.25, $\Delta_k \cap A^{\complement} \neq \varnothing$ for $k = 1, \cdots, N$; thus we must have $\inf\limits_{x\in\Delta_k} \bar{f}(x) \leqslant 0$ and $\sup\limits_{x\in\Delta_k} \bar{f}(x) \geqslant 0$. As a consequence, if $\mathcal{P}$ is a partition of A,

$$L(f,\mathcal{P}) = \sum_{k=1}^{N} \inf_{x\in\Delta_k} \bar{f}(x)\nu(\Delta_k) \leqslant 0 \quad \text{and} \quad U(f,\mathcal{P}) = \sum_{k=1}^{N} \sup_{x\in\Delta_k} \bar{f}(x)\nu(\Delta_k) \geqslant 0\,;$$

thus $\underline{\int}_A f(x)dx \leqslant 0 \leqslant \overline{\int}_A f(x)dx$. Since f is integrable on A, $\int_A f(x)dx = 0$.

2. Let $A_k = \{x \in A \,|\, f(x) \geqslant \frac{1}{k}\}$. We claim that A_k has measure zero for all $k \in \mathbb{N}$.

   Let $\varepsilon > 0$ be given. Since $\int_A f(x)dx = 0$, there exists a partition $\mathcal{P}$ of A such that $U(f,\mathcal{P}) < \frac{\varepsilon}{k}$. Let $C = \{\Delta \in \mathcal{P} \,|\, \Delta \cap A_k \neq \varnothing\}$. Then $A_k \subseteq \bigcup\limits_{\Delta\in C} \Delta$, and

$$\frac{1}{k}\sum_{\Delta\in C} \nu(C) \leqslant \sum_{\Delta\in C} \sup_{x\in\Delta} \bar{f}(x)\nu(\Delta) \leqslant \sum_{\Delta\in\mathcal{P}} \sup_{x\in\Delta} \bar{f}(x)\nu(\Delta) = U(f,\mathcal{P}) < \frac{\varepsilon}{k}$$

   which implies that $\sum\limits_{\Delta\in C} \nu(\Delta) < \varepsilon$. Therefore, A_k has measure zero; thus Theorem 6.26 implies that $A = \bigcup\limits_{k=1}^{\infty} A_k$ also has measure zero. □

Remark 6.46. Combining Corollary 6.36 and Theorem 6.45, we conclude that the integral of a bounded function on a compact set of measure zero is zero.

Remark 6.47. Let $A = \mathbb{Q} \cap [0,1]$ and $f : A \to \mathbb{R}$ be the constant function $f \equiv 1$. We have shown in Example 6.33 that f is not Riemann integrable. We note that A has no volume since $\partial A = [0,1]$ which is not a set of measure zero. However, A has measure zero since it consists of countable number of points.

1. Since f is continuous on A, the condition that A has volume in Corollary 6.35 cannot be removed.

2. Since A has measure zero, the condition that f is Riemann integrable in Theorem 6.45 cannot be removed.

Definition 6.48. Let $A \subseteq \mathbb{R}^n$ be a set and $f : A \to \mathbb{R}$ be a function. For $B \subseteq A$, the ***restriction of*** f ***to*** B is the function $f\big|_B : A \to \mathbb{R}$ given by $f|_B = f\mathbf{1}_B$. In other words,

$$f\big|_B(x) = \begin{cases} f(x) & \text{if } x \in B\,, \\ 0 & \text{if } x \in A\backslash B\,. \end{cases}$$

The following two theorems are direct consequences of (a) of Proposition 6.41 and Corollary 6.43.

Theorem 6.49. *Let A, B be bounded subsets of $\mathbb{R}^n$, $B \subseteq A$, and $f : A \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable on B if and only if $f|_B$ is Riemann integrable on A. In either cases,*

$$\int_A f\big|_B(x)\,dx = \int_B f(x)\,dx\,.$$

Theorem 6.50. *Let A, B be bounded subsets of $\mathbb{R}^n$ be such that $A \cap B$ has volume zero, and $f : A \cup B \to \mathbb{R}$ be bounded such that $f\big|_A$ and $f\big|_B$ are all Riemann integrable on $A \cup B$. Then f is Riemann integrable on $A \cup B$, and*

$$\int_{A\cup B} f(x)\,dx = \int_A f(x)\,dx + \int_B f(x)\,dx\,.$$

6.4 The Fubini Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous, the fundamental theorem of Calculus can be applied to computed the integral of f on $[a, b]$. In this section, we focus on how the integral of f on $A \subseteq \mathbb{R}^n$, where $n \geqslant 2$, can be computed if the integral exists.

Definition 6.51. Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be bounded sets, and $f : A \times B \to \mathbb{R}$ be a bounded function. For each fixed $x \in A$, the lower integral of the function $f(x, \cdot) : B \to \mathbb{R}$ is denoted by $\underline{\int_B} f(x, y)\,dy$, and the upper integral of $f(x, \cdot) : B \to \mathbb{R}$ is denoted by $\overline{\int_B} f(x, y)\,dy$. If the upper integral and the lower integral of $f(x, \cdot) : B \to \mathbb{R}$ are the same at $x \in A$, we simply write $\displaystyle\int_B f(x,y)\,dy$ for the integrals of $f(x,\cdot)$ on B. The integrals $\displaystyle\overline{\int_A} f(x,y)\,dx$, $\displaystyle\underline{\int_A} f(x,y)\,dx$ and $\displaystyle\int_A f(x,y)\,dx$ are defined similarly.

Theorem 6.52 (Fubini's Theorem)**.** *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be bounded sets, and $f : A \times B \to \mathbb{R}$ be a bounded function. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, write $z = (x,y)$. Then*

$$\underline{\int_{A\times B}} f(z)\,dz \leqslant \underline{\int_A} \Big(\underline{\int_B} f(x,y)dy\Big)dx \leqslant \overline{\int_A} \Big(\overline{\int_B} f(x,y)dy\Big)dx \leqslant \overline{\int_{A\times B}} f(z)\,dz\,, \tag{6.4.1a}$$

$$\underline{\int_{A\times B}} f(z)\,dz \leqslant \underline{\int_B} \Big(\underline{\int_A} f(x,y)dx\Big)dy \leqslant \overline{\int_B} \Big(\overline{\int_A} f(x,y)dx\Big)dy \leqslant \overline{\int_{A\times B}} f(z)\,dz\,. \tag{6.4.1b}$$

In particular, if $f : A \times B \to \mathbb{R}$ is Riemann integrable, then

$$\begin{aligned}\int_{A\times B} f(z)\,dz &= \int_A \Big(\underline{\int_B} f(x,y)dy\Big)dx = \int_A \Big(\overline{\int_B} f(x,y)dy\Big)dx\\ &= \int_B \Big(\underline{\int_A} f(x,y)dx\Big)dy = \int_B \Big(\overline{\int_A} f(x,y)dx\Big)dy\,.\end{aligned}$$

Proof. It suffices to prove (6.4.1a). Let $\varepsilon > 0$ be given. Choose a partition $\mathcal{P}$ of $A \times B$ such that $L(f,\mathcal{P}) > \displaystyle\underline{\int_{A\times B}} f(z)\,dz - \varepsilon$. Since $\mathcal{P}$ is a partition of $A \times B$, there exist partition $\mathcal{P}_A$ of A and partition $\mathcal{P}_B$ of B such that $\mathcal{P} = \big\{\Delta = R \times S \,\big|\, R \in \mathcal{P}_A, S \in \mathcal{P}_B\big\}$. By Proposition 6.41 and Corollary 6.43, we find that

$$\begin{aligned}\underline{\int_A} \Big(\underline{\int_B} f(x,y)\,dy\Big)dx &= \underline{\int_{\bigcup_{R\in\mathcal{P}_A} R}} \mathbf{1}_A(x)\Big(\underline{\int_{\bigcup_{S\in\mathcal{P}_B} S}} f(x,y)\mathbf{1}_B(y)\,dy\Big)dx\\ &\geqslant \sum_{R\in\mathcal{P}_A} \underline{\int_R} \Big(\sum_{S\in\mathcal{P}_B} \underline{\int_S} \overline{f}^{A\times B}(x,y)\,dy\Big)dx\\ &\geqslant \sum_{R\in\mathcal{P}_A}\sum_{S\in\mathcal{P}_B} \underline{\int_R} \Big(\underline{\int_S} \overline{f}^{A\times B}(x,y)\,dy\Big)dx\\ &\geqslant \sum_{R\in\mathcal{P}_A, S\in\mathcal{P}_B} \inf_{(x,y)\in R\times S} \overline{f}^{A\times B}(x,y)\nu_m(S)\nu_n(R)\\ &= \sum_{\Delta\in\mathcal{P}} \inf_{(x,y)\in\Delta} \overline{f}^{A\times B}(x,y)\nu_{n+m}(\Delta) = L(f,\mathcal{P}) > \underline{\int_{A\times B}} f(z)dz - \varepsilon\,.\end{aligned}$$

Since $\varepsilon > 0$ is given arbitrarily, we conclude that

$$\underline{\int_{A\times B}} f(z)\,dz \leqslant \underline{\int_B} \Big(\underline{\int_A} f(x,y)dx\Big)dy\,.$$

Similarly, $\displaystyle\overline{\int_A} \Big(\overline{\int_B} f(x,y)dy\Big)dx \leqslant \overline{\int_{A\times B}} f(z)\,dz$; thus (6.4.1a) is concluded. □

Corollary 6.53. *Let $S \subseteq \mathbb{R}^n$ be a bounded set with volume, $\varphi_1, \varphi_2 : S \to \mathbb{R}$ be continuous maps such that $\varphi_1(x) \leqslant \varphi_2(x)$ for all $x \in S$, $A = \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R} \,\big|\, x \in S, \varphi_1(x) \leqslant y \leqslant \varphi_2(x)\big\}$, and $f : A \to \mathbb{R}$ be continuous. Then f is Riemann integrable on A, and*

$$\int_A f(x,y)\,d(x,y) = \int_S \Big(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\Big)dx\,. \tag{6.4.2}$$

Proof. Since ∂A has measure zero, and f is continuous on A, Corollary 6.35 implies that f is Riemann integrable on A. Let $m = \min\limits_{x\in S} \varphi_1(x)$ and $M = \max\limits_{x\in S} \varphi_2(x)$. Then $A \subseteq S \times [m,M]$; thus Theorem 6.50 and the Fubini Theorem imply that

$$\begin{aligned}\int_A f(x,y)\,d(x,y) &= \int_{S\times[m,M]} \overline{f}^A(x,y)\,d(x,y) = \underline{\int}_S \Big(\underline{\int}_m^M \overline{f}^A(x,y)\,dy\Big)dx \\ &= \overline{\int}_S \Big(\overline{\int}_m^M \overline{f}^A(x,y)\,dy\Big)dx\,.\end{aligned}$$

Noting that $[m,M]$ has a boundary of volume zero in $\mathbb{R}$, and for each $x \in S$, $\overline{f}^A(x,\cdot)$ is continuous except perhaps at $y = \varphi_1(x)$ and $y = \varphi_2(x)$, Corollary 6.35 implies that $\overline{f}^A(x,\cdot)$ is Riemann integrable on $[m,M]$ for each $x \in S$. Therefore, $\underline{\int}_m^M \overline{f}^A(x,y)\,dy = \overline{\int}_m^M \overline{f}^A(x,y)\,dy$ which further implies that

$$\int_A f(x,y)\,d(x,y) = \int_S \Big(\int_m^M \overline{f}^A(x,y)\,dy\Big)dx\,. \tag{6.4.3}$$

For each fixed $x \in S$, let $A_x = \big\{y \in \mathbb{R} \,\big|\, \varphi_1(x) \leqslant y \leqslant \varphi_2(x)\big\}$. Then $\overline{f}^A(x,y) = f(x,y)\mathbf{1}_{A_x}(y)$ for all $(x,y) \in S \times [m,M]$ or equivalently, $\overline{f}^A(x,\cdot) = f(x,\cdot)|_{A_x}$ for all $x \in S$; thus Proposition 6.41 (a) implies that

$$\int_m^M \overline{f}^A(x,y)\,dy = \int_{A_x} f(x,y)\,dy = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \qquad \forall\, x \in S\,. \tag{6.4.4}$$

Combining (6.4.3) and (6.4.4), we conclude (6.4.2). □

Corollary 6.54. 1. *Let $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ be continuous maps such that $\varphi_1(x) \leqslant \varphi_2(x)$ for all $x \in [a,b]$, $A = \big\{(x,y) \,\big|\, a \leqslant x \leqslant b, \varphi_1(x) \leqslant y \leqslant \varphi_2(x)\big\}$, and $f : A \to \mathbb{R}$ be continuous. Then f is Riemann integrable on A, and*

$$\int_A f(x,y)\,d\mathbb{A} = \int_a^b \Big(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\Big)dx\,.$$

2. *Let* $\psi_1, \psi_2 : [c,d] \to \mathbb{R}$ *be continuous maps such that* $\psi_1(y) \leqslant \psi_2(y)$ *for all* $y \in [c,d]$, $A = \{(x,y) \,|\, c \leqslant y \leqslant d, \psi_1(y) \leqslant x \leqslant \psi_2(y)\}$, *and* $f : A \to \mathbb{R}$ *be continuous. Then* f *is Riemann integrable on* A*, and*

$$\int_A f(x,y)\, d\mathbb{A} = \int_c^d \Big(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx \Big) dy\,.$$

Remark 6.55. To simplify the notation, sometimes we use $\int_S \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dydx$ to denote the iterated integral the iterated integral $\int_S \Big(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy \Big) dx$. Similar notation applies to the upper and the lower integrals. For example, we also have $\overline{\int_a^b} \underline{\int_c^d} f(x,y)\, dydx = \overline{\int_a^b} \Big(\underline{\int_c^d} f(x,y)\, dy \Big) dx$.

Remark 6.56. For each $x \in [a,b]$, define $\varphi(x) = \underline{\int_c^d} f(x,y)\, dy$ and $\psi(x) = \overline{\int_c^d} f(x,y)\, dy$. Then $\varphi(x) \leqslant \psi(x)$ for all $x \in [a,b]$, and the Fubini Theorem implies that

$$\int_a^b \big[\psi(x) - \varphi(x)\big]\, dx = 0\,.$$

By Theorem 6.45, the set $\{x \in [a,b] \,|\, \psi(x) - \varphi(x) \neq 0\}$ has measure zero. In other words, except on a set of measure zero, $f(x,\cdot)$ is Riemann integrable on $[c,d]$ if f is Riemann integrable on $[a,b] \times [c,d]$. This property can be rephrased as that "$f(x,\cdot)$ is Riemann integrable on $[c,d]$ for **almost every** $x \in [a,b]$ if f is Riemann integrable on the rectangle $[a,b] \times [c,d]$". Similarly, $f(\cdot,y)$ is Riemann integrable for almost every $y \in [c,d]$ if f is Riemann integrable on $[a,b] \times [c,d]$.

Remark 6.57. The integrability of f does not guarantee that $f(x,\cdot)$ or $f(\cdot,y)$ is Riemann integrable. In fact, there exists a function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that f is Riemann integrable, $f(\cdot,y)$ is Riemann integrable for each $y \in [0,1]$, but $f(x,\cdot)$ is not Riemann integrable for infinitely many $x \in [0,1]$. For example, let

$$f(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ or } y \text{ is irrational}\,, \\ \dfrac{1}{p} & \text{if } x,y \in \mathbb{Q} \text{ and } x = \dfrac{q}{p} \text{ with } (p,q) = 1\,. \end{cases}$$

Then

1. For each $y \in [0,1]$, $f(\cdot, y)$ is continuous at all irrational numbers. Therefore, $f(\cdot, y)$ is Riemann integrable, and $\overline{\int_0^1} f(x,y)\,dx = \underline{\int_0^1} f(x,y)\,dx = 0$.

2. For $x = 0$ or $x \notin \mathbb{Q}$, $f(x, \cdot)$ is Riemann integrable, and $\int_0^1 f(x,y)\,dy = 0$.

3. If $x = \dfrac{q}{p}$ with $(p,q) = 1$, $f(x,\cdot)$ is nowhere continuous in $[0,1]$. In fact, for each $y_0 \in [0,1]$,
$$\lim_{\substack{y \to y_0 \\ y \in \mathbb{Q}}} f(x,y) = \frac{1}{p} \quad \text{while} \quad \lim_{\substack{y \to y_0 \\ y \notin \mathbb{Q}}} f(x,y) = 0\,;$$
thus the limit of $f(x,y)$ as $y \to y_0$ does not exist. Therefore, the Lebesgue theorem implies that $f(x,\cdot)$ is not Riemann integrable if $x \in \mathbb{Q} \cap (0,1]$. On the other hand, for $x = \dfrac{q}{p}$ with $(p,q) = 1$ we have
$$\underline{\int_0^1} f(x,y)\,dy = 0 \quad \text{and} \quad \overline{\int_0^1} f(x,y)\,dy = \frac{1}{p}\,.$$

4. Define $\varphi(x) = \underline{\int_0^1} f(x,y)\,dy$ and $\psi(x) = \overline{\int_0^1} f(x,y)\,dy$. Then 2 and 3 imply that φ and ψ are Riemann integrable on $[0,1]$, and $\int_0^1 \varphi(x)dx = \int_0^1 \psi(x)dx = 0$.

5. For each $a \notin \mathbb{Q} \cap [0,1]$ and $b \in [0,1]$, f is continuous at (a,b). In fact, for any given $\varepsilon > 0$, there exists a prime number p such that $\dfrac{1}{p} < \varepsilon$. Let
$$\delta = \min\Big\{\big|a - \frac{\ell}{k}\big| \,\Big|\, 0 \leqslant \ell \leqslant k \leqslant p, k \in \mathbb{N}, \ell \in \mathbb{N} \cup \{0\}\Big\}\,.$$
Then $\delta > 0$, and if $(x,y) \in B\big((a,b),\delta\big) \cap ([0,1] \times [0,1])$, we have
$$\big|f(x,y) - f(a,b)\big| = \big|f(x,y)\big| < \frac{1}{p} < \varepsilon\,,$$
where we use the fact that if $(x,y) \in B\big((a,b),\delta\big)$ and $x \in \mathbb{Q}$, then $x = \dfrac{\ell}{k}$ (in reduced form) for some $k > p$.

As a consequence, $\big\{(a,b) \in \mathbb{R}^2 \,\big|\, \bar{f} \text{ is discontinuous at } (a,b)\big\} \subseteq \mathbb{Q} \times [0,1]$. Since $\mathbb{Q} \times [0,1]$ is a countable union of measure zero sets, it has measure zero; thus f is Riemann integrable by the Lebesgue theorem. The Fubini theorem then implies that
$$\int_{[0,1]\times[0,1]} f(x,y)\,d\mathbb{A} = \int_0^1 \int_0^1 f(x,y)\,dxdy = 0\,.$$

Remark 6.58. The integrability of $f(x,\cdot)$ and $f(\cdot,y)$ does not guarantee the integrability of f. In fact, there exists a bounded function $f:[0,1]\times[0,1]\to\mathbb{R}$ such that $f(x,\cdot)$ and $f(\cdot,y)$ are both Riemann integrable on $[0,1]$, but f is not Riemann integrable on $[0,1]\times[0,1]$. For example, let

$$f(x,y)=\begin{cases} 1 & \text{if } (x,y)=\big(\frac{k}{2^n},\frac{\ell}{2^n}\big),\ 0<k,\ell<2^n \text{ odd numbers, } n\in\mathbb{N}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then for each $x\in[0,1]$, $f(x,\cdot)$ only has finite number of discontinuities; thus $f(x,\cdot)$ is Riemann integrable, and

$$\int_0^1 f(x,y)\,dy=0\,.$$

Similarly, $f(\cdot,y)$ is Riemann integrable, and $\int_0^1 f(x,y)\,dx=0$. As a consequence,

$$\int_0^1\int_0^1 f(x,y)\,dydx=\int_0^1\int_0^1 f(x,y)\,dxdy=0\,.$$

However, note that f is nowhere continuous on $[0,1]\times[0,1]$; thus the Lebesgue theorem implies that f is not Riemann integrable. One can also see this by the fact that $U(f,\mathcal{P})=1$ and $L(f,\mathcal{P})=0$ for all partition of $[0,1]\times[0,1]$.

Example 6.59. Let $A=\big\{(x,y)\in\mathbb{R}^2\,\big|\,0\leqslant x\leqslant 1, x\leqslant y\leqslant 1\big\}$, and $f:A\to\mathbb{R}$ be given by $f(x,y)=xy$. Then Corollary 6.54 implies that

$$\int_A f(x,y)\,d\mathbb{A}=\int_0^1\Big(\int_x^1 xy\,dy\Big)dx=\int_0^1\frac{xy^2}{2}\Big|_{y=x}^{y=1}dx=\int_0^1\Big(\frac{x}{2}-\frac{x^3}{2}\Big)dx=\frac{1}{4}-\frac{1}{8}=\frac{1}{8}\,.$$

On the other hand, since $A=\big\{(x,y)\in\mathbb{R}^2\,\big|\,0\leqslant y\leqslant 1, 0\leqslant x\leqslant y\big\}$, we can also evaluate the integral of f on A by

$$\int_A xy\,d\mathbb{A}=\int_0^1\Big(\int_0^y xy\,dx\Big)dy=\int_0^1\frac{x^2y}{2}\Big|_{x=0}^{x=y}dy=\int_0^1\frac{y^3}{2}dy=\frac{1}{8}\,.$$

Example 6.60. Let $A=\big\{(x,y)\in\mathbb{R}^2\,\big|\,0\leqslant x\leqslant 1,\sqrt{x}\leqslant y\leqslant 1\big\}$, and $f:A\to\mathbb{R}$ be given by $f(x,y)=e^{y^3}$. Then Corollary 6.54 implies that

$$\int_A f(x,y)\,d\mathbb{A}=\int_0^1\Big(\int_{\sqrt{x}}^1 e^{y^3}\,dy\Big)dx\,.$$

Since we do not know how to compute the inner integral, we look for another way of finding the integral. Observing that $A = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leqslant y \leqslant 1, 0 \leqslant x \leqslant y^2\}$, we have

$$\int_A f(x,y)\,d\mathbb{A} = \int_0^1 \Big(\int_0^{y^2} e^{y^3}\,dx\Big)dy = \int_0^1 y^2 e^{y^3} dy = \frac{1}{3}e^{y^3}\Big|_{y=0}^{y=1} = \frac{e-1}{3}\,.$$

Example 6.61. Let $A \subseteq \mathbb{R}^3$ be the set $\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, \text{and } x_1 + x_2 + x_3 \leqslant 1\}$, and $f : A \to \mathbb{R}$ be given by $f(x_1,x_2,x_3) = (x_1+x_2+x_3)^2$. Let $S = [0,1] \times [0,1] \times [0,1]$, and $\bar{f} : \mathbb{R}^3 \to \mathbb{R}$ be the extension of f by zero outside A. Then Corollary 6.35 implies that f is Riemann integrable (since ∂A has measure zero). Write $\widehat{x}_1 = (x_2,x_3)$, $\widehat{x}_2 = (x_1,x_3)$ and $\widehat{x}_3 = (x_1,x_2)$. Theorem 6.49 implies that

$$\int_A f(x)dx = \int_S \bar{f}(x)dx\,,$$

and Theorem 6.52 implies that

$$\int_S \bar{f}(x)dx = \int_{[0,1]} \Big(\int_{[0,1]\times[0,1]} \bar{f}(\widehat{x}_3,x_3)d\widehat{x}_3\Big)dx_3\,.$$

Let $A_{x_3} = \{(x_1,x_2) \in \mathbb{R}^2 \,|\, x_1 \geqslant 0, x_2 \geqslant 0, x_1+x_2 \leqslant 1-x_3\}$. Then for each $x_3 \in [0,1]$,

$$\int_{[0,1]\times[0,1]} \bar{f}(\widehat{x}_3,x_3)d\widehat{x}_3 = \int_{A_{x_3}} f(\widehat{x}_3,x_3)d\widehat{x}_3 = \int_0^{1-x_3} \Big(\int_0^{1-x_3-x_2} f(x_1,x_2,x_3)dx_1\Big)dx_2\,.$$

Computing the iterated integral, we find that

$$\begin{aligned}
\int_A f(x)dx &= \int_0^1 \Big[\int_0^{1-x_3} \Big(\int_0^{1-x_3-x_2} (x_1+x_2+x_3)^2 dx_1\Big)dx_2\Big]dx_3 \\
&= \int_0^1 \Big[\int_0^{1-x_3} \frac{(x_1+x_2+x_3)^3}{3}\Big|_{x_1=0}^{x_1=1-x_3-x_2} dx_2\Big]dx_3 \\
&= \int_0^1 \Big[\int_0^{1-x_3} \Big(\frac{1}{3} - \frac{(x_2+x_3)^3}{3}\Big)dx_2\Big]dx_3 \\
&= \int_0^1 \Big(\frac{1}{4} - \frac{x_3}{3} + \frac{x_3^4}{12}\Big)dx_3 = \frac{1}{4} - \frac{1}{6} + \frac{1}{60} = \frac{15-10+1}{60} = \frac{1}{10}\,.
\end{aligned}$$

Example 6.62. In this example we compute the volume of the n-dimensional unit ball ω_n. By the Fubini theorem,

$$\omega_n = \int_{-1}^1 \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \cdots \int_{-\sqrt{1-x_1^2-\cdots-x_{n-1}^2}}^{\sqrt{1-x_1^2-\cdots-x_{n-1}^2}} dx_n \cdots dx_1\,.$$

Note that the integral $\displaystyle\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}}\cdots\int_{-\sqrt{1-x_1^2-\cdots-x_{n-1}^2}}^{\sqrt{1-x_1^2-\cdots-x_{n-1}^2}} dx_n\cdots dx_2$ is in fact $\omega_{n-1}(1-x_1^2)^{\frac{n-1}{2}}$; thus

$$\omega_n = \int_{-1}^{1}\omega_{n-1}(1-x^2)^{\frac{n-1}{2}}\,dx = 2\,\omega_{n-1}\int_0^{\frac{\pi}{2}}\cos^n\theta d\theta\,. \tag{6.4.5}$$

Integrating by parts,

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\cos^n\theta\,d\theta &= \int_0^{\frac{\pi}{2}}\cos^{n-1}\theta\,d(\sin\theta) = \cos^{n-1}\theta\sin\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}\theta\sin^2\theta\,d\theta\\ &= (n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}\theta(1-\cos^2\theta)\,d\theta\end{aligned}$$

which implies that

$$\int_0^{\frac{\pi}{2}}\cos^n\theta\,d\theta = \frac{n-1}{n}\int_0^{\frac{\pi}{2}}\cos^{n-2}\theta\,d\theta\,.$$

As a consequence,

$$\int_0^{\frac{\pi}{2}}\cos^n\theta\,d\theta = \begin{cases}\dfrac{(n-1)(n-3)\cdots 2}{n(n-2)\cdots 3}\displaystyle\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta & \text{if } n \text{ is odd}\,,\\[2ex] \dfrac{(n-1)(n-3)\cdots 1}{n(n-2)\cdots 2}\displaystyle\int_0^{\frac{\pi}{2}}d\theta & \text{if } n \text{ is even}\,;\end{cases}$$

thus the recursive formula (6.4.5) implies that $\omega_n = \dfrac{2\,\omega_{n-2}}{n}\pi$. Further computations shows that

$$\omega_n = \begin{cases}\dfrac{(2\pi)^{\frac{n-1}{2}}}{n(n-2)\cdots 3}\omega_1 & \text{if } n \text{ is odd}\,,\\[2ex] \dfrac{(2\pi)^{\frac{n-2}{2}}}{n(n-2)\cdots 4}\omega_2 & \text{if } n \text{ is even}\,.\end{cases}$$

Let Γ be the Gamma function defined by $\Gamma(t) = \displaystyle\int_0^\infty x^{t-1}e^{-x}\,dx$ for $t>0$. Then $\Gamma(x+1) = x\Gamma(x)$ for all $x>0$, $\Gamma(1)=1$ and $\Gamma\big(\frac{1}{2}\big)=\sqrt{\pi}$. By the fact that $\omega_1 = 2$ and $\omega_2=\pi$, we can express ω_n as

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\big(\frac{n+2}{2}\big)}\,.$$

6.5 The Monotone and Bounded Convergence Theorems

In the following, we introduce two very important theorems in the theory of integration of functions. Before proceeding, we establish the following two lemmas.

Lemma 6.63. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then for each $\varepsilon > 0$, there exist continuous functions $g, h : [a,b] \to \mathbb{R}$ such that $\inf_{x\in[a,b]} f(x) \leqslant g \leqslant f \leqslant h \leqslant \sup_{x\in[a,b]} f(x)$ and*

$$\underline{\int_a^b} f(x)\,dx < \int_a^b g(x)\,dx + \varepsilon \qquad \textit{and} \qquad \overline{\int_a^b} f(x)\,dx > \int_a^b h(x)\,dx - \varepsilon\,.$$

Proof. We only prove the case of lower integral since the proof of the counter-part is similar.

Let $\varepsilon > 0$ be given, and $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$ such that $L(f,\mathcal{P}) > \underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2}$. Let $s(x)$ be the step function given by

$$s(x) = \sum_{k=1}^{n-1} \Big(\inf_{x\in[x_{k-1},x_k]} f(x)\Big)\mathbf{1}_{[x_k-1,x_k)}(x) + \Big(\inf_{x\in[x_{n-1},b]} f(x)\Big)\mathbf{1}_{[x_{n-1},b]}(x)$$

which is a linear combination of characteristic functions. Then $\inf_{x\in[a,b]} f(x) \leqslant s(x) \leqslant f(x)$ for all $x \in [a,b]$ and

$$\underline{\int_a^b} f(x)\,dx < \int_a^b s(x)\,dx + \frac{\varepsilon}{2} \tag{6.5.1}$$

since the integral of s on $[a,b]$ is exactly the lower sum $L(f,\mathcal{P})$. On the other hand, for such a simple function s we can always find a continuous function $g : [a,b] \to \mathbb{R}$ (for example, g can be a trapezoidal function) such that $\inf_{x\in[a,b]} f(x) \leqslant g(x) \leqslant s(x)$ for all $x \in [a,b]$ and

$$\int_a^b s(x)\,dx < \int_a^b g(x)\,dx + \frac{\varepsilon}{2}\,. \tag{6.5.2}$$

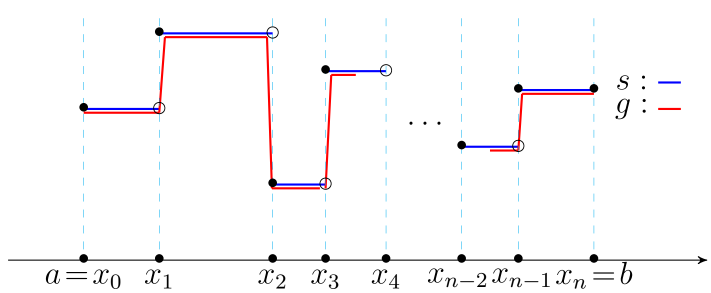


Figure 6.3: One way of constructing g given simple function s

The combination of (6.5.1) and (6.5.2) then concludes the lemma. □

Lemma 6.64. *Let $h_k : [a,b] \to \mathbb{R}$ be continuous for each $k \in \mathbb{N}$, and $h_k(x) \geqslant h_{k+1}(x)$ for each $k \in \mathbb{N}$ and $x \in [a,b]$. If $\lim_{k\to\infty} h_k(x) = 0$ for each $x \in [a,b]$, then $\lim_{k\to\infty} \int_a^b h_k(x)\,dx = 0$.*

Proof. For each $k \in \mathbb{N}$ define $c_k = \max\limits_{x\in[a,b]} h_k(x)$. Then $\{c_k\}_{k=1}^\infty$ is a decreasing sequence of non-negative real numbers, so $\lim\limits_{k\to\infty} c_k$ exists and is non-negative. Since $\int_a^b h_k(x)\,dx \leqslant c_k(b-a)$, it suffices to show that $\lim\limits_{k\to\infty} c_k = 0$.

Suppose the contrary that $\lim\limits_{k\to\infty} c_k = 2\delta$ for some $\delta > 0$. Then there exists $N > 0$ such that $c_k \geqslant \delta$ for all $k \geqslant N$. Define $F_k = \big\{x \in [a,b] \,\big|\, h_k(x) \geqslant \delta\big\}$. Then

1. F_k is closed for each $k \in \mathbb{N}$ by Theorem 4.14.
2. $F_k \supseteq F_{k+1}$ for each $k \in \mathbb{N}$.
3. $F_k \neq \varnothing$ for each $k \geqslant N$.

Therefore, by the nested set property we have $\bigcap\limits_{k=N}^\infty F_k \neq \varnothing$ $\Big($or otherwise $\bigcup\limits_{k=N}^\infty F_k^\complement$ is an open cover of compact set $[a,b]$ which, using the finite subcover property, implies that there exists $F_m \subseteq [a,b]^\complement$, a contradiction$\Big)$. This then implies that there exists $c \in [a,b]$ such that $h_k(c) \geqslant \delta$ for all $k \geqslant N$ which contradicts to the condition that $\lim\limits_{k\to\infty} h_k(x) = 0$ for all $x \in [a,b]$. Therefore, $\lim\limits_{k\to\infty} c_k = 0$. □

Theorem 6.65. *Let $\{f_k\}_{k=1}^\infty$ be a decreasing sequence of bounded functions on $[a,b]$; that is, for each $k \in \mathbb{N}$, $f_k(x) \geqslant f_{k+1}(x)$ for all $x \in [a,b]$ and f_k is bounded. If $\lim\limits_{k\to\infty} f_k(x) = 0$ for all $x \in [a,b]$, then*

$$\lim_{k\to\infty} \underline{\int_a^b} f_k(x)\,dx = 0 \left(= \underline{\int_a^b} \lim_{k\to\infty} f_k(x)\,dx\right).$$

Proof. Let $\varepsilon > 0$ be given. By Lemma 6.63, for each $k \in \mathbb{N}$ there exists a continuous function $g_k : [a,b] \to \mathbb{R}$ such that $0 \leqslant g_k \leqslant f_k$ and

$$\underline{\int_a^b} f_k(x)\,dx < \int_a^b g_k(x)\,dx + \frac{\varepsilon}{2^{k+1}}. \tag{6.5.3}$$

Define $h_k = \min\{g_1, \cdots, g_k\}$. Then h_k is continuous on $[a,b]$, $h_k \geqslant h_{k+1}$ (that is, $\{h_k\}_{k=1}^\infty$ is a decreasing sequence of funtions), $0 \leqslant h_k \leqslant g_k \leqslant f_k$ for all $k \in \mathbb{N}$, and $\lim\limits_{k\to\infty} h_k(x) = 0$ for all $x \in [a,b]$. Therefore, Lemma 6.64 implies that there exists $N > 0$ such that

$$\int_a^b h_k(x)\,dx < \frac{\varepsilon}{4} \qquad \forall\, k \geqslant N. \tag{6.5.4}$$

On the other hand, for $1 \leqslant \ell \leqslant k$, $\max\{g_\ell, \cdots, g_k\} \leqslant \max\{f_\ell, \cdots, f_k\} = f_\ell$; thus

$$\int_a^b \big(\max\{g_\ell, \cdots, g_k\}(x) - g_\ell(x)\big)\,dx \leqslant \underline{\int_a^b} f_\ell(x)\,dx - \int_a^b g_\ell(x)\,dx < \frac{\varepsilon}{2^{\ell+1}}.$$

Moreover, for each $1 \leqslant j \leqslant k$ and $x \in [a, b]$,

$$\begin{aligned} 0 \leqslant g_k(x) = g_j(x) + \big(g_k(x) - g_j(x)\big) &\leqslant g_j(x) + \big(\max\{g_j(x), \cdots, g_k(x)\} - g_j\big) \\ &\leqslant g_j(x) + \sum_{\ell=1}^{k-1} \big(\max\{g_\ell, \cdots, g_k\}(x) - g_\ell(x)\big), \end{aligned}$$

so minimizing the right-hand side over all $1 \leqslant j \leqslant k$ implies that

$$0 \leqslant g_k(x) \leqslant h_k(x) + \sum_{\ell=1}^{k-1} \big(\max\{g_\ell, \cdots, g_k\}(x) - g_\ell(x)\big) \qquad \forall\, x \in [a, b].$$

As a consequence,

$$0 \leqslant \int_a^b g_k(x)\,dx \leqslant \int_a^b h_k(x)\,dx + \sum_{\ell=1}^{k-1} \frac{\varepsilon}{2^{\ell+1}} \leqslant \int_a^b h_k(x)\,dx + \frac{\varepsilon}{2};$$

thus (6.5.3) and (6.5.4) imply that

$$0 \leqslant \underline{\int_a^b} f_k(x)dx < \varepsilon \qquad \forall\, k \geqslant N. \qquad \square$$

Example 6.66. Let $\{q_1, q_2, \cdots, q_n, \cdots\}$ denote the rational numbers in $[0, 1]$, and $f_k : [0, 1] \to \mathbb{R}$ be defined by

$$f_k(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^\complement \cup \{q_1, q_2, \cdots, q_k\}, \\ 1 & \text{otherwise}. \end{cases}$$

Then $f_k \geqslant f_{k+1}$ and $\lim\limits_{k\to\infty} f_k(x) = 0$ for all $x \in [0, 1]$. Note that f_k is not Riemann integrable on $[0, 1]$ but $\underline{\int_0^1} f_k(x)\,dx = 0$ for all $k \in \mathbb{N}$; thus

$$\lim_{k\to\infty} \underline{\int_0^1} f_k(x)\,dx = 0.$$

Note that $\overline{\int_0^1} f_k(x)\,dx = 1$ for all $k \in \mathbb{N}$.

Corollary 6.67 (Monotone Convergence Theorem)**.** *Let $f_k, f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$, and $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in [a, b]$. Suppose that $f_k \leqslant f_{k+1}$ for all $k \in \mathbb{N}$. Then*

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} \int_a^b f_k(x)\,dx.$$

Proof. Let $g_k = f - f_k$. Then $\{g_k\}_{k=1}^\infty$ is a decreasing sequence of bounded functions on $[a, b]$ (since $0 \leqslant g_k \leqslant f - f_1$) and $\lim\limits_{k\to\infty} g_k(x) = 0$ for all $x \in [a, b]$. Therefore, the integrability of f_k and f, as well as Theorem 6.65, imply that

$$\begin{aligned} 0 = \lim_{k\to\infty} \underline{\int_a^b} g_k(x)\,dx = \lim_{k\to\infty} \underline{\int_a^b} (f - f_k)(x)\,dx = \lim_{k\to\infty} \int_a^b (f - f_k)(x)dx \\ = \int_a^b f(x)\,dx - \lim_{k\to\infty} \int_a^b f_k(x)\,dx\,. \end{aligned}$$ □

Corollary 6.68 (Arzelà's Bounded Convergence Theorem)**.** *Let $f_k, f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$, and $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in [a, b]$. Suppose that there exists a constant $M > 0$ such that $|f_k(x)| \leqslant M$ for all $x \in [a, b]$ and $k \in \mathbb{N}$. Then*

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} \int_a^b f_k(x)\,dx\,.$$

Proof. Let $\varepsilon > 0$ be given. For each $k \in \mathbb{N}$, define $g_k(x) = \sup\limits_{\ell \geqslant k} |f_\ell(x) - f(x)|$. Then $\{g_k\}_{k=1}^\infty$ is a decreasing sequence of bounded functions on $[a, b]$ and $\lim\limits_{k\to\infty} g_k(x) = 0$ for all $x \in [a, b]$. Therefore, Theorem 6.65 implies that there exists $N > 0$ such that

$$\underline{\int_a^b} g_k(x)\,dx < \varepsilon \qquad \forall\, k \geqslant N\,.$$

Therefore, by observing that $0 \leqslant |f_k(x) - f(x)| \leqslant g_k(x)$ for all $k \in \mathbb{N}$, by the integrability of f_k and f we conclude that

$$\int_a^b |f_k(x) - f(x)|\,dx = \underline{\int_a^b} |f_k(x) - f(x)|\,dx \leqslant \underline{\int_a^b} g_k(x)\,dx < \varepsilon \qquad \forall\, k \geqslant N\,.$$ □

Theorem 6.69 (Monotone Convergence Theorem)**.** *Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a rectangle in $\mathbb{R}^n$, $f_k, f : A \to \mathbb{R}$ be Riemann integrable on A and $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in A$. Suppose that $\{f_k\}_{k=1}^\infty$ is a monotone sequence of functions; that is, $f_k \leqslant f_{k+1}$ or $f_k \geqslant f_{k+1}$ for all $k \in \mathbb{N}$. Then*

$$\lim_{k\to\infty} \int_A f_k(x)\,dx = \int_A f(x)\,dx\,.$$

Proof. W.L.O.G. we assume that $f_k \geqslant f_{k+1}$ for all $k \in \mathbb{N}$. We first prove the case $n = 2$ and write $A = [a, b] \times [c, d]$. Define $g_k(x) = \underline{\int_c^d} \big(f_k(x, y) - f(x, y)\big)dy$. Then $g_k \geqslant g_{k+1}$ for all $k \in \mathbb{N}$. Moreover, Theorem 6.65 implies that $\lim\limits_{k\to\infty} g_k(x) = 0$, and the Fubini theorem (Theorem 6.52) implies that g_k is Riemann integrable on $[a,b]$ for all $k \in \mathbb{N}$. Therefore, by the monotone convergence theorem for functions of one variable (Corollary 6.67) we find that

$$\begin{aligned}0 = \lim_{k\to\infty} \int_a^b g_k(x)\,dx &= \lim_{k\to\infty} \int_a^b \Big(\underline{\int_c^d} \big(f_k(x,y) - f(x,y)\big) dy \Big) dx \\ &= \lim_{k\to\infty} \int_A \big(f_k(x,y) - f(x,y)\big)\, d(x,y)\,.\end{aligned}$$

Now suppose that the conclusion holds for the case $n = N$. Then for $n = N+1$, write $A = R \times [c,d]$ for some rectangle R in $\mathbb{R}^N$, and define g_k by

$$g_k(x_1, \cdots, x_N) = \underline{\int_c^d} \big(f_k(x_1, \cdots, x_{N+1}) - f(x_1, \cdots, x_{N+1})\big) dx_{N+1}\,.$$

Then Theorem 6.65 again implies that $\{g_k\}_{k=1}^\infty$ converges monotonically to 0 on R, and the Fubini theorem (Theorem 6.52) implies that g_k is Riemann integrable on R for all $k \in \mathbb{N}$. Write $x' = (x_1, \cdots, x_N)$. Then the validity of the monotone convergence theorem for N-tuple integrals implies that

$$\begin{aligned}0 = \lim_{k\to\infty} \int_R g_k(x')\,dx' &= \lim_{k\to\infty} \int_R \Big(\underline{\int_c^d} \big(f_k(x', x_{N+1}) - f(x', x_{N+1})\big) dx_{N+1} \Big)\, dx' \\ &= \lim_{k\to\infty} \int_A \big(f_k(x) - f(x)\big)\, dx\,.\end{aligned}$$

□

Theorem 6.70 (Bounded Convergence Theorem)**.** *Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a rectangle in $\mathbb{R}^n$, $f_k, f : A \to \mathbb{R}$ be Riemann integrable on A and $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in A$. Suppose that there exists a constant $M > 0$ such that $|f_k(x)| \leqslant M$ for all $x \in A$ and $k \in \mathbb{N}$. Then*

$$\lim_{k\to\infty} \int_A f_k(x)\,dx = \int_A f(x)\,dx\,.$$

Proof. For $1 \leqslant j \leqslant n$, let R_j be the rectangle $[a_j, b_j] \times \cdots \times [a_n, b_n]$, and define $g_k^{(j)} : R_j \to \mathbb{R}$ iteratively as follows:

1. $g_k^{(1)}(x) = \sup\limits_{\ell \geqslant k} |f_\ell(x) - f(x)|$;

2. for $1 \leqslant j \leqslant n-1$, $g_k^{(j+1)}(x_{j+1}, \cdots, x_n) = \underline{\int_{a_j}^{b_j}} g_k^{(j)}(x_j, x_{j+1} \cdots, x_n)\, dx_j\,.$

Then for each $1 \leqslant j \leqslant n$, $\{g_k^{(j)}\}_{k=1}^\infty$ is a decreasing sequence of bounded functions; that is,

$$g_k^{(j)}(x_j, \cdots, x_n) \geqslant g_{k+1}^{(j)}(x_j, \cdots, x_n) \qquad \forall\, k \in \mathbb{N}, 1 \leqslant j \leqslant n, (x_j, \cdots, x_n) \in R_j\,.$$

Moreover, for each $1 \leqslant j \leqslant n$, $\lim\limits_{k\to\infty} g_k^{(j)}(x) = 0$ for all $x \in R_j$. To see this, we note that by the fact that $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in A$,

$$\lim_{k\to\infty} g_k^{(1)}(x) = \lim_{k\to\infty} \sup_{\ell \geqslant k} \big|f_\ell(x) - f(x)\big| = \limsup_{k\to\infty} \big|f_\ell(x) - f(x)\big| = 0\,.$$

Assume that for some $1 \leqslant j \leqslant n-1$ such that $\lim\limits_{k\to\infty} g_k^{(j)}(x) = 0$ for all $x \in R_j$. Then

$$\lim_{k\to\infty} g_k^{(j+1)}(x_{j+1}, \cdots, x_n) = \lim_{k\to\infty} \underline{\int_{a_j}^{b_j}} g_k^{(j)}(x_j, x_{j+1}, \cdots, x_n)\, dx_j = 0$$

for all $(x_{j+1}, \cdots, x_n) \in R_{j+1}$ which shows that $\lim\limits_{k\to\infty} g_k^{(j+1)}(x) = 0$ for all $x \in R_{j+1}$. By induction we conclude that $\lim\limits_{k\to\infty} g_k^{(j)}(x) = 0$ for all $x \in R_j$; thus Theorem 6.65 and the Fubini theorem (Theorem 6.52) imply that

$$\begin{aligned}
0 &= \lim_{k\to\infty} \underline{\int_{a_n}^{b_n}} g_k^{(n)}(x_n)\, dx_n = \lim_{k\to\infty} \underline{\int_{a_n}^{b_n}} \cdots \underline{\int_{a_1}^{b_1}} \sup_{\ell \geqslant k} \big|f_\ell(x_1, \cdots, x_n) - f(x_1, \cdots, x_n)\big|\, dx_1 \cdots dx_n \\
&\geqslant \limsup_{k\to\infty} \underline{\int_{a_n}^{b_n}} \cdots \underline{\int_{a_1}^{b_1}} \big|f_k(x_1, \cdots, x_n) - f(x_1, \cdots, x_n)\big|\, dx_1 \cdots dx_n \\
&\geqslant \limsup_{k\to\infty} \underline{\int_A} \big|f_k(x) - f(x)\big| dx = \limsup_{k\to\infty} \int_A \big|f_k(x) - f(x)\big| dx\,.
\end{aligned}$$

□

Remark 6.71. 1. If A is a bounded set with volume, we can choose a rectangle $S \supseteq A$ and consider $g_k = \overline{f}_k^A$ as well as $g = \overline{f}^A$. Then $g_k, g : S \to \mathbb{R}$ satisfy the assumptions in Theorem 6.69 and 6.70; thus Proposition 6.41 implies that

$$\lim_{k\to\infty} \int_A f_k(x)\, dx = \lim_{k\to\infty} \int_R g_k(x)\, dx = \int_R g(x)\, dx = \int_A f(x)\, dx\,.$$

In other words, the Monotone Convergence Theorem and the Bounded Convergence Theorem also hold for more general domain A, or to be more precise, for bounded set A with volume.

2. The Monotone Convergence Theorem (MCT) can be viewed as a corollary of the Bounded Convergence Theorem (BCT) since under the assumptions of MCT, we can apply BCT $\big($choose $M = \max\big\{\sup\limits_{x\in A} f(x), \sup\limits_{x\in A} f_1(x)\big\}\big)$ directly to conclude the MCT. Here we prove MCT without the help of BCT to demonstrate the power of the Fubini Theorem.

6.6 Improper Integrals

The Riemann integral deals with the "integrals" of bounded functions on bounded sets; however, often times we need to integrate unbounded functions on unbounded sets, such as finding the area under an unbounded function above x-axis. The improper integral is an answer to this particular situation. We first consider improper integrals of non-negative functions. Let $A \subseteq \mathbb{R}^n$ be a set and $f : A \to \mathbb{R}$ be a non-negative function. If f is bounded but A is unbounded, to define the integral of f on A, it is natural to consider the limit

$$\lim_{k\to\infty} \int_{A\cap B(0,k)} f(x)\, dx\,.$$

We note that for this limit to make sense, it is required that the integral $\displaystyle\int_{A\cap B(0,k)} f(x)\, dx$ exists for all $k \in \mathbb{N}$. On the other hand, if A is bounded and f is unbounded, to define the integral of f on A it is also natural to consider the limit

$$\lim_{k\to\infty} \int_A (f \wedge k)(x)\, dx\,, \tag{6.6.1}$$

where $(f \wedge k)(x) = \min\{f(x), k\}$. Again, for the limit above to make sense, it is required that the integral $\displaystyle\int_A (f \wedge k)(x)\, dx$ exists for all $k \in \mathbb{N}$. In both cases, we look for generic conditions (independent of k) that f and A have to satisfy so that

$$\int_{A\cap B(0,k)} f(x)\, dx \quad \text{and} \quad \int_A (f \wedge k)(x)\, dx$$

are well-defined for all $k \in \mathbb{N}$, and we have the following

Definition 6.72 (Riemann measurable sets and functions)**.**

1. A set $A \subseteq \mathbb{R}^n$ is said to be ***Riemann measurable*** if ∂A has measure zero.
2. A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be ***Riemann measurable*** if

$$\text{the set } \big\{x \in A \,\big|\, f \text{ is discontinuous at } x\big\} \text{ has measure zero in } \mathbb{R}^n.$$

Adopting this definition, the Lebesgue theorem shows that if A is bounded and Riemann measurable, then $f : A \to \mathbb{R}$ is Riemann integrable on A if and only if f is bounded and Riemann measurable. Since the improper integrals deal with integrals of possibly unbounded functions on possibly unbounded sets, in view of the Lebesgue theorem it is quite natural to consider removing the boundedness but keeping the Riemann measurability of A and f in order to define the improper integrals. Observe that

1. If A is Riemann measure, then $A \cap B(0,k)$ is Riemann measurable for all $k \in \mathbb{N}$ since

$$\partial(A \cap B(0,k)) \subseteq \partial A \cup \partial B(0,k)\,.$$

2. If f is Riemann measure, then $f \wedge k$ is Riemann measurable for all $k \in \mathbb{N}$. In fact, since the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x,y) = \max\{x,y\}$ is continuous, if f is continuous at x, then for all $k \in \mathbb{N}$,

$$\lim_{y\to x}(f \wedge k)(y) = \lim_{y\to x} F(f(y),k) = F(f(x),k) = (f \wedge k)(x)\,.$$

Therefore, for all $k \in \mathbb{N}$,

$$\big\{x \in A \,\big|\, f \wedge k \text{ is discontinuous at } x\big\} \subseteq \big\{x \in A \,\big|\, f \text{ is discontinuous at } x\big\}\,.$$

In other words, if $A \subseteq \mathbb{R}^n$ is a Riemann measurable set and $f : A \to \mathbb{R}$ is a Riemann measurable function.

1. If f is bounded, then $\displaystyle\int_{A\cap B(0,k)} f(x)\,dx$ exists for all $k \in \mathbb{N}$.
2. If A is bounded, then $\displaystyle\int_A (f \wedge k)(x)\,dx$ exists for all $k \in \mathbb{N}$.

How about the case that A is an unbounded set and f is an unbounded function? From the discussion above, it is natural to consider the limit of $\displaystyle\int_{A\cap B(0,k)} (f \wedge k)(x)\,dx$ as $k \to \infty$. We note that if $A \subseteq \mathbb{R}^n$ is a Riemann measurable set and $f : A \to \mathbb{R}$ is a Riemann measurable function, then $\displaystyle\int_{A\cap B(0,k)} (f \wedge k)(x)\,dx$ exists for all $k \in \mathbb{N}$. This motivates the following

Definition 6.73. Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f : A \to \mathbb{R}$ be a non-negative Riemann measurable function. f is said to be ***integrable*** on A if the limit

$$\int_A f(x)\,dx \equiv \lim_{k\to\infty} \int_{A\cap B(0,k)} (f \wedge k)(x)\,dx \tag{6.6.2}$$

is finite, and in such a case $\displaystyle\int_A f(x)\,dx$ is called the integral of f on A.

Remark 6.74. 1. For non-negative function $f : A \to \mathbb{R}$ (with f and A satisfying assumptions in Definition 6.73), if the limit $\displaystyle\int_{A\cap B(0,k)} f(x)\,dx$ is infinite, we still call $\displaystyle\int_A f(x)\,dx$ the integral of f on A. However, in this case f is not integrable on A.

2. Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$ be given in Definition 6.73. If $F \subseteq A$ is a Riemann measurable set with measure zero, then
$$\int_F f(x)\,dx = \lim_{k\to\infty} \int_{F\cap B(0,k)} (f \wedge k)(x)\,dx = 0\,,$$
where Theorem 6.45 is used to evaluate the integral.

3. By the Monotone Convergence Theorem (Theorem 6.69), (6.6.2) always holds if $f : A \to \mathbb{R}$ is Riemann integrable; thus Riemann integrable functions are integrable. **From now on, when talking about integrability, it could refer to Riemann integrable functions as well.**

Remark 6.75. When $f : A \to \mathbb{R}$ is unbounded, instead of (6.6.1) one might want to define the improper integral of f on A as
$$\lim_{k\to\infty} \int_A f_k(x)\,dx\,,$$
where
$$f_k(x) = \big(f\mathbf{1}_{\{f\leqslant k\}}\big)(x) = \begin{cases} f(x) & \text{if } f(x) \leqslant k\,, \\ 0 & \text{otherwise}\,. \end{cases}$$
The sequence $\{f_k\}_{k=1}^{\infty}$ still monotonically converges to f; however, it is not easy to see if the collection of points of discontinuity of f_k has measure zero since the set $\{x \in A \,|\, f(x) = k\}$ could be large. In other words, by defining f_k in this way we do not know the integrability of f_k on A; thus it is meaningless to define the improper integral as the limit of $\int_A f_k(x)\,dx$.

Example 6.76. Let $f : [1,\infty) \to \mathbb{R}$ be given by $f(x) = x^p$ for some $p \in \mathbb{R}$. If $p > 0$, then f is unbounded, and in this case
$$(f \wedge k)(x) = \begin{cases} x^p & \text{if } 1 \leqslant x \leqslant k^{\frac{1}{p}}\,, \\ k & \text{if } x > k^{\frac{1}{p}}\,; \end{cases}$$
thus for $p \geqslant 1$
$$\int_{[1,\infty)\cap(-k,k)} (f \wedge k)(x)\,dx = \int_1^{k^{\frac{1}{p}}} x^p dx + \int_{k^{\frac{1}{p}}}^k k dx = \frac{1}{p+1}(k^{1+\frac{1}{p}} - 1) + k(k - k^{\frac{1}{p}})$$
and for $0 < p < 1$,
$$\int_{[1,\infty)\cap(-k,k)} (f \wedge k)(x)\,dx = \int_1^k x^p dx = \frac{1}{p+1}(k^{p+1} - 1)\,.$$

In both cases, the limit (as $k \to \infty$) do not exist.

When $p \leqslant 0$, f is bounded by 1 on $[1,\infty)$. Therefore,

$$\int_{[1,\infty)\cap(-k,k)} (f \wedge k)(x)\,dx = \int_1^k f(x)dx = \begin{cases} \dfrac{1}{p+1}(k^{p+1}-1) & \text{if } p \neq -1\,, \\ \log k & \text{if } p = -1\,. \end{cases}$$

It is easy to see that the limit (as $k \to \infty$) exists only when $p < -1$. Therefore, f is integrable on $(0,1)$ if and only if $p < -1$, and in this case

$$\int_{[1,\infty)} f(x)\,dx = \lim_{k\to\infty} \frac{1}{p+1}(k^{p+1}-1) = -\frac{1}{p+1}\,.$$

Example 6.77. Let $f : (0,1) \to \mathbb{R}$ be given by $f(x) = x^p$ for some $p \in \mathbb{R}$. If $p \geqslant 0$, f is continuous on $(0,1)$, so f is Riemann integrable on $(0,1)$. If $p < 0$, f is unbounded on $(0,1)$, so the Riemann integral of f no longer makes sense. Nevertheless, we can find the improper integral of f using (6.6.2): for each $k \in \mathbb{N}$,

$$(f \wedge k)(x) = \begin{cases} x^p & \text{if } x \geqslant k^{\frac{1}{p}}\,, \\ k & \text{if } 0 < x < k^{\frac{1}{p}}\,; \end{cases}$$

thus

$$\int_0^1 (f \wedge k)(x)\,dx = \int_0^{k^{\frac{1}{p}}} k dx + \int_{k^{\frac{1}{p}}}^1 x^p dx = \begin{cases} \dfrac{1}{p+1}(pk^{1+\frac{1}{p}}+1) & \text{if } p \neq -1\,, \\ 1+\log k & \text{if } p = -1\,. \end{cases}$$

It is easy to see that the limit (as $k \to \infty$) exists only when $p > -1$. Therefore, f is integrable on $(0,1)$ if $p > -1$, and in this case

$$\int_{(0,1]} f(x)dx = \lim_{k\to\infty} \frac{1}{p+1}(pk^{1+\frac{1}{p}}+1) = \frac{1}{p+1}\,.$$

The following three propositions are generalization of corresponding results in Riemann integrals.

Proposition 6.78.

1. *Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f, g : A \to \mathbb{R}$ be non-negative, Riemann measurable functions. If $f \leqslant g$, then*

$$\int_A f(x)\,dx \leqslant \int_A g(x)\,dx\,.$$

2. *Let $A, B \subseteq \mathbb{R}^n$ be Riemann measurable sets, and $f : A \cup B \to \mathbb{R}$ be a non-negative, Riemann measurable function. If $A \subseteq B$, then*

$$\int_A f(x)\,dx \leqslant \int_B f(x)\,dx\,.$$

Proof. The first case follows from that $\displaystyle\int_{A\cap B(0,k)} (f \wedge k)(x)\,dx \leqslant \int_{A\cap B(0,k)} (g \wedge k)(x)\,dx$, while the second case follows from that $\displaystyle\int_{A\cap B(0,k)} (f \wedge k)(x)\,dx \leqslant \int_{B\cap B(0,k)} (f \wedge k)(x)\,dx$. □

Corollary 6.79. *Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f : A \to \mathbb{R}$ be a non-negative Riemann measurable function. Then*

$$\int_A f(x)\,dx = \lim_{k\to\infty} \int_{A\cap B(0,k)} f(x)\,dx = \lim_{k\to\infty} \int_A (f \wedge k)(x)\,dx\,. \tag{6.6.3}$$

Proof. For each $k \in \mathbb{N}$, Proposition 6.78 implies that

$$\int_{A\cap B(0,k)} (f \wedge k)(x)\,dx \leqslant \int_{A\cap B(0,k)} f(x)\,dx \leqslant \int_A f(x)\,dx\,,$$
$$\int_{A\cap B(0,k)} (f \wedge k)(x)\,dx \leqslant \int_A (f \wedge k)(x)\,dx \leqslant \int_A f(x)\,dx\,.$$

The conclusion follows from passing to the limit as $k \to \infty$. □

Corollary 6.80. *Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f : A \to \mathbb{R}$ be a non-negative Riemann measurable function. Then for all $\alpha > 0$,*

$$\int_A (\alpha f)(x)\,dx = \alpha \int_A f(x)\,dx\,.$$

Proof. By Corollary 6.79,

$$\begin{aligned}\int_A (\alpha f)(x)\,dx &= \lim_{k\to\infty} \int_{A\cap B(0,\alpha k)} \big[(\alpha f) \wedge (\alpha k)\big](x)\,dx = \lim_{k\to\infty} \int_{A\cap B(0,\alpha k)} \alpha (f \wedge k)(x)\,dx \\ &= \alpha \lim_{k\to\infty} \int_{A\cap B(0,\alpha k)} (f \wedge k)(x)\,dx = \alpha \int_A f(x)\,dx\,.\end{aligned}$$ □

Proposition 6.81. *Let $A, B \subseteq \mathbb{R}^n$ be Riemann measurable sets, and $f : A \cup B \to \mathbb{R}$ be a non-negative, Riemann measurable function. If $A \cap B$ has measure zero, then*

$$\int_{A\cup B} f(x)\,dx = \int_A f(x)\,dx + \int_B f(x)\,dx\,.$$

Proof. To simplify the notation, for each $k \in \mathbb{N}$ we let $f_k = f \wedge k$, and $A_k = A \cap B(0,k)$ as well as $B_k = B \cap B(0,k)$. Then

$$\partial(A_k \cup B_k) = \partial\big((A \cup B) \cap B(0,k)\big) \subseteq \partial(A \cup B) \cup \partial B(0,k) \subseteq \partial A \cup \partial B \cup \partial B(0,k)\,,$$
$$\partial(A_k \cap B_k) = \partial\big((A \cap B) \cap B(0,k)\big) \subseteq \partial(A \cap B) \cup \partial B(0,k) \subseteq \partial A \cup \partial B \cup \partial B(0,k)\,;$$

thus under the assumptions of this proposition, $A_k \cup B_k$ and $A_k \cap B_k$ have volume for each $k \in \mathbb{N}$. Therefore, Corollary 6.35 implies that for each $k \in \mathbb{N}$, $f_k 1_{A_k}$, $f_k 1_{B_k}$ and $f_k 1_{A_k \cap B_k}$ are all Riemann integrable on $A_k \cup B_k$. Since $A_k \cap B_k$ has measure zero, Theorem 6.45 and 6.50 imply that

$$\begin{aligned}\int_{(A\cup B)\cap B(0,k)} (f \wedge k)\, dx &= \int_{A_k \cup B_k} f_k(x)\, dx = \int_{A_k} f_k(x)\, dx + \int_{B_k} f_k(x)\, dx \\ &= \int_{A \cap B(0,k)} (f \wedge k)\, dx + \int_{B \cap B(0,k)} (f \wedge k)\, dx\,,\end{aligned}$$

and the theorem is concluded by passing to the limit as $k \to \infty$. □

Proposition 6.82. *Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f, g : A \to \mathbb{R}$ be non-negative, Riemann measurable functions. Then*

$$\int_A (f+g)(x)\, dx = \int_A f(x)\, dx + \int_A g(x)\, dx\,.$$

Proof. Note that if f, g are non-negative functions, then for all $k \in \mathbb{N}$,

$$\big[(f+g) \wedge k\big](x) \leqslant (f \wedge k)(x) + (g \wedge k)(x) \leqslant \big[(f+g) \wedge (2k)\big](x) \qquad \forall\, x \in A\,.$$

Therefore, Proposition 6.78 implies that for all $k \in \mathbb{N}$,

$$\begin{aligned}&\int_{A\cap B(0,k)} \big[(f+g) \wedge k\big](x)\, dx \leqslant \int_{A \cap B(0,k)} \big[(f \wedge k)(x) + (g \wedge k)(x)\big]\, dx \\ &\qquad \leqslant \int_{A \cap B(0,2k)} \big[(f \wedge k)(x) + (g \wedge k)(x)\big]\, dx \leqslant \int_{B(0,2k)} \big[(f+g) \wedge (2k)\big](x)\, dx\,.\end{aligned}$$

By Theorem 6.44, we obtain that

$$\begin{aligned}\int_{A\cap B(0,k)} \big[(f+g) \wedge k\big](x)\, dx &\leqslant \int_{A \cap B(0,k)} (f \wedge k)(x)\, dx + \int_{A \cap B(0,k)} (g \wedge k)(x)\, dx \\ &\leqslant \int_{B(0,2k)} \big[(f+g) \wedge (2k)\big](x)\, dx\,,\end{aligned}$$

and the conclusion follows from passing to the limit as $k \to \infty$. □

Those who are familiar with the improper integrals introduced in Calculus might be confused with the way we compute the improper integrals in Example 6.76 and 6.77. In fact, there are other ways of evaluating the improper integrals for functions of one variable, and the following theorem is useful for this particular purpose.

Theorem 6.83. 1. *Let $f : [a, \infty) \to \mathbb{R}$ be bounded, non-negative, and continuous except perhaps on a set of measure zero. Then*

$$\int_{[a,\infty)} f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx\,. \tag{6.6.4}$$

2. *Let $a \in \mathbb{R}$, $f : (a, b] \to \mathbb{R}$ be non-negative, bounded on $[a+\varepsilon, b]$ for all $\varepsilon > 0$, and continuous except perhaps on a set of measure zero. Then*

$$\int_{(a,b]} f(x)\,dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)\,dx\,. \tag{6.6.5}$$

Proof. 1. Note that for $k \geqslant \max\big\{|a|, \sup_{x\in[a,\infty)} f(x)\big\}$,

$$\int_{[a,\infty)\cap(-k,k)} (f\wedge k)(x)\,dx = \int_a^k f(x)\,dx\,;$$

thus (6.6.4) is obtained by passing to the limit as $k \to \infty$.

2. For each $\varepsilon > 0$ sufficiently small,

$$\int_{a+\varepsilon}^b f(x)\,dx = \int_{(a,b]} (f\mathbf{1}_{[a+\varepsilon,b]})(x)\,dx \leqslant \int_{(a,b]} f(x)\,dx\,;$$

thus passing to the limit as $\varepsilon \to 0^+$, we find that

$$\lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)\,dx \leqslant \int_{(a,b]} f(x)\,dx\,. \tag{6.6.6}$$

On the other hand, note that the Monotone Convergence Theorem for Riemann integrable sequence of functions (Theorem 6.69) implies that

$$\int_a^b (f\wedge k)(x)\,dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b (f\wedge k)(x)\,dx \leqslant \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)\,dx\,.$$

Passing to the limit as $k \to \infty$, we find that

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+} \int_{(a,b]\cap(-k,k)} (f\wedge k)(x)\,dx \leqslant \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)\,dx\,. \tag{6.6.7}$$

Combining (6.6.6) and (6.6.7), we concluded (6.6.5). □

Corollary 6.84. *Let $a \in \mathbb{R}$, $f : (a,\infty) \to \mathbb{R}$ be non-negative, bounded on $[a+\varepsilon,\infty)$ for all $\varepsilon > 0$, and continuous except perhaps on a set of measure zero. Then*

$$\int_{(a,\infty)} f(x)\,dx = \lim_{\substack{R\to\infty \\ \varepsilon\to 0^+}} \int_{a+\varepsilon}^{R} f(x)\,dx\,. \tag{6.6.8}$$

Proof. Let $a < b < \infty$. Then Theorem 6.81 implies that

$$\int_{(a,\infty)} f(x)\,dx = \int_{(a,b]} f(x)\,dx + \int_{[b,\infty)} f(x)\,dx\,;$$

thus we conclude from Theorem 6.83 that

$$\begin{aligned}\int_{(a,\infty)} f(x)\,dx &= \lim_{\varepsilon\to 0^+} \int_{[a+\varepsilon,b]} f(x)\,dx + \lim_{R\to\infty} \int_{[b,R]} f(x)\,dx \\ &= \lim_{\substack{R\to\infty \\ \varepsilon\to 0^+}} \int_{[a+\varepsilon,b]} f(x)\,dx + \lim_{\substack{R\to\infty \\ \varepsilon\to 0^+}} \int_{[b,R]} f(x)\,dx \\ &= \lim_{\substack{R\to\infty \\ \varepsilon\to 0^+}} \Big(\int_{[a+\varepsilon,b]} f(x)\,dx + \int_{[b,R]} f(x)\,dx\Big) = \lim_{\substack{R\to\infty \\ \varepsilon\to 0^+}} \int_{a+\varepsilon}^{R} f(x)\,dx\,,\end{aligned}$$

where the sum of the limits of two integrals is the same as the limit of sums of integrals since both integrals are increasing as $R \to \infty$ and $\varepsilon \to 0^+$. □

Remark 6.85. In view of (6.6.4) and (6.6.5), we also have the following notation for improper integrals for functions of one variable:

$$\int_a^\infty f(x)\,dx \equiv \int_{[a,\infty)} f(x)\,dx \qquad \text{and} \qquad \int_a^b f(x)\,dx = \int_{(a,b]} f(x)\,dx\,.$$

Example 6.86. Let $f(x) = x^p$ as in Example 6.76 and 6.77. Since

$$\int_1^R x^p\,dx = \begin{cases} \dfrac{1}{p+1}(R^{p+1}-1) & \text{if } p \neq -1\,, \\ \log R & \text{if } p = -1\,, \end{cases}$$

and

$$\int_\varepsilon^1 x^p\,dx = \begin{cases} \dfrac{1}{p+1}(1-\varepsilon^{1+p}) & \text{if } p \neq -1\,, \\ -\log \varepsilon & \text{if } p = -1\,, \end{cases}$$

by Theorem 6.83 we find that f is integrable on $[1,\infty)$ if and only if $p < -1$ and f is integrable on $(0,1]$ if and only if $p > -1$. These are the conclusions that we have obtained in Example 6.76 and 6.77.

Example 6.87 (The Gamma function)**.** For each $t > 0$, define $\Gamma(t) = \displaystyle\int_0^\infty x^{t-1}e^{-x}\,dx$.

1. For $1 \leqslant t < \infty$, the integrand is bounded and non-negative. In fact, $x^{t-1}e^{-x} \leqslant M_t e^{-\frac{x}{2}}$ for some constant $M_t > 0$ (we can choose $M_t = \sup\limits_{x\in[0,\infty)} x^{t-1}e^{-\frac{x}{2}}$). Since
$$\int_0^R x^{t-1}e^{-x}\,dx \leqslant \int_0^R M_t e^{-\frac{x}{2}}dx \leqslant -2M_t e^{-\frac{x}{2}}\Big|_{x=0}^{x=R} \leqslant 2M_t < \infty\,;$$
we find that $\Gamma(t)$ is well-defined for $1 \leqslant t < \infty$.

2. For $0 < t < 1$, the integrand is unbounded near 0; thus by Theorem 6.81 we rewrite
$$\int_0^\infty x^{t-1}e^{-x}\,dx = \int_0^1 x^{t-1}e^{-x}\,dx + \int_1^\infty x^{t-1}e^{-\frac{x}{2}}e^{-\frac{x}{2}}dx\,.$$
Since $x^{t-1}e^{-x} \leqslant x^{t-1}$ on $(0,1]$ and $x^{t-1}e^{-x} \leqslant e^{-x}$ on $[1,\infty)$, for $\varepsilon > 0$ we have
$$\int_\varepsilon^1 x^{t-1}e^{-x}dx \leqslant \int_\varepsilon^1 x^{t-1}dx = \frac{1}{t}x^t\Big|_{x=\varepsilon}^1 = \frac{1}{t}(1-\varepsilon^t) \leqslant \frac{1}{t}$$
and for $R > 1$,
$$\int_1^R x^{t-1}e^{-x}dx \leqslant \int_1^R e^{-x}dx = -e^{-x}\Big|_{x=1}^{x=R} = e^{-1} - e^{-R} \leqslant e^{-1}\,.$$
Therefore, $\Gamma(t)$ is also well-defined for $0 < t < 1$.

The following theorem provides different ways of computing the improper (multiple) integrals.

Theorem 6.88. *Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f : A \to \mathbb{R}$ be a non-negative, Riemann measurable function. Then f is integrable on A if and only if for each sequence $\{B_k\}_{k=1}^\infty \subseteq \mathbb{R}^n$ of bounded sets with volume satisfying*

1. *$B_k \subseteq B_{k+1}$ for all $k \in \mathbb{N}$;*

2. *for all $R > 0$ we have $B(0,R) \subseteq B_k$ for sufficient large $k \in \mathbb{N}$;*

the limit $\displaystyle\lim_{k\to\infty}\int_{A\cap B_k}(f \wedge k)(x)\,dx$ exists.

Proof. "$\Leftarrow$" Simply choose $B_k = B(0,k)$ to conclude the integrability of f on A.

"⇒" For each $\ell \in \mathbb{N}$, there exists $N(\ell) \geqslant \ell$ such that $B(0,\ell) \subseteq B_k$ for all $k \geqslant N(\ell)$. Then

$$\int_{A\cap B(0,\ell)} (f\wedge \ell)(x)\,dx \leqslant \int_{A\cap B_k} (f\wedge \ell)(x)\,dx \leqslant \int_{A\cap B_k} (f\wedge k)(x)\,dx \qquad \forall\, k \geqslant N(\ell)\,.$$

Since $\displaystyle\int_{A\cap B_k} (f\wedge k)(x)\,dx = \int_A \big((f\wedge k)1_{B_k}\big)(x)\,dx \leqslant \int_A f(x)\,dx$, by the sandwich lemma we conclude that

$$\int_A f(x)\,dx = \lim_{\ell\to\infty} \int_{A\cap B(0,\ell)} (f\wedge \ell)(x)\,dx = \lim_{k\to\infty} \int_{A\cap B_k} (f\wedge k)(x)\,dx\,. \qquad \square$$

In other words, as long as $\{B_k\}_{k=1}^\infty$ "expands to the whole space", one can evaluate the improper integral using

$$\int_A f(x)\,dx = \lim_{k\to\infty} \int_{A\cap B_k} (f\wedge k)(x)\,dx\,.$$

One particular sequence of sets $\{B_k\}_{k=1}^\infty$ is given by $B_k = [-k,k]\times\cdots\times[-k,k]$.

Example 6.89. Consider the improper integral $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$. Instead of evaluating this improper integral directly, we consider the improper integral $\displaystyle\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,d\mathbb{A}$. Note that Theorem 6.88 implies that

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,d\mathbb{A} = \lim_{k\to\infty} \int_{[-k,k]\times[-k,k]} e^{-(x^2+y^2)}\,d\mathbb{A} = \lim_{k\to\infty} \int_{B(0,k)} e^{-(x^2+y^2)}\,d\mathbb{A}\,.$$

By the Fubini theorem,

$$\lim_{k\to\infty} \int_{[-k,k]\times[-k,k]} e^{-(x^2+y^2)}\,d\mathbb{A} = \lim_{k\to\infty} \int_{-k}^{k} \Big(\int_{-k}^{k} e^{-(x^2+y^2)}dy\Big)dx = \Big(\int_{-\infty}^{\infty} e^{-x^2}dx\Big)^2,$$

while the change of variables formula (with $(x,y) = (r\cos\theta, r\sin\theta)$) implies that

$$\begin{aligned}\lim_{k\to\infty} \int_{B(0,k)} e^{-(x^2+y^2)}\,d\mathbb{A} &= \lim_{k\to\infty} \int_{[0,k]\times[0,2\pi]} e^{-r^2} r\,d(r,\theta) = \lim_{k\to\infty} \int_0^{2\pi} \Big(\int_0^k e^{-r^2} r\,dr\Big)d\theta \\ &= \lim_{k\to\infty} \int_0^{2\pi} \Big(\frac{e^{-r^2}}{-2}\Big|_{r=0}^{r=k}\Big)d\theta = \lim_{k\to\infty} \pi(1-e^{-k^2}) = \pi\,.\end{aligned}$$

Since $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx \geqslant 0$, we must have

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}\,.$$

Now we define the improper integrals for general functions. Let $\vee$ be an operation which outputs the maximum of values from both sides of $\vee$; that is,

$$(f \vee g)(x) = \max\big\{f(x), g(x)\big\}\,.$$

Define the positive and negative parts, denoted by f^+ and f^- respective, by $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. Since f^+, f^- are non-negative and $f = f^+ - f^-$, to defined the integral of f it is natural to consider the difference of the integrals $\displaystyle\int_A f^+(x)\,dx$ and $\displaystyle\int_A f^-(x)\,dx$. Note that if the collection of discontinuities of f has measure zero, the collections of discontinuities of both f^+ and f^- are sets of measure zero; thus the integrals $\displaystyle\int_A f^\pm(x)\,dx$ makes sense.

The discussion above motivates the following

Definition 6.90. Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f : A \to \mathbb{R}$ be a Riemann measurable function. f is said to be ***integrable*** on A if both integrals

$$\int_A f^+(x)\,dx \qquad \text{and} \qquad \int_A f^-(x)\,dx$$

are finite, where f^+ and f^- are the positive and negative parts of f defined by

$$f^+ = f \vee 0 \quad \text{and} \quad f^- = (-f) \vee 0\,.$$

If f is integrable on A, the integral of f on A, denoted by $\displaystyle\int_A f(x)\,dx$, is the number $\displaystyle\int_A f^+(x)\,dx - \int_A f^-(x)\,dx$.

Remark 6.91. 1. If f is integrable on A, then

$$\int_A \big|f(x)\big|\,dx = \int_A f^+(x)\,dx + \int_A f^-(x)\,dx < \infty\,;$$

thus the integrability of f on A sometimes is also called the absolute integrability of f on A or that the integral $\displaystyle\int_A f(x)\,dx$ is absolutely convergent.

2. For integrable function $f : A \to \mathbb{R}$, one can compute the integral of f on A by

$$\begin{aligned}\int_A f(x)\,dx &= \lim_{k\to\infty} \int_{A\cap B(0,k)} (f^+ \wedge k)(x)\,dx - \lim_{k\to\infty} \int_{A\cap B(0,k)} (f^- \wedge k)(x)\,dx \\ &= \lim_{k\to\infty} \int_{A\cap B(0,k)} f^+(x)\,dx - \lim_{k\to\infty} \int_{A\cap B(0,k)} f^-(x)\,dx \\ &= \lim_{k\to\infty} \int_{A\cap B(0,k)} \big(f^+(x) - f^-(x)\big)\,dx = \lim_{k\to\infty} \int_{A\cap B(0,k)} f(x)\,dx\,. \qquad (6.6.9)\end{aligned}$$

where (6.6.3) is used to conclude the third equality. In (6.6.9), the set $B(0,k)$ can also be replaced by increasing sequence of set $\{B_k\}_{k=1}^{\infty}$ as introduced in Theorem 6.88.

By Proposition 6.78, 6.81, 6.82 and Corollary 6.80, we can also establish the following

Theorem 6.92. *Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f, g : A \to \mathbb{R}$ be integrable functions. If $f \leqslant g$, then*

$$\int_A f(x)\,dx \leqslant \int_A g(x)\,dx\,.$$

Theorem 6.93. *Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f : A \to \mathbb{R}$ be an integrable function. Then for all $\alpha \in \mathbb{R}$,*

$$\int_A (\alpha f)(x)\,dx = \alpha \int_A f(x)\,dx\,.$$

Theorem 6.94. *Let $A, B \subseteq \mathbb{R}^n$ be Riemann measurable sets, and $f : A \cup B \to \mathbb{R}$ be an integrable function. If $A \cap B$ has measure zero, then*

$$\int_{A\cup B} f(x)\,dx = \int_A f(x)\,dx + \int_B f(x)\,dx\,.$$

Theorem 6.95. *Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f, g : A \to \mathbb{R}$ be integrable functions. Then*

$$\int_A (f+g)(x)\,dx = \int_A f(x)\,dx + \int_A g(x)\,dx\,.$$

The proofs for the theorems above are left to the readers as exercises.

Remark 6.96. 1. If f is not integrable on A but one of the integrals $\int_A f^+(x)\,dx$ or $\int_A f^-(x)\,dx$ is finite, the number $\int_A f^+(x)\,dx - \int_A f^-(x)\,dx$ is still well-understood, and we still call this difference as the integral of f on A.

2. When at least one of the integrals $\int_A f^+(x)\,dx$ or $\int_A f^-(x)\,dx$ is finite,

$$\begin{aligned}\int_A f(x)\,dx &= \lim_{k\to\infty} \int_{A\cap B(0,k)} (f^+ \wedge k)(x)\,dx - \lim_{k\to\infty} \int_{A\cap B(0,k)} (f^- \wedge k)(x)\,dx \\ &= \lim_{k\to\infty} \Big(\int_{A\cap B(0,k)} (f^+ \wedge k)(x)\,dx - \int_{A\cap B(0,k)} (f^- \wedge k)(x)\,dx \Big) \\ &= \lim_{k\to\infty} \int_{A\cap B(0,k)} (-k) \vee (f \wedge k)(x)\,dx\,.\end{aligned}$$

Therefore, it is tempting to define the integrability of f on A by the existence of the limit

$$\lim_{k\to\infty}\int_{A\cap B(0,k)}\big[(-k)\vee(f\wedge k)\big](x)\,dx\,.$$

However, this cannot be the correct definition since if we adapt this definition, then the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\dfrac{1}{x}$ for $x\neq 0$ (and $f(0)$ is given arbitrarily) will be integrable on $\mathbb{R}$ (and the integral is 0 by symmetry), while Theorem 6.94, a should-have theorem for integrable functions, fails to hold for this particular function since

$$0=\int_{\mathbb{R}}f(x)\,dx\neq\int_{[0,\infty)}f(x)\,dx+\int_{(-\infty,0]}f(x)\,dx\,.$$

Theorem 6.97 (Comparison Test)**.** *Let $A\subseteq\mathbb{R}^n$ be a Riemann measurable set, $f,g:A\to\mathbb{R}$ be Riemann measurable functions. If $|f|\leqslant g$ on A and g is integrable on A, then f is integrable on A.*

Proof. Since $|f|=f^+ + f^-$, the condition that $|f|\leqslant g$ implies that $f^+\leqslant g$ and $f^-\leqslant g$; thus

$$\int_{A\cap B(0,k)}(f^\pm\wedge k)(x)\,dx\leqslant\int_{A\cap B(0,k)}(g\wedge k)(x)\,dx\leqslant\int_A g(x)\,dx<\infty\,.$$

Since $\displaystyle\int_{A\cap B(0,k)}(f^\pm\wedge k)(x)\,dx$ are increasing in k, both limits

$$\lim_{k\to\infty}\int_{A\cap B(0,k)}(f^\pm\wedge k)(x)\,dx$$

must exist (and are finite). Therefore, f is integrable on A. □

Example 6.98. Let $f:[0,\infty)\to\mathbb{R}$ be given by $f(x)=\dfrac{\sin x}{x^2+1}$. Then $\big|f(x)\big|\leqslant\dfrac{1}{x^2+1}$ and the function $y=\dfrac{1}{x^2+1}$ is integrable on $[0,\infty)$ since

$$\lim_{R\to\infty}\int_0^R\frac{1}{x^2+1}dx=\lim_{R\to\infty}\tan^{-1}x\Big|_{x=0}^{x=R}=\lim_{R\to\infty}\tan^{-1}R=\frac{\pi}{2}\,.$$

Let $\{a_k\}_{k=1}^\infty$ be a non-negative sequence. Then the series $\sum\limits_{k=1}^{\infty}a_k$ can be viewed as the integral of the piecewise constant function

$$f(x)=a_k\quad\text{if}\quad k-1\leqslant x<k \tag{6.6.10}$$

over the set $\mathbb{R}^+ \equiv \{x \geqslant 0\}$. Now suppose that $\{a_k\}_{k=1}^\infty$ is a general sequence in $\mathbb{R}$. Define $a_k^+ = \max\{a_k, 0\}$, $a_k^- = \max\{-a_k, 0\}$ for each $k \in \mathbb{N}$, and let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by (6.6.10). Recall that a series $\sum\limits_{k=1}^\infty a_k$ is absolutely convergent if $\sum\limits_{k=1}^\infty |a_k| < \infty$ and this is equivalent to that

$$\sum_{k=1}^\infty a_k^+ < \infty \quad \text{and} \quad \sum_{k=1}^\infty a_k^- < \infty\,.$$

Since $\displaystyle\int_{\mathbb{R}^+} f^+(x)\,dx = \sum_{k=1}^\infty a_k^+$ and $\displaystyle\int_{\mathbb{R}^+} f^-(x)\,dx = \sum_{k=1}^\infty a_k^-$, we find that the series $\sum\limits_{k=1}^\infty a_k$ is absolutely convergent if and only if f given by (6.6.10) is integrable on $\mathbb{R}^+$.

There is another concept of convergence of series, called the conditional convergence. Recall that a series $\sum\limits_{k=1}^\infty a_k$ is said to be conditionally convergent if the limit $\lim\limits_{\ell\to\infty} \sum\limits_{k=1}^{\ell} a_k$ exists but $\sum\limits_{k=1}^\infty |a_k| = \infty$. Let $\sum\limits_{k=1}^\infty a_k$ be a conditionally convergent series, and $f : \mathbb{R}^+ \to \mathbb{R}$ be given by (6.6.10). Then

$$\int_{\mathbb{R}^+} f^+(x)\,dx = \sum_{k=1}^\infty a_k^+ = \infty \quad \text{and} \quad \int_{\mathbb{R}^+} f^-(x)\,dx = \sum_{k=1}^\infty a_k^- = \infty\,,$$

while the limit $\lim\limits_{\ell\to\infty} \displaystyle\int_0^\ell f(x)\,dx = \lim\limits_{\ell\to\infty} \sum\limits_{k=1}^{\ell} a_k$ exists. The connection between the two kinds of convergence of series and integrals motivates the following

Definition 6.99. Let $A \subseteq \mathbb{R}$ be a Riemann measurable set, and $f : A \to \mathbb{R}$ be a Riemann measurable function. The improper integral $\displaystyle\int_A f(x)\,dx$ is said to be ***conditionally convergent*** if f is not integrable on A but the limit $\lim\limits_{k,\ell\to\infty} \displaystyle\int_{A\cap(-\ell,k)} f(x)\,dx$ exists.

Remark 6.100. Suppose that the series $\sum\limits_{k=1}^\infty a_k$ is conditionally convergent. Then for each $r \in \mathbb{R}$, there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$r = \sum_{k=1}^\infty a_{\pi(k)} = a_{\pi(1)} + a_{\pi(2)} + a_{\pi(3)} + \cdots\,.$$

In other words, the order of the summation matters in a conditional convergent series; thus in general it will not be possible to talk about conditionally convergent integrals of functions on subsets of $\mathbb{R}^n$ if $n \geqslant 2$ since it usually requires the Fubini theorem to evaluate the multiple integrals while the Fubini theorem involves evaluating integrals in different orders.

Example 6.101. Let $f : [0,\infty) \to \mathbb{R}$ be given by $f(x) = \dfrac{\sin x}{x}$. Then for all $0 < r < R$, using the integration by parts formula we obtain that

$$\Big|\int_r^R \frac{\sin x}{x} dx\Big| = \Big|\frac{1-\cos x}{x}\Big|_{x=r}^{x=R} + \int_r^R \frac{1-\cos x}{x^2} dx\Big| \leqslant \frac{2}{R} + \frac{2}{r} + \int_r^R \frac{2}{x^2} dx = \frac{4}{r}\,.$$

Let $I_k = \displaystyle\int_0^k \frac{\sin x}{x} dx$. Then the inequality above implies that $\{I_k\}_{k=1}^\infty$ is Cauchy in $\mathbb{R}$, so the limit $\displaystyle\int_0^\infty \frac{\sin x}{x} dx = \lim_{k\to\infty} I_k$ exists. However,

$$\int_0^\infty f^+(x)\, dx = \sum_{k=1}^\infty \int_{(2k-2)\pi}^{(2k-1)\pi} \frac{\sin x}{x} dx \geqslant \sum_{k=1}^\infty \frac{1}{(2k-1)\pi} \int_{(2k-2)\pi}^{(2k-1)\pi} \sin x dx = \frac{2}{\pi} \sum_{k=1}^\infty \frac{1}{2k-1} = \infty$$

and

$$\int_0^\infty f^-(x)\, dx = \sum_{k=1}^\infty \int_{(2k-1)\pi}^{2k\pi} \frac{-\sin x}{x} dx \geqslant \sum_{k=1}^\infty \frac{1}{2k\pi} \int_{(2k-1)\pi}^{2k\pi} (-\sin x) dx = \frac{1}{\pi} \sum_{k=1}^\infty \frac{1}{k} = \infty\,.$$

Therefore, the improper integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$ is conditionally convergent.

6.6.1 The monotone convergence theorem and the dominated convergence theorem

In the remaining part of this section, we present some important theorems introduced in Section 6.4 under the new settings of improper integrals.

Theorem 6.102 (Dominated Convergence Theorem)**.** *Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f_k, f : A \to \mathbb{R}$ be Riemann measurable functions such that $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in A$. Suppose that there exists an integrable function g such that $|f_k| \leqslant g$ for all $k \in \mathbb{N}$. Then f is integrable on A, and*

$$\int_A f(x)\, dx = \lim_{k\to\infty} \int_A f_k(x)\, dx\,.$$

Proof. Since $\big|f_k(x)\big| \leqslant g(x)$ for all $x \in A$ and $k \in \mathbb{N}$, $\big|f(x)\big| \leqslant g(x)$ for all $x \in A$. By the integrability of g, the comparison test (Theorem 6.97) implies that f_k and f are also integrable on A.

Let $\varepsilon > 0$ be given. Since f, g are integrable on A, there exists $L > 0$ such that

$$0 \leqslant \int_A g(x)\, dx - \int_{A\cap B(0,\ell)} (g \wedge \ell)(x)\, dx < \frac{\varepsilon}{3} \qquad \forall\, \ell \geqslant L \tag{6.6.11}$$

and

$$\Big|\int_A f(x)\,dx - \int_{A\cap B(0,\ell)} \big((-\ell)\vee(f\wedge\ell)\big)(x)\,dx\Big| < \frac{\varepsilon}{3} \qquad \forall\,\ell \geqslant L\,.$$

Moreover, since $(-L)\vee(f_k\wedge L) \to (-L)\vee(f\wedge L)$ p.w. as $k\to\infty$ (due to the pointwise convergence of $\{f_k\}_{n=1}^\infty$ to f), and $\big|(-L)\vee(f_k\wedge L)\big| \leqslant L$ on $A\cap B(0,L)$, the Bounded Convergence Theorem (Theorem 6.70) implies that there exists $K>0$ such that

$$\Big|\int_{A\cap B(0,L)} \big((-L)\vee(f_k\wedge L)\big)(x)\,dx - \int_{A\cap B(0,L)} \big((-L)\vee(f\wedge L)\big)(x)\,dx\Big| < \frac{\varepsilon}{3} \quad \forall\,k\geqslant K\,.$$

Note that Theorem 6.81 implies that

$$\begin{aligned}\Big|\int_A f(x)\,dx - \int_A f_k(x)\,dx\Big| &\leqslant \Big|\int_A f(x)\,dx - \int_{A\cap B(0,L)} \big((-L)\vee(f\wedge L)\big)(x)\,dx\Big| \\ &+ \Big|\int_{A\cap B(0,L)} \big((-L)\vee(f\wedge L)\big)(x)\,dx - \int_{A\cap B(0,L)} \big((-L)\vee(f_k\wedge L)\big)(x)\,dx\Big| \\ &+ \Big|\int_{A\cap B(0,L)} \big((-L)\vee(f_k\wedge L)\big)(x)\,dx - \int_{A\cap B(0,L)} f_k(x)\,dx\Big| + \Big|\int_{A\cap B(0,L)^\complement} f_k(x)\,dx\Big|,\end{aligned}$$

and the fact that $|f_k|\leqslant g$ implies that

$$\big|\big((-L)\vee(f_k\wedge L)\big)(x) - f_k(x)\big| \leqslant g(x) - (g\wedge L)(x)\,.$$

Therefore, for $k\geqslant K$,

$$\begin{aligned}&\Big|\int_A f(x)\,dx - \int_A f_k(x)\,dx\Big| \\ &\qquad < \frac{2\varepsilon}{3} + \int_{A\cap B(0,L)} \big(g(x) - (g\wedge L)(x)\big)\,dx + \int_{A\cap B(0,L)^\complement} g(x)\,dx \\ &\qquad \leqslant \frac{2\varepsilon}{3} + \int_A g(x)\,dx - \int_{A\cap B(0,L)} (g\wedge L)(x)\,dx < \varepsilon\,.\end{aligned}$$ □

The Monotone Convergence Theorem for improper integrals, unlike the case in the Riemann integrals, is no longer an immediate consequence of the Dominated Convergence Theorem since the "integral" of the limit function might be infinite. It requires a little bit more attention to get proved.

Theorem 6.103 (Monotone Convergence Theorem for Improper Integrals)**.** *Let $A\subseteq\mathbb{R}^n$ be a Riemann measurable set, and $f_k, f: A\to\mathbb{R}$ be non-negative, Riemann measurable* *functions such that* $\lim\limits_{k\to\infty} f_k(x) = f(x)$ *for all* $x \in A$*. Suppose that* $\{f_k\}_{k=1}^\infty$ *is a monotone increasing sequence of functions; that is,* $f_k \leqslant f_{k+1}$ *for all* $k \in \mathbb{N}$*. Then*

$$\int_A f(x)\,dx = \lim_{k\to\infty} \int_A f_k(x)\,dx\,. \tag{6.6.12}$$

Proof. By the Dominated Convergence Theorem (Theorem 6.102), we only need to consider the case that $\displaystyle\int_A f(x)\,dx = \infty$ and show that $\displaystyle\lim_{k\to\infty}\int_A f_k(x)\,dx = \infty$. We also assume the non-trivial case that $\displaystyle\int_A f_k(x)\,dx < \infty$ for all $k \in \mathbb{N}$.

Let $M > 0$ be given. Since $\displaystyle\int_A f(x)\,dx = \infty$, there exists $L > 0$ such that

$$\int_{A\cap B(0,\ell)} (f \wedge \ell)(x)\,dx \geqslant 2M \qquad \forall\, \ell \geqslant L\,.$$

By the Monotone Convergence Theorem for Riemann integrals (Theorem 6.69), there exists $K > 0$ such that

$$-M \leqslant \int_{A\cap B(0,L)} (f_k \wedge L)(x)\,dx - \int_{A\cap B(0,L)} (f \wedge L)(x)\,dx \leqslant 0 \qquad \forall\, k \geqslant K\,.$$

Therefore, for all $k \geqslant K$,

$$\begin{aligned}
\int_A f_k(x)\,dx &= \int_A f_k(x)\,dx - \int_{A\cap B(0,L)} (f_k \wedge L)(x)\,dx + \int_{A\cap B(0,L)} (f_k \wedge L)(x)\,dx \\
&\quad - \int_{A\cap B(0,L)} (f \wedge L)(x)\,dx + \int_{A\cap B(0,L)} (f \wedge L)(x)\,dx \\
&\geqslant \int_A f_k(x)\,dx - \int_{A\cap B(0,L)} (f_k \wedge L)(x)\,dx + M \geqslant M\,.
\end{aligned}$$

□

When non-negativity of functions is removed from the condition, for (6.6.12) to hold it is required that the sequence of functions has an integrable lower bound. To be more precise, we have the following

Corollary 6.104. *Let* $A \subseteq \mathbb{R}^n$ *be a Riemann measurable set, and* $f_k, f : A \to \mathbb{R}$ *be Riemann measurable functions such that* $\lim\limits_{k\to\infty} f_k(x) = f(x)$ *for all* $x \in A$*. Suppose that there exists an integrable function* $g : A \to \mathbb{R}$ *such that* $f_k(x) \geqslant g(x)$ *for all* $x \in A$ *and* $k \in \mathbb{N}$*, and* $\{f_k\}_{k=1}^\infty$ *is a monotone increasing sequence of functions; that is,* $f_k \leqslant f_{k+1}$ *for all* $k \in \mathbb{N}$*. Then*

$$\int_A f(x)\,dx = \lim_{k\to\infty} \int_A f_k(x)\,dx\,.$$

Proof. Consider the new sequence of functions $\{h_k\}_{k=1}^{\infty}$ defined by $h_k = f_k - g$ and apply the Monotone Convergence Theorem (Theorem 6.103). □

For monotone decreasing sequences of functions, we have the following

Corollary 6.105. *Let $A \subseteq \mathbb{R}^n$ be a Riemann measurable set, and $f_k, f : A \to \mathbb{R}$ be Riemann measurable functions such that $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in A$. Suppose that f_1 is integrable on A and $\{f_k\}_{k=1}^{\infty}$ is a monotone decreasing sequence of functions; that is, $f_k \geqslant f_{k+1}$ for all $k \in \mathbb{N}$. Then*

$$\int_A f(x)\,dx = \lim_{k\to\infty} \int_A f_k(x)\,dx\,.$$

Proof. Consider the new sequence of functions $\{h_k\}_{k=1}^{\infty}$ defined by $h_k = f_1 - f_k$ and apply the Monotone Convergence Theorem (Theorem 6.103). □

6.6.2 The Fubini theorem and the Tonelli theorem

In this section we present the Fubini theorem for improper integrals. The Fubini theorem for improper integrals takes the form

$$\int_{A\times B} f(x,y)d(x,y) = \int_A \Big(\int_B f(x,y)dy\Big)dx$$

or

$$\int_{A\times B} f(x,y)d(x,y) = \int_B \Big(\int_A f(x,y)dx\Big)dy\,.$$

as long as f satisfies certain conditions. However, the iterated integrals on the right-hand side will be meaningless if the functions $F(x) \equiv \int_B f(\cdot,y)dy$ and $G(y) \equiv \int_A f(x,\cdot)\,dx$ are not Riemann measurable; thus in general we need to impose the condition that F and G are Riemann measurable. In fact, even if $f : A \times B \to \mathbb{R}$ is continuous, F might still be discontinuous at some points. For example, let $A = [-1,1]$, $B = [1,\infty)$ and $f(x,y) = |x|y^{-1-|x|}$. Then

$$F(x) = \begin{cases} 0 & \text{if } x = 0\,, \\ 1 & \text{otherwise}\,, \end{cases}$$

which is discontinuous at $x = 0$. In other words, the collection of discontinuities of F might not be empty even if f is continuous on $A \times B$ since "partial integration" might produce extra discontinuities; thus in general we do not know if F is Riemann measurable even if f is continuous.

Before proceeding to the Fubini theorem, we first establish the following

Theorem 6.106 (Tonelli). *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be Riemann measurable sets such that $A \times B$ is Riemann measurable, and $f : A \times B \to \mathbb{R}$ be a non-negative Riemann measurable function.*

1. *If for all $x \in A$, $f(x, \cdot)$ is integrable on B and the function $\displaystyle\int_B f(\cdot, y)\, dy : A \to \mathbb{R}$ is Riemann measurable, then*
$$\int_{A\times B} f(x,y)d(x,y) = \int_A \Big(\int_B f(x,y)dy\Big)dx\,.$$

2. *If for all $y \in B$, $f(\cdot, y)$ is integrable on A and the function $\displaystyle\int_A f(x, \cdot)\, dx : B \to \mathbb{R}$ is Riemann measurable, then*
$$\int_{A\times B} f(x,y)d(x,y) = \int_B \Big(\int_A f(x,y)dx\Big)dy\,.$$

Proof. It suffices to show the first case since the proof of the other case is similar.

We show that for each $k \in \mathbb{N}$, the function $g_k : A \to \mathbb{R}$ defined by
$$g_k(x) = \int_{B\cap[-k,k]^m} (f \wedge k)(x,y)dy$$
is Riemann measurable. We note that the fact that $f(x, \cdot)$ is integrable on B for each $x \in A$ implies that g_k is well-defined for all $x \in A$.

For each $\ell \in \mathbb{N}$, the Fubini theorem for Riemann integrals provides that
$$\int_{(A\cap[-\ell,\ell]^n)\times(B\cap[-k,k]^m)} (f\wedge k)(x,y)d(x,y) = \underline{\int}_{A\cap[-\ell,\ell]^n} g_k(x)\,dx = \overline{\int}_{A\cap[-\ell,\ell]^n} g_k(x)\,dx\,.$$

Therefore, g_k is Riemann integrable on $A \cap [-\ell, \ell]^n$ for all $\ell \in \mathbb{N}$, and the Lebesgue theorem (Theorem 6.32) implies that the collection of discontinuities of g_k in $A \cap [-\ell, \ell]^n$ has measure zero. By Theorem 6.26 and the fact that $A = \bigcup_{\ell=1}^{\infty} (A \cap [-\ell, \ell]^n)$, the collection of discontinuities of g_k in A has measure zero; thus g_k is Riemann measurable.

For each $k \in \mathbb{N}$, define $f_k(x, y) = \mathbf{1}_{[-k,k]^{n+m}}(x, y)(f \wedge k)(x, y)$ and $h_k(x) = \mathbf{1}_{[-k,k]^n}(x)g_k(x)$. Then $\{f_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ are non-negative monotone increasing sequences. Moreover, it is clear that $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f (on A) and $\{h_k\}_{k=1}^{\infty}$ converges pointwise to the function $\displaystyle\int_B f(\cdot,y)\,dy$. By the Fubini theorem for Riemann integrals again,

$$\begin{aligned}\int_{A\times B} f_k(x,y)d(x,y) &= \int_{(A\cap[-k,k]^n)\times(B\cap[-k,k]^m)} (f\wedge k)(x,y)d(x,y)\\ &= \int_{A\cap[-k,k]^n} g_k(x)\,dx = \int_A h_k(x)\,dx\,;\end{aligned}$$

thus the Monotone Convergence Theorem (Theorem 6.103) implies that

$$\begin{aligned}\int_{A\times B} f(x,y)d(x,y) &= \lim_{k\to\infty}\int_{A\times B} f_k(x,y)d(x,y) = \lim_{k\to\infty}\int_A h_k(x)\,dx\\ &= \int_A\Big(\int_B f(x,y)dy\Big)dx\,.\end{aligned}$$

□

Now we can present the Fubini theorem.

Theorem 6.107 (Fubini)**.** *Let $A\subseteq\mathbb{R}^n$ and $B\subseteq\mathbb{R}^m$ be Riemann measurable sets such that $A\times B$ is Riemann measurable, and $f: A\times B\to\mathbb{R}$ be an integrable function.*

1. *If for all $x\in A$, $f(x,\cdot)$ is integrable on B, and the functions $\displaystyle\int_B f(\cdot,y)dy : A\to\mathbb{R}$ and $\displaystyle\int_B \big|f(\cdot,y)\big|dy : A\to\mathbb{R}$ are Riemann measurable, then*
$$\int_{A\times B} f(x,y)d(x,y) = \int_A\Big(\int_B f(x,y)dy\Big)dx\,.$$

2. *If for all $y\in B$, $f(\cdot,y)$ is integrable on A, and the functions $\displaystyle\int_A f(x,\cdot)dx : B\to\mathbb{R}$ and $\displaystyle\int_A \big|f(x,\cdot)\big|dx : B\to\mathbb{R}$ are Riemann measurable, then*
$$\int_{A\times B} f(x,y)d(x,y) = \int_B\Big(\int_A f(x,y)dx\Big)dy\,.$$

Proof. It suffices to prove the first case.

Since $f^{\pm} = \dfrac{1}{2}(|f| + f)$, by assumption the functions $\displaystyle\int_B f^+(\cdot,y)dy : A\to\mathbb{R}$ and $\displaystyle\int_B f^-(\cdot,y)dy : A\to\mathbb{R}$ are Riemann measurable functions. Therefore, the Tonelli theorem implies that

$$\int_{A\times B} f^{\pm}(x,y)d(x,y) = \int_A\Big(\int_B f^{\pm}(x,y)\,dy\Big)dx\,;$$

thus

$$\begin{aligned}\int_{A\times B} f(x,y)d(x,y) &= \int_{A\times B} f^+(x,y)d(x,y) - \int_{A\times B} f^-(x,y)d(x,y)\\ &= \int_A \Big(\int_B f^+(x,y)\,dy\Big)dx - \int_A \Big(\int_B f^-(x,y)\,dy\Big)dx\\ &= \int_A \Big(\int_B f^+(x,y)\,dy - \int_A\int_B f^-(x,y)\,dy\Big)dx = \int_A \Big(\int_B f(x,y)\,dy\Big)dx\,. \end{aligned}$$ □

Corollary 6.108. *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be Riemann measurable sets such that $A \times B$ is Riemann measurable, and $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be integrable functions. Then the function $h : A \times B \to \mathbb{R}$ given by $h(x,y) = f(x)g(y)$ is integrable, and*

$$\int_{A\times B} h(x,y)d(x,y) = \Big(\int_A f(x)\,dx\Big)\Big(\int_B g(y)\,dy\Big)\,.$$

Proof. By Theorem 6.28, $|h|$ is Riemann measurable. Moreover, by the integrability of g we find that for each $x \in A$ the function $\big|h(x,\cdot)\big| : B \to \mathbb{R}$ is Riemann measurable. Since $|f|$ is integrable on A and

$$\int_B \big|h(x,y)\big|\,dy = \big|f(x)\big| \int_B \big|g(y)\big|\,dy\,,$$

the function $\displaystyle\int_B |h(\cdot,y)|dy : A \to \mathbb{R}$ is Riemann measurable. In other words, h satisfies conditions in the Tonelli theorem (Theorem 6.106); thus we have

$$\int_{A\times B} \big|h(x,y)\big|d(x,y) = \int_A \Big(\int_B \big|h(x,y)\big|dy\Big)dx = \Big(\int_A \big|f(x)\big|\,dx\Big)\Big(\int_B \big|g(y)\big|\,dy\Big) < \infty\,.$$

Therefore, h is integrable on $A \times B$. Since $h(x,\cdot)$ is integrable on B for all $x \in A$ and $h(\cdot,y)$ is integrable on A for all $y \in B$, the Fubini theorem (Theorem 6.107) further implies that

$$\int_{A\times B} h(x,y)d(x,y) = \int_A \Big(\int_B h(x,y)dy\Big)dx = \Big(\int_A f(x)\,dx\Big)\Big(\int_B g(y)\,dy\Big)\,.$$ □

Chapter 7

Uniform Convergence and the Space of Continuous Functions

7.1 Pointwise Convergence and Uniform Convergence (逐點收斂與均勻收斂)

Definition 7.1. Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k : A \to N$ be a function for each $k \in \mathbb{N}$. The sequence of functions $\{f_k\}_{k=1}^{\infty}$ is said to ***converge pointwise*** if $\big\{f_k(a)\big\}_{k=1}^{\infty}$ converges for all $a \in A$. In other words, $\{f_k\}_{k=1}^{\infty}$ converges pointwise if there exists a function $f : A \to N$ such that

$$\lim_{k\to\infty} \rho\big(f_k(a), f(a)\big) = 0 \qquad \forall\, a \in A.$$

In this case, $\{f_k\}_{k=1}^{\infty}$ is said to converge pointwise to f and is denoted by $f_k \to f$ p.w. or $f_k \overset{\text{p.w.}}{\to} f$.

Let $B \subseteq A$ be a subset. The sequence of functions $\{f_k\}_{k=1}^{\infty}$ is said to ***converge uniformly*** on B if there exists $f : B \to N$ such that

$$\lim_{k\to\infty} \sup_{x\in B} \rho\big(f_k(x), f(x)\big) = 0\,.$$

In this case, $\{f_k\}_{k=1}^{\infty}$ is said to converge uniformly to f on B (or converge to f uniformly on B) and is denoted by $f_k \to f$ unif. or $f_k \overset{\text{unif.}}{\to} f$. In other words, $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on B if for every $\varepsilon > 0$, there exists $N > 0$ such that

$$\rho\big(f_k(x), f(x)\big) < \varepsilon \qquad \forall\, k \geqslant N \text{ and } x \in B\,.$$

The sequence of functions $\{f_k\}_{k=1}^{\infty}$ is said to converge uniformly (to f) if $\{f_k\}_{k=1}^{\infty}$ converges uniformly (to f) on A.

Example 7.2. Let $f_k, f : [0,1] \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 0 & \text{if } \dfrac{1}{k} \leqslant x \leqslant 1\,, \\ -kx+1 & \text{if } 0 \leqslant x < \dfrac{1}{k}. \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 0 & \text{if } x \in (0,1], \\ 1 & \text{if } x = 0. \end{cases}$$

Then $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f. To see this, fix $x \in [0,1]$.

1. Case $x \neq 0$: Let $\varepsilon > 0$ be given, take $N \geqslant \dfrac{1}{x}$. If $k \geqslant N$, $x \geqslant \dfrac{1}{k}$ so that $f_k(x) = 0$; thus
$$\big|f_k(x) - f(x)\big| = \big|f_k(x) - 0\big| = |0 - 0| < \varepsilon\,.$$

2. Case $x = 0$: For any $\varepsilon > 0$, $k = 1, 2, 3, \dots$, $\big|f_k(0) - f(0)\big| = |1 - 1| = 0 < \varepsilon$.

However, $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to f on $[0,1]$ because

$$\lim_{k\to\infty} \sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| = \lim_{k\to\infty} \sup_{x\in(0,1]} \big|f_k(x)\big| = 1 \neq 0\,.$$

Nevertheless, if $0 < a < 1$, then by the fact that f_k is decreasing for each $k \in \mathbb{N}$ and $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f,

$$\lim_{k\to\infty} \sup_{x\in[a,1]} \big|f_k(x) - f(x)\big| = \lim_{k\to\infty} \sup_{x\in[a,1]} \big|f_k(x)\big| = \lim_{k\to\infty} \big|f_k(a)\big| = \big|f(a)\big| = 0$$

which implies that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on $[a,1]$ for all $a \in (0,1)$.

Example 7.3. Let $f_k : [0,1] \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 0 & \text{if } \dfrac{2}{k} \leqslant x \leqslant 1\,, \\ -k^2x + 2k & \text{if } \dfrac{1}{k} \leqslant x < \dfrac{2}{k}\,, \\ k^2x & \text{if } 0 \leqslant x < \dfrac{1}{k}. \end{cases}$$

Then similar to the previous example, we have

1. $\{f_k\}_{k=1}^{\infty}$ converges pointwise to the zero function.

2. $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to the zero function on $[0,1]$.

3. $\{f_k\}_{k=1}^{\infty}$ converges uniformly to the zero function on $[a,1]$ for all $a \in (0,1)$.

Example 7.4. Let $f_k : [0,1] \to \mathbb{R}$ be given by $f_k(x) = x^k$. Then for each $a \in [0,1)$, $f_k(a) \to 0$ as $k \to \infty$, while if $a = 1$, $f_k(a) = 1$ for all k. Therefore, if $f(x) = \begin{cases} 0 & \text{if } x \in [0,1)\,, \\ 1 & \text{if } x = 1\,, \end{cases}$ then $f_k \to f$ p.w.. However, since

$$\sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| = \sup_{x\in[0,1)} \big|f_k(x)\big| = 1\,,$$

we must have

$$\lim_{k\to\infty} \sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| = 1 \neq 0\,.$$

Therefore, $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to f on $[0,1]$.

On the other hand, if $0 < a < 1$, then the fact that f_k is increasing for all $k \in \mathbb{N}$ implies that

$$\sup_{x\in[0,a]} \big|f_k(x) - f(x)\big| \leqslant a^k\,;$$

thus by the Sandwich lemma,

$$\lim_{k\to\infty} \sup_{x\in[0,a]} \big|f_k(x) - f(x)\big| = 0\,.$$

Therefore, $\{f_k\}_{k=1}^{\infty}$ converges to uniformly f on $[0,a]$ if $0 < a < 1$.

Example 7.5. Let $f_k : \mathbb{R} \to \mathbb{R}$ be given by $f_k(x) = \dfrac{\sin x}{k}$. Then for each $x \in \mathbb{R}$, $|f_k(x)| \leqslant \dfrac{1}{k}$ which converges to 0 as $k \to \infty$. By the Sandwich lemma,

$$\lim_{k\to\infty} \big|f_k(x)\big| = 0 \qquad \forall\, x \in \mathbb{R}\,.$$

Therefore, $f_k \to 0$ p.w.. Moreover, since $\sup\limits_{x\in\mathbb{R}} \big|f_k(x)\big| \leqslant \dfrac{1}{k}$, $\lim\limits_{k\to\infty} \sup\limits_{x\in\mathbb{R}} \big|f_k(x)\big| = 0$. Therefore, $\{f_k\}_{k=1}^{\infty}$ converges uniformly to 0 on $\mathbb{R}$.

Proposition 7.6. *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k, f : A \to N$ be functions for $k = 1, 2, \cdots$. If $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on A, then $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f.*

Proof. For each $a \in A$, $\rho\big(f_k(a), f(a)\big) \leqslant \sup_{x\in A} \rho\big(f_k(x), f(x)\big)$; thus the Sandwich lemma shows that

$$\lim_{k\to\infty} \rho\big(f_k(a), f(a)\big) = 0$$

since $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on A. □

Proposition 7.7 (Cauchy criterion for uniform convergence)**.** *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k : A \to N$ be a sequence of functions. Suppose that (N,ρ) is complete. Then $\{f_k\}_{k=1}^{\infty}$ converges uniformly on $B \subseteq A$ if and only if for every $\varepsilon > 0$, there exists $N > 0$ such that*

$$\rho\big(f_k(x), f_\ell(x)\big) < \varepsilon \qquad \forall\, k,\ell \geqslant N \text{ and } x \in B\,.$$

Proof. "$\Rightarrow$" Suppose that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on B. Let $\varepsilon > 0$ be given. Then there exists $N > 0$ such that

$$\rho\big(f_k(x), f(x)\big) < \frac{\varepsilon}{2} \qquad \forall\, k \geqslant N \text{ and } x \in B\,.$$

Then if $k,\ell \geqslant N$ and $x \in B$,

$$\rho\big(f_k(x), f_\ell(x)\big) \leqslant \rho\big(f_k(x), f(x)\big) + \rho\big(f(x), f_\ell(x)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

"$\Leftarrow$" Let $b \in B$. By assumption, $\big\{f_k(b)\big\}_{k=1}^{\infty}$ is a Cauchy sequence in (N,ρ); thus is convergent by the completeness of (N,ρ). Therefore, we establish a map $f : B \to N$ defined by $f(b) = \lim\limits_{k\to\infty} f_k(b)$. We claim that $\{f_k\}_{k=1}^{\infty}$ convergence uniformly to f on B.

Let $\varepsilon > 0$ be given. Then there exists $N > 0$ such that

$$\rho\big(f_k(x), f_\ell(x)\big) < \frac{\varepsilon}{2} \qquad \forall\, k,\ell \geqslant N \text{ and } x \in B\,.$$

Moreover, for each $x \in B$ there exists $N_x > 0$ such that

$$\rho\big(f_\ell(x), f(x)\big) < \frac{\varepsilon}{2} \qquad \forall\, \ell \geqslant N_x\,.$$

Then for all $k \geqslant N$ and $x \in B$,

$$\rho\big(f_k(x), f(x)\big) \leqslant \rho\big(f_k(x), f_\ell(x)\big) + \rho\big(f_\ell(x), f(x)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

in which we choose $\ell \geqslant \max\{N, N_x\}$ to conclude the inequality. $\square$

Theorem 7.8. *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k : A \to N$ be a sequence of continuous functions converging to $f : A \to N$ uniformly on A. Then f is continuous on A; that is,*

$$\lim_{x\to a} f(x) = \lim_{x\to a}\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty}\lim_{x\to a} f_k(x) = f(a)\,.$$

Proof. Let $a \in A$ and $\varepsilon > 0$ be given. Since $\{f_k\}_{k=1}^\infty$ converges uniformly to f on A, there exists $N > 0$ such that

$$\rho\big(f_k(x), f(x)\big) < \frac{\varepsilon}{3} \qquad \forall\, k \geqslant N \text{ and } x \in A\,.$$

By the continuity of f_N, there exists $\delta > 0$ such that

$$\rho\big(f_N(x), f_N(a)\big) < \frac{\varepsilon}{3} \quad \text{whenever} \quad x \in B_M(a,\delta) \cap A\,.$$

Therefore, if $x \in B_M(a,\delta) \cap A$, by the triangle inequality

$$\begin{aligned}\rho\big(f(x), f(a)\big) &\leqslant \rho\big(f(x), f_N(x)\big) + \rho\big(f_N(x), f_N(a)\big) + \rho\big(f_N(a), f(a)\big)\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,;\end{aligned}$$

thus f is continuous at a. □

Example 7.9. Let $f_k : [0,2] \to \mathbb{R}$ be given by $f_k(x) = \dfrac{x^k}{1+x^k}$. Then

1. For each $a \in [0,1)$, $f_k(a) \to 0$ as $k \to \infty$;
2. For each $a \in (1,2]$, $f_k(a) \to 1$ as $k \to \infty$;
3. If $a = 1$, then $f_k(a) = \dfrac{1}{2}$ for all k.

Let $f(x) = \begin{cases} 0 & \text{if } x \in [0,1)\,, \\ \dfrac{1}{2} & \text{if } x = 1\,, \\ 1 & \text{if } x \in (1,2]\,. \end{cases}$ Then $\{f_k\}_{k=1}^\infty$ converges pointwise to f. However, $\{f_k\}_{k=1}^\infty$ does not converge uniformly to f on $[0,2]$ since f_k are continuous functions for all $k \in \mathbb{N}$ but f is not.

Remark 7.10. The uniform limit of sequence of continuous function might not be uniformly continuous. For example, let $A = (0,1)$ and $f_k(x) = \dfrac{1}{x}$ for all $k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^\infty$ converges uniformly to $f(x) = \dfrac{1}{x}$, but the limit function is not uniformly continuous on A.

Theorem 7.11. *Let $I \subseteq \mathbb{R}$ be a finite interval, $f_k : I \to \mathbb{R}$ be a sequence of differentiable functions, and $g : I \to \mathbb{R}$ be a function. Suppose that $\big\{f_k(a)\big\}_{k=1}^\infty$ converges for some $a \in I$, and $\{f_k'\}_{k=1}^\infty$ converges uniformly to g on I. Then*

1. *$\{f_k\}_{k=1}^\infty$ converges uniformly to some function f on I.*

2. *The limit function f is differentiable on I, and $f'(x) = g(x)$ for all $x \in I$; that is,*

$$\lim_{k\to\infty} f_k'(x) = \lim_{k\to\infty} \frac{d}{dx} f_k(x) = \frac{d}{dx} \lim_{k\to\infty} f_k(x) = f'(x)\,.$$

Proof. 1. Let $\varepsilon > 0$ be given. Since $\big\{f_k(a)\big\}_{k=1}^{\infty}$ converges to $f(a)$, $\big\{f_k(a)\big\}_{k=1}^{\infty}$ is a Cauchy sequence. Therefore, there exists $N_1 > 0$ such that

$$\big|f_k(a) - f_\ell(a)\big| < \frac{\varepsilon}{2} \qquad \forall\, k, \ell \geqslant N_1\,.$$

By the uniform convergence of $\{f_k'\}_{k=1}^{\infty}$ on I and Proposition 7.7, there exists $N_2 > 0$ such that

$$\big|f_k'(x) - f_\ell'(x)\big| < \frac{\varepsilon}{2|I|} \qquad \forall\, k, \ell \geqslant N_2 \text{ and } x \in I\,,$$

where $|I|$ is the length of the interval.

Let $N = \max\{N_1, N_2\}$. By the mean value theorem, for all $k, \ell \geqslant N$ and $x \in I$, there exists ξ in between x and a such that

$$\big|f_k(x) - f_\ell(x) - f_k(a) + f_\ell(a)\big| = |f_k'(\xi) - f_\ell'(\xi)||x-a| \leqslant \frac{\varepsilon|x-a|}{2|I|} \leqslant \frac{\varepsilon}{2}\,;$$

thus for all $k, \ell \geqslant N$ and $x \in I$,

$$\big|f_k(x) - f_\ell(x)\big| \leqslant \big|f_k(a) - f_\ell(a)\big| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, Proposition 7.7 implies that $\{f_k\}_{k=1}^{\infty}$ converges uniformly on I.

2. Suppose that the uniform limit of $\{f_k\}_{k=1}^{\infty}$ is f. For any given point $c \in I$, define

$$\phi_k(x) = \begin{cases} \dfrac{f_k(x) - f_k(c)}{x - c} & \text{if } x \in I,\ x \neq c\,, \\ f_k'(c) & \text{if } x = c\,, \end{cases} \quad \text{and} \quad \phi(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c} & \text{if } x \in I,\ x \neq c\,, \\ g(c) & \text{if } x = c\,. \end{cases}$$

Then ϕ_k is continuous on I for all $k \in \mathbb{N}$, and $\{\phi_k\}_{k=1}^{\infty}$ converges pointwise to ϕ.

Claim: $\{\phi_k\}_{k=1}^{\infty}$ converges uniformly to ϕ on I.

Proof of claim: Let $\varepsilon > 0$ be given. Since $\{f_k'\}_{k=1}^{\infty}$ converges uniformly on I, there exists $N > 0$ such that

$$\sup_{s\in I} \big|f_k'(s) - f_\ell'(s)\big| < \varepsilon \qquad \forall\, k, \ell \geqslant N\,.$$

Since

$$\big|\phi_k(x) - \phi_\ell(x)\big| = \begin{cases} \dfrac{\big|f_k(x) - f_k(c) - f_\ell(x) + f_\ell(c)\big|}{|x-c|} & \text{if } x \neq c,\, x \in I\,, \\ \big|f_k'(c) - f_\ell'(c)\big| & \text{if } x = c\,, \end{cases}$$

by the mean value theorem we obtain that

$$\big|\phi_k(x) - \phi_\ell(x)\big| \leqslant \sup_{s\in I}\big|f_k'(s) - f_\ell'(s)\big| < \varepsilon \qquad \forall\, k, \ell \geqslant N \text{ and } x \in I\,.$$

Therefore, the Cauchy criterion shows that $\{\phi_k\}_{k=1}^\infty$ converges uniformly to ϕ on I, and Theorem 7.8 further shows that ϕ is continuous on I; thus

$$f'(c) = \lim_{x\to c}\phi(x) = \phi(c) = g(c)\,. \qquad \square$$

Example 7.12. Assume that $f_k : I \to \mathbb{R}$ is differentiable for all $k \in \mathbb{N}$, and $\{f_k'\}_{k=1}^\infty$ converges uniformly to g on I. Then $\{f_k\}_{k=1}^\infty$ might **NOT** converge. For example, consider $f_k(x) = k$. Then $f_k' \equiv 0$ but $\{f_k\}_{k=1}^\infty$ does not converge.

Example 7.13. For each $k \in \mathbb{N}$, let $f_k : [0,1] \to \mathbb{R}$ be defined by $f_k(x) = \dfrac{\sin(k^2x)}{k}$. Then $\{f_k\}_{k=1}^\infty$ converges uniformly to the zero function on $[0,1]$ since

$$\sup_{x\in[0,1]}\big|f_k(x) - 0\big| = \sup_{x\in[0,1]}\Big|\frac{\sin(k^2x)}{k}\Big| \leqslant \frac{1}{k} \Rightarrow \lim_{k\to\infty}\sup_{x\in[0,1]}\big|f_k(x) - 0\big| = 0\,.$$

However, note that $f_k'(x) = k\cos(k^2x)$ so that $\{f_k'(0) = k$ which diverges to ∞ (so that $\{f_k'\}_{k=1}^\infty$ does not even converge pointwise). Therefore, even if a differentiable sequence $\{f_k\}_{k=1}^\infty$ converges uniformly, it does not implies that $\{f_k'\}_{k=1}^\infty$ converges (pointwise).

Example 7.14. Assume that $f_k : I \to \mathbb{R}$ is differentiable for all $k \in \mathbb{N}$, and $\{f_k\}_{k=1}^\infty$ converges uniformly to f on I. Then f might **NOT** be differentiable. In fact, there are differentiable functions $f_k : [a,b] \to \mathbb{R}$ such that f_k converges uniformly to f on $[a,b]$ but f is not differentiable. For example, consider

$$f_k(x) = \begin{cases} \dfrac{k}{2}x^2 & \text{if } |x| \leqslant \dfrac{1}{k}, \\ |x| - \dfrac{1}{2k} & \text{if } \dfrac{1}{k} \leqslant |x| \leqslant 1. \end{cases}$$

Observe that $f_k(-x) = f_k(x)$, so it suffices to consider $x \geqslant 0$.

1. Let $f(x) = |x|$. Then $f_k \to f$ uniformly:

$$\begin{aligned}
&\sup_{x\in[-1,1]} \big|f_k(x) - f(x)\big| \\
&\qquad = \sup_{x\in[0,1]} \big|f_k(x) - x\big| = \max\Big\{ \sup_{x\in[0,\frac{1}{k}]} \big|f_k(x) - x\big|, \sup_{x\in[\frac{1}{k},1]} \big|f_k(x) - x\big| \Big\} \\
&\qquad = \max\Big\{ \sup_{x\in[0,\frac{1}{k}]} \big|\frac{kx^2}{2} - x\big|, \sup_{x\in[\frac{1}{k},1]} \big|x - \frac{1}{2k} - x\big| \Big\} \\
&\qquad \leqslant \sup_{x\in[0,\frac{1}{k}]} \big|\frac{kx^2}{2}\big| + \big|x\big| \leqslant \frac{k}{2}(\frac{1}{k})^2 + \frac{1}{k} = \frac{3}{2k} \to 0 \text{ as } k \to \infty\,.
\end{aligned}$$

2. To see if f_k are differentiable, it suffices to show $f_k'(\frac{1}{k})$ exists.

$$f_k'(\frac{1}{k}) = \lim_{h\to 0} \frac{f_k(\frac{1}{k}+h) - f_k(\frac{1}{k})}{h} = \lim_{h\to 0} \frac{1}{h} \begin{cases} (\frac{1}{k} + h) - \frac{1}{2k} - \frac{1}{2k} & \text{if } h > 0 \\ \frac{k}{2}(\frac{1}{k} + h)^2 - \frac{1}{2k} & \text{if } h < 0 \end{cases}$$

$$= \lim_{h\to 0} \frac{1}{h} \begin{cases} h & \text{if } h > 0 \\ h + \frac{k}{2}h^2 & \text{if } h < 0 \end{cases} = 1\,.$$

Example 7.15. Assume that $f_k : [-1,1] \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 0 & \text{if } x \in [-1,0]\,, \\ \frac{k^2}{2}x^2 & \text{if } x \in \big(0, \frac{1}{k}\big]\,, \\ 1 - \frac{k^2}{2}\big(x - \frac{2}{k}\big)^2 & \text{if } x \in \big(\frac{1}{k}, \frac{2}{k}\big]\,, \\ 1 & \text{if } x \in \big(\frac{2}{k}, 1\big]\,. \end{cases}$$

Then $f_k'(x) = \begin{cases} 0 & \text{if } x \in [-1,0]\,, \\ k^2 x & \text{if } x \in \big(0, \frac{1}{k}\big]\,, \\ -k^2\big(x - \frac{2}{k}\big) & \text{if } x \in \big(\frac{1}{k}, \frac{2}{k}\big]\,, \\ 0 & \text{if } x \in \big(\frac{2}{k}, 1\big]\,, \end{cases}$ and $\{f_k'\}_{k=1}^\infty$ converges pointwise to 0 but not uniformly on $[-1,1]$. We note that $\{f_k\}_{k=1}^\infty$ converges to a discontinuous function

$$f(x) = \begin{cases} 0 & \text{if } x \in [-1,0]\,, \\ 1 & \text{if } x \in (0,1]\,, \end{cases}$$

so the convergence of $\{f_k\}_{k=1}^\infty$ cannot be uniform on $[-1,1]$.

Example 7.16. There are differentiable functions $f_k : [a,b] \to \mathbb{R}$ such that f_k converges uniformly to f on $[a,b]$ but $\lim\limits_{k\to\infty} f_k' \neq (\lim\limits_{k\to\infty} f_k)'$. For example, take $f_k(x) = \dfrac{x}{1+k^2x^2}$ on $[-1,1]$. Then $f_k'(x) = \dfrac{1-k^2x^2}{(1+k^2x^2)^2}$.

1. Since $\lim\limits_{k\to\infty} \sup\limits_{x\in[-1,1]} \Big|\dfrac{x}{1+k^2x^2} - 0\Big| = \lim\limits_{k\to\infty} \dfrac{1}{2k} = 0$, f_k converges uniformly to 0 on $[-1,1]$.
2. $\big(\lim\limits_{k\to\infty} f_k(x)\big)' = 0' = 0$.
3. $\lim\limits_{k\to\infty} f_k'(x) = \lim\limits_{k\to\infty} \dfrac{1-k^2x^2}{(1+k^2x^2)^2} = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, |x| < 1. \end{cases}$ Note that f_k' does not converge uniformly.

Theorem 7.17. *Let $f_k : A \to \mathbb{R}$ be a sequence of Riemann integrable functions which converges uniformly to f on A. Then f is Riemann integrable on A, and*

$$\lim_{k\to\infty} \int_A f_k(x)\,dx = \int_A \lim_{k\to\infty} f_k(x)\,dx = \int_A f(x)\,dx\,. \tag{7.1.1}$$

Proof. Let R be a rectangle such that $A \subseteq R$ and $\nu(R) > 0$, and $\varepsilon > 0$ be given. Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on A, there exists $N > 0$ such that

$$\big|f_k(x) - f(x)\big| < \frac{\varepsilon}{4\nu(R)} \qquad \forall\, k \geqslant N \text{ and } x \in A\,. \tag{7.1.2}$$

Since f_N is Riemann integrable on A, by Riemann's condition there exists a partition $\mathcal{P}$ of A such that

$$U(f_N, \mathcal{P}) - L(f_N, \mathcal{P}) < \frac{\varepsilon}{2}\,.$$

By Proposition 6.8, we find that

$$\begin{aligned} U(f,\mathcal{P}) - L(f,\mathcal{P}) &= U(f - f_N + f_N, \mathcal{P}) - L(f - f_N + f_N, \mathcal{P}) \\ &\leqslant U(f - f_N, \mathcal{P}) + U(f_N, \mathcal{P}) - L(f - f_N, \mathcal{P}) - L(f_N, \mathcal{P}) \\ &\leqslant \frac{\varepsilon}{4\nu(R)}\nu(R) + \frac{\varepsilon}{4\nu(R)}\nu(R) + U(f_N, \mathcal{P}) - L(f_N, \mathcal{P}) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon\,; \end{aligned}$$

thus by Riemann's condition f is Riemann integrable on A.

Now, if $k \geqslant N$, (7.1.2) implies that

$$\begin{aligned}\Big|\int_A f_k(x)dx - \int_A f(x)dx\Big| &= \Big|\int_A \big(f_k(x) - f(x)\big)dx\Big| \leqslant \int_A \big|f_k(x) - f(x)\big|dx \\ &\leqslant \frac{\varepsilon}{4\nu(R)}\nu(R) = \frac{\varepsilon}{4} < \varepsilon\end{aligned}$$

which shows (7.1.1). □

Example 7.18. In this example we provide a sequence of integrable functions converges pointwise to a limit function which is not integrable. Let $\{q_k\}_{k=1}^{\infty}$ be the rational numbers in $[0,1]$, and

$$f_k(x) = \begin{cases} 0 & \text{if } x \in \{q_1, q_2, \cdots, q_k\}\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Then f_k converges pointwise to the Dirichlet function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 1 & \text{if } x \in [0,1]\backslash\mathbb{Q}\,. \end{cases}$$

It is well-known that the Dirichlet function is not integrable. However, $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to f since f_k are Riemann integrable on $[0,1]$ for all $k \in \mathbb{N}$ but f is not.

7.2 Series of Functions and The Weierstrass M-Test

Definition 7.19. Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed space, $A \subseteq M$ be a subset, and $g_k, g : A \to \mathcal{V}$ be functions. We say that the series $\sum\limits_{k=1}^{\infty} g_k$ converges pointwise if the sequence of partial sum $\{s_n\}_{n=1}^{\infty}$ given by

$$s_n = \sum_{k=1}^{n} g_k$$

converges pointwise. We use $\sum\limits_{k=1}^{\infty} g_k = g$ p.w. to denote that the series $\sum\limits_{k=1}^{\infty} g_k$ converges pointwise to g. The series $\sum\limits_{k=1}^{\infty} g_k$ is said to converge uniformly on $B \subseteq A$ if $\{s_n\}_{n=1}^{\infty}$ converges uniformly on B.

Example 7.20. Consider the geometric series $\sum\limits_{k=0}^{\infty} x^k$. The partial sum s_n is given by

$$s_n(x) = \begin{cases} \dfrac{1 - x^{n+1}}{1 - x} & \text{if } x \neq 1\,, \\ n+1 & \text{if } x = 1\,. \end{cases}$$

Then

1. $\sum\limits_{k=0}^{\infty} x^k$ converges pointwise to $g(x) = \dfrac{1}{1-x}$ in $(-1,1)$.

2. $\sum\limits_{k=0}^{\infty} x^k$ does not converge pointwise in $(-\infty,-1] \cup [1,\infty)$.

3. $\sum\limits_{k=0}^{\infty} x^k$ converges uniformly on $(-a,a)$ if $0 < a < 1$ since
$$\sup_{x\in(-a,a)} |s_n(x) - g(x)| = \sup_{x\in(-a,a)} \frac{|x|^{n+1}}{1-x} \leqslant \frac{|a|^{n+1}}{1-a} \to 0 \text{ as } n \to \infty.$$

4. $\sum\limits_{k=0}^{\infty} x^k$ does not converge uniformly on $(-1,1)$ since $\sup\limits_{x\in(-1,1)} |s_n(x) - g(x)| = \infty$.

The following two corollaries are direct consequences of Proposition 7.7 and Theorem 7.8.

Corollary 7.21. *Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a complete normed vector space, $A \subseteq M$ be a subset, and $g_k : A \to \mathcal{V}$ be functions. Then $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on A if and only if*
$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni \Big\| \sum_{k=m+1}^{n} g_k(x) \Big\| < \varepsilon \qquad \forall\, n > m \geqslant N \text{ and } x \in A\,.$$

Corollary 7.22. *Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, $A \subseteq M$ be a subset, and $g_k, g : A \to \mathcal{V}$ be functions. If $g_k : A \to \mathcal{V}$ is continuous for all $k \in \mathbb{N}$ and $\sum\limits_{k=1}^{\infty} g_k(x)$ converges to g uniformly on A, then g is continuous.*

Theorem 7.23. *Let $f : (a,b) \to \mathbb{R}$ be an infinitely differentiable functions; that is, $f^{(k)}(x)$ exists for all $k \in \mathbb{N}$ and $x \in (a,b)$. Let $c \in (a,b)$ and suppose that for some $0 < h < \infty$, $\big|f^{(k)}(x)\big| \leqslant M$ for all $x \in (c-h, c+h) \subseteq (a,b)$ and $k \in \mathbb{N}$. Then*
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad \forall\, x \in (c-h, c+h)$$
and the convergence is uniform.

Proof. First, we claim that
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^n \int_c^x \frac{(y-x)^n}{n!} f^{(n+1)}(y)dy \qquad \forall\, x \in (a,b)\,. \tag{7.2.1}$$

By the fundamental theorem of Calculus it is clear that (7.2.1) holds for $n = 0$. Suppose that (7.2.1) holds for $n = m$. Then

$$\begin{aligned} f(x) &= \sum_{k=0}^{m} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^m \Big[\frac{(y-x)^{m+1}}{(m+1)!} f^{(m+1)}(y)\Big|_{y=c}^{y=x} - \int_c^x \frac{(y-x)^{m+1}}{(m+1)!} f^{(m+2)}(y)dy \Big] \\ &= \sum_{k=0}^{m+1} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^{m+1} \int_c^x \frac{(y-x)^{m+1}}{(m+1)!} f^{(m+2)}(y)dy \end{aligned}$$

which implies that (7.2.1) also holds for $n = m+1$. By induction (7.2.1) holds for all $n \in \mathbb{N}$.

Define $s_n(x) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$. Our goal is to show that $\{s_n\}_{n=1}^{\infty}$ converges uniformly to f on $(c-h, c+h)$. Nevertheless, note that if $x \in (c-h, c+h)$,

$$\big|s_n(x) - f(x)\big| \leqslant \Big| \int_c^x \frac{h^n}{n!} M dy \Big| \leqslant \frac{h^{n+1}}{n!} M .$$

By the fact that $\lim\limits_{n\to\infty} \frac{h^{n+1}}{n!} M = 0$, we conclude that

$$\lim_{n\to\infty} \sup_{x\in(c-h,c+h)} \big|s_n(x) - f(x)\big| = 0 . \qquad \square$$

Remark 7.24. Assume the conditions in Theorem 7.23. Then applying Theorem 7.23 to the function $g = f'$ shows that

$$\frac{d}{dx} \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = \sum_{k=0}^{\infty} \frac{d}{dx} \Big[\frac{f^{(k)}(c)}{k!}(x-c)^k \Big] \qquad \forall\, x \in (c-h, c+h) .$$

This is a special case of Corollary 7.38.

Example 7.25. The series $\sum\limits_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ converges to $\sin x$ uniformly on any bounded subset of $\mathbb{R}$.

Theorem 7.26 (Weierstrass M-test)**.** *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a complete normed vector space, $A \subseteq M$ be a subset, and $g_k : A \to \mathcal{V}$ be a sequence of functions. Suppose that there exists $M_k > 0$ such that $\sup\limits_{x\in A} \|g_k(x)\| \leqslant M_k$ for all $k \in \mathbb{N}$ and $\sum\limits_{k=1}^{\infty} M_k$ converges. Then $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly and absolutely (that is, $\sum\limits_{k=1}^{\infty} \|g_k\|$ converges uniformly) on A.*

Proof. We show that the partial sum $s_n = \sum\limits_{k=1}^{n} g_k$ satisfies the Cauchy criterion. Let $\varepsilon > 0$ be given. Since $\sum\limits_{k=1}^{\infty} M_k$ converges (which means $\sum\limits_{k=1}^{n} M_k$ converges as $n \to \infty$), the Cauchy criterion for the convergence of series of vectors (Theorem 2.65) implies that there exists $N > 0$ such that

$$\sum_{k=m+1}^{n} M_k = \Big| \sum_{k=m+1}^{n} M_k \Big| < \varepsilon \quad \forall\, n > m \geqslant N\,.$$

Therefore,

$$\Big\| \sum_{k=m+1}^{n} g_k(x) \Big\| \leqslant \sum_{k=m+1}^{n} \|g_k(x)\| \leqslant \sum_{k=m+1}^{n} M_k < \varepsilon \quad \forall\, n > m \geqslant N \text{ and } x \in A$$

and the desired result follows from the Cauchy criterion for the uniform convergence of series of functions (Corollary 7.21). □

Theorem 7.8 and 7.26 together imply the following

Corollary 7.27. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a complete normed vector space, $A \subseteq M$ be a subset, and $g_k : A \to \mathcal{V}$ be a sequence of continuous functions. Suppose that there exists $M_k > 0$ such that $\sup\limits_{x \in A} \|g_k(x)\| \leqslant M_k$ for all $k \in \mathbb{N}$ and $\sum\limits_{k=1}^{\infty} M_k$ converges. Then $\sum\limits_{k=1}^{\infty} g_k$ is continuous on A.*

Example 7.28. Consider the series $f(x) = \sum\limits_{k=0}^{\infty} \Big(\dfrac{x^k}{k!}\Big)^2$. For all $x \in [-R, R]$, $\Big(\dfrac{x^k}{k!}\Big)^2 \leqslant \dfrac{R^{2k}}{(k!)^2}$. Moreover,

$$\limsup_{k\to\infty} \frac{R^{2(k+1)}}{((k+1)!)^2} \Big/ \frac{R^{2k}}{(k!)^2} = \limsup_{k\to\infty} \frac{R^2}{(k+1)^2} = 0\,;$$

thus the ratio test and the Weierstrass M-test imply that the series $\sum\limits_{k=0}^{\infty} \Big(\dfrac{x^k}{k!}\Big)^2$ converges uniformly on $[-R, R]$. Theorem 7.8 then shows that f is continuous on $[-R, R]$. Since R is arbitrary, we find that f is continuous on $\mathbb{R}$.

Example 7.29. Let $\{a_k\}_{k=0}^{\infty}$ be a bounded sequence. Then $\sum\limits_{k=0}^{\infty} \dfrac{a_k}{k!} x^k$ converges to a continuous function.

Example 7.30. Consider the function $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi} \sum\limits_{k=0}^{\infty} \dfrac{\cos(2k+1)x}{(2k+1)^2}$. We can in fact show (later) that $f(x) = |x|$ for all $x \in [-\pi, \pi]$, and by the Weierstrass M-test it is easy to see that the convergence is uniform on $\mathbb{R}$.

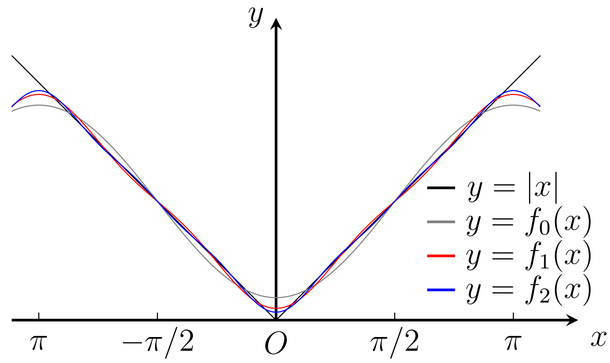


Figure 7.1: The graph of some partial sums

7.3 Integration and Differentiation of Series

The following two theorems are direct consequences of Theorem 7.11 and 7.17.

Theorem 7.31. *Let $g_k : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable functions. If $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on $[a,b]$, then*

$$\int_a^b \sum_{k=1}^{\infty} g_k(x)dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)dx\,.$$

Theorem 7.32. *Let $g_k : (a,b) \to \mathbb{R}$ be a sequence of differentiable functions. Suppose that $\sum\limits_{k=1}^{\infty} g_k$ converges for some $c \in (a,b)$, and $\sum\limits_{k=1}^{\infty} g_k'$ converges uniformly on (a,b). Then*

$$\sum_{k=1}^{\infty} g_k'(x) = \frac{d}{dx} \sum_{k=1}^{\infty} g_k(x)\,.$$

Definition 7.33. A series is called a ***power series about*** c or ***centered at*** c if it is of the form $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ for some sequence $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ (or $\mathbb{C}$) and $c \in \mathbb{R}$ (or $\mathbb{C}$).

Proposition 7.34. *If a power series centered at c is convergent at some point $b \neq c$, then the power series converges pointwise on $B(c, |b-c|)$, and converges uniformly on any compact subsets of $B(c, |b-c|)$.*

Proof. Since the series $\sum\limits_{k=0}^{\infty} a_k(b-c)^k$ converges, $|a_k||b-c|^k \to 0$ as $k \to \infty$; thus there exists $M > 0$ such that $|a_k||b-c|^k \leqslant M$ for all k.

1. $x \in B(c, |b-c|)$, the series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely since
$$\sum_{k=0}^{\infty} |a_k(x-c)^k| \leqslant \sum_{k=0}^{\infty} |a_k||x-c|^k = \sum_{k=0}^{\infty} |a_k||b-c|^k \frac{|x-c|^k}{|b-c|^k} \leqslant M \sum_{k=0}^{\infty} \Big(\frac{|x-c|}{|b-c|}\Big)^k$$
which converges (because of the geometric series test or ratio test).

2. Let $K \subseteq B(c, |b-c|)$ be a compact set. Then
$$\text{dist}(K, \partial B(c, |b-c|)) \equiv \inf\big\{|x-y| \,\big|\, x \in K, |y-c| = |b-c|\big\} > 0\,.$$
Define $r = \dfrac{|b-c| - \text{dist}(K, \partial B(c, |b-c|))}{|b-c|}$. Then $0 \leqslant r < 1$, and $|x-c| \leqslant r|b-c|$ for all $x \in K$. Therefore, $|a_k(x-c)^k| \leqslant Mr^k$ if $x \in K$; thus the Weierstrass M-test implies that the series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges uniformly on K. □

By the proposition above, we immediately conclude that the collection of all x at which the power series converges must be connected and symmetric; thus is a disc or a point. This observation induce the following

Definition 7.35. A non-negative number R is called the ***radius of convergence*** of the power series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ if the series converges for all $x \in B(c, R)$ but diverges if $x \notin B[c, R]$. In other words,
$$R = \sup\Big\{r \geqslant 0 \,\Big|\, \sum_{k=0}^{\infty} a_k(x-c)^k \text{ converges in } B[c, R]\Big\}\,.$$
The ***interval of convergence*** or ***convergence interval*** of a power series is the collection of all x at which the power series converges.

Remark 7.36. A power series converges pointwise on its interval of convergence.

Theorem 7.37. *Let* $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{C}$, $c \in \mathbb{C}$, $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ *be a power series with radius of convergence* $R > 0$, *and* $K \subseteq B(c, R)$ *be a compact set. Then*

1. *The power series* $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ *converges uniformly on* K.

2. *The power series* $\sum\limits_{k=0}^{\infty} (k+1)a_{k+1}(x-c)^k$ *converges pointwise on* $B(c, R)$, *and converges uniformly on* K.

Proof. 1. It is simply a restatement of Proposition 7.34.

2. By 1, it suffices to show that the power series $\sum_{k=0}^{\infty}(k+1)a_{k+1}(x-c)^k$ converges pointwise on $B(c,R)$. Clearly the series converges at $x=c$. Let $x \in B(c,R)$ and $x \neq c$. Since $|x-c|<R$, there exists $b \in B(c,R)$ such that

$$|b-c| = \frac{R+|x-c|}{2}\,.$$

Then if $r = \frac{|x-c|}{|b-c|}$, $0<r<1$ and

$$\sum_{k=0}^{\infty}(k+1)|a_{k+1}||x-c|^k \leqslant \sum_{k=0}^{\infty}(k+1)|a_{k+1}||b-c|^k\Big(\frac{|x-c|}{|b-c|}\Big)^k \leqslant M\sum_{k=0}^{\infty}(k+1)r^k$$

for some $M>0$. Note that the ratio test implies that the series $\sum_{k=0}^{\infty}(k+1)r^k$ converges if $0<r<1$; thus $\sum_{k=0}^{\infty}(k+1)|a_{k+1}||x-c|^k$ converges by the comparison test. □

Corollary 7.38. *Let $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, and $\sum_{k=0}^{\infty} a_k(x-c)^k$ be a power series with radius of convergence $R>0$. Then $\sum_{k=0}^{\infty} a_k(x-c)^k$ is differentiable in $(c-R,c+R)$ and is Riemann integrable over any closed intervals $[\alpha,\beta] \subseteq (c-R,c+R)$. Moreover,*

$$\frac{d}{dx}\sum_{k=0}^{\infty} a_k(x-c)^k = \sum_{k=1}^{\infty} k a_k(x-c)^{k-1} \qquad \forall\, x \in (c-R,c+R)$$

and

$$\int_{\alpha}^{\beta}\sum_{k=0}^{\infty} a_k(x-c)^k dx = \sum_{k=0}^{\infty} a_k\int_{\alpha}^{\beta}(x-c)^k dx\,.$$

Example 7.39. Let $\{a_k\}_{k=0}^{\infty}$ be a bounded sequence. Then

$$\frac{d}{dx}\Big(\sum_{k=0}^{\infty}\frac{a_k}{k!}x^k\Big) = \sum_{k=1}^{\infty}\frac{a_k}{(k-1)!}x^{k-1} = \sum_{k=0}^{\infty}\frac{a_{k+1}}{k!}x^k\,.$$

Example 7.40. We show $\int_0^t e^x dx = e^t - 1$ as follows. By Theorem 7.23, $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ and the convergence is uniform on any bounded sets of $\mathbb{R}$; thus Corollary 7.38 implies that

$$\int_0^t e^x dx = \int_0^t \sum_{k=0}^{\infty}\frac{x^k}{k!}dx = \sum_{k=0}^{\infty}\int_0^t \frac{x^k}{k!}dx = \sum_{k=0}^{\infty}\frac{t^{k+1}}{(k+1)!} = \sum_{k=1}^{\infty}\frac{t^k}{k!} = e^t - 1\,.$$

Example 7.41. $\frac{d}{dx}\Big(\sum\limits_{k=1}^{\infty}\frac{x^k}{k}\Big) = \sum\limits_{k=1}^{\infty} x^{k-1} = \sum\limits_{k=0}^{\infty} x^k$ for all $x \in (-1,1)$; thus

$$\frac{d}{dx}\Big(\sum_{k=1}^{\infty}\frac{x^k}{k}\Big) = \frac{1}{1-x} \qquad \forall\, x \in (-1,1)\,.$$

As a consequence,

$$\sum_{k=1}^{\infty}\frac{t^k}{k} = \int_0^t \frac{d}{dx}\Big(\sum_{k=1}^{\infty}\frac{x^k}{k}\Big)dx = -\log(1-t) \qquad \forall\, t \in (-1,1)\,. \tag{7.3.1}$$

Using the alternating series test, it is clear that the left-hand side of (7.3.1) converges at $t=-1$. What is the value of

$$-\sum_{k=1}^{\infty}\frac{(-1)^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots ?$$

Consider the partial sum $\frac{d}{dx}\Big(\sum\limits_{k=1}^{n}\frac{x^k}{k}\Big) = \sum\limits_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$. Integrating both sides over $[-1,0]$,

$$\Big|\sum_{k=1}^{n}\frac{(-1)^k}{k} + \log 2\Big| \leqslant \int_{-1}^{0}\frac{|x|^n}{1-x}dx \leqslant \int_{-1}^{0}(-x)^n dx = \frac{1}{n+1} \to 0 \text{ as } n \to \infty;$$

thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log 2\,.$$

In other words,

$$\sum_{k=1}^{\infty}\frac{t^k}{k} = -\log(1-t) \qquad \forall\, t \in [-1,1)\,.$$

Example 7.42. It is clear that $\frac{1}{1+x^2} = \sum\limits_{k=0}^{\infty}(-x^2)^k = \sum\limits_{k=0}^{\infty}(-1)^k x^{2k}$ for all $x \in (-1,1)$. So if $x \in (-1,1)$,

$$\begin{aligned}\tan^{-1} x &= \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{k=0}^{\infty}(-1)^k t^{2k} dt = \sum_{k=0}^{\infty}\int_0^x (-1)^k t^{2k} dt \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} t^{2k+1}\Big|_{t=0}^{t=x} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\,.\end{aligned}$$

The right-hand side of the identity above converges at $x=1$. What is the value of

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots ?$$

Mimic the previous example, we consider

$$\begin{aligned}\tan^{-1} x &= \int_0^x \frac{dt}{1+t^2} = \int_0^x \frac{1-(-t^2)^{n+1}}{1+t^2}dt + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2}dt \\ &= \int_0^x \sum_{k=0}^{n} (-1)^k t^{2k} dt + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2}dt \\ &= \sum_{k=0}^{n} \int_0^x (-1)^k t^{2k} dt + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2}dt = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1} + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2}dt\,;\end{aligned}$$

thus plugging $x = 1$,

$$\Big|\tan^{-1} 1 - \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\Big| \leqslant \int_0^1 \frac{t^{2(n+1)}}{1+t^2}dt \leqslant \int_0^1 t^{2(n+1)}dt = \frac{1}{2n+3} \to 0 \text{ as } n \to \infty.$$

Therefore,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \tan^{-1} 1 = \frac{\pi}{4}\,.$$

7.4 The Space of Continuous Functions

Definition 7.43. Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. We define $\mathscr{C}(A; \mathcal{V})$ as the collection of all continuous functions on A with value in $\mathcal{V}$; that is,

$$\mathscr{C}(A;\mathcal{V}) = \big\{ f : A \to \mathcal{V} \,\big|\, f \text{ is continuous on } A \big\}\,.$$

Let $\mathscr{C}_b(A; \mathcal{V})$ be the subspace of $\mathscr{C}(A; \mathcal{V})$ which consists of all bounded continuous functions on A; that is,

$$\mathscr{C}_b(A;\mathcal{V}) = \big\{ f \in \mathscr{C}(A;\mathcal{V}) \,\big|\, f \text{ is bounded} \big\}\,.$$

Every $f \in \mathscr{C}_b(A; \mathcal{V})$ is associated with a non-negative real number $\|f\|_\infty$ given by

$$\|f\|_\infty = \sup \big\{ \|f(x)\| \,\big|\, x \in A \big\} = \sup_{x \in A} \|f(x)\|\,.$$

The number $\|f\|_\infty$ is called the ***sup-norm*** of f.

Proposition 7.44. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, $A \subseteq M$ be a subset.*

1. *$\mathscr{C}(A; \mathcal{V})$ and $\mathscr{C}_b(A; \mathcal{V})$ are vector spaces.*

2. $\big(\mathscr{C}_b(A;\mathcal{V}), \|\cdot\|_\infty\big)$ *is a normed vector space.*

3. *If* $K \subseteq M$ *is compact, then* $\mathscr{C}(K;\mathcal{V}) = \mathscr{C}_b(K;\mathcal{V})$.

Proof. 1 and 2 are trivial, and 3 is concluded by Theorem 4.25. □

Remark 7.45. In general $\|\cdot\|_\infty$ is not a "norm" on $\mathscr{C}(A;\mathcal{V})$. For example, the function $f(x) = \dfrac{1}{x}$ belongs to $\mathscr{C}((0,1);\mathbb{R})$ and $\|f\|_\infty = \infty$. Note that to be a norm $\|f\|_\infty$ has to take values in $\mathbb{R}$, and $\infty \notin \mathbb{R}$.

Proposition 7.46. *Let* (M,d) *be a metric space,* $(\mathcal{V}, \|\cdot\|)$ *be a normed vector space,* $A \subseteq M$ *be a subset, and* $f_k \in \mathscr{C}_b(A;\mathcal{V})$ *for all* $k \in \mathbb{N}$. *Then* $\{f_k\}_{k=1}^\infty$ *converges uniformly on* A *if and only if* $\{f_k\}_{k=1}^\infty$ *converges in* $\big(\mathscr{C}_b(A;\mathcal{V}), \|\cdot\|_\infty\big)$.

Proof. (⇐) Suppose that $\{f_k\}_{k=1}^\infty$ converges in $\big(\mathscr{C}_b(A;\mathcal{V}), \|\cdot\|_\infty\big)$. Then there exists $f \in \big(\mathscr{C}_b(A;\mathcal{V}), \|\cdot\|_\infty\big)$ such that $\lim\limits_{k\to\infty} \|f_k - f\|_\infty = 0$, and by the definition of the sup-norm,

$$\lim_{k\to\infty} \sup_{x\in A} \|f_k(x) - f(x)\| = 0\,.$$

Therefore, $\{f_k\}_{k=1}^\infty$ converges to f uniformly on A.

(⇒) Suppose that $\{f_k\}_{k=1}^\infty$ converges uniformly on A. Then there exists a function $f : A \to \mathcal{V}$ such that

$$\lim_{k\to\infty} \sup_{x\in A} \|f_k(x) - f(x)\| = 0\,.$$

By the definition of the sup-norm, it suffices to show that $f \in \mathscr{C}_b(A;\mathcal{V})$ in order to conclude the proposition. By Theorem 7.8, we obtain that $f \in \mathscr{C}(A;\mathcal{V})$. Moreover, the uniform convergence implies that there exists $N > 0$ such that

$$\|f_k(x) - f(x)\| < 1 \qquad \forall\, k \geqslant N \text{ and } x \in A\,.$$

In particular, the boundedness of f_N provides $M > 0$ such that $\|f_N(x)\| \leqslant M$ for all $x \in A$; thus

$$\|f(x)\| \leqslant \|f_N(x)\| + \|f(x) - f_N(x)\| \leqslant M + 1 \qquad \forall\, x \in A\,.$$

This implies that f is bounded; thus $f \in \mathscr{C}_b(A;\mathcal{V})$. □

Theorem 7.47. *Let* (M,d) *be a metric space,* $(\mathcal{V}, \|\cdot\|)$ *be a normed vector space, and* $A \subseteq M$ *be a subset. If* $(\mathcal{V}, \|\cdot\|)$ *is complete, so is* $\big(\mathscr{C}_b(A;\mathcal{V}), \|\cdot\|_\infty\big)$.

Proof. Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $\big(\mathscr{C}_b(A;\mathcal{V}), \|\cdot\|_\infty\big)$. Then

$$\forall\,\varepsilon > 0, \exists\, N > 0 \ni \|f_k - f_\ell\|_\infty < \varepsilon \quad \text{whenever} \quad k, \ell \geqslant N\,.$$

By the definition of the sup-norm, the statement above implies that

$$\forall\,\varepsilon > 0, \exists\, N > 0 \ni \|f_k(x) - f_\ell(x)\| < \varepsilon \quad \text{whenever} \quad k, \ell \geqslant N \text{ and } x \in A$$

which shows that $\{f_k\}_{k=1}^\infty$ converges uniformly on A because of the Cauchy criterion. Proposition 7.46 then implies that $\{f_k\}_{k=1}^\infty$ converges in $\big(\mathscr{C}_b(A;\mathcal{V}), \|\cdot\|_\infty\big)$. □

Example 7.48. In this example we try to visualize a ball in $\mathscr{C}_b(A, \mathcal{V})$. Note that if $f \in \mathscr{C}_b(A, \mathcal{V})$ and $\varepsilon > 0$. Then

$$B(f, \varepsilon) = \big\{ g \in \mathscr{C}_b(A, \mathcal{V}) \,\big|\, \|f - g\|_\infty < \varepsilon \big\}\,.$$

In particular, if $A = [a, b]$ and $\mathcal{V} = \mathbb{R}$, then $g \in B(f, \varepsilon)$ if and only if $|f(x) - g(x)| < \varepsilon$ for all $x \in [a, b]$ which means that the graph of g lies between the graph of $y = f(x) + \varepsilon$ and $y = f(x) - \varepsilon$.

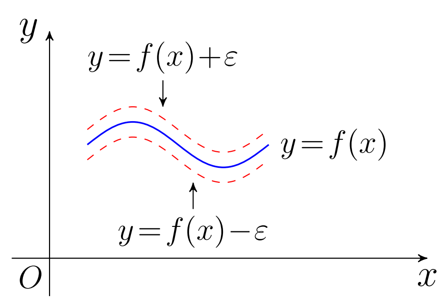


Figure 7.2: $g \in B(f, \varepsilon)$ if the graph of g lies in between the two red dash lines

Example 7.49. The set $B = \big\{ f \in \mathscr{C}([0,1];\mathbb{R}) \,\big|\, f(x) > 0 \text{ for all } x \in [0,1] \big\}$ is open in $\big(\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty\big)$.

Reason: Let $f \in B$ be given. Since $[0,1]$ is compact and f is continuous, by the extreme value theorem there exists $x_0 \in [0,1]$ so that $\inf\limits_{x\in[0,1]} f(x) = f(x_0) > 0$. Take $\varepsilon = \dfrac{f(x_0)}{2}$. Now if g is such that $\|g - f\|_\infty = \sup\limits_{x\in[0,1]} \big|g(x) - f(x)\big| < \varepsilon = \dfrac{f(x_0)}{2}$, we have for any $y \in [0,1]$,

$$\big|g(y) - f(y)\big| \leqslant \sup_{x\in[0,1]} \big|g(x) - f(x)\big| < \frac{f(x_0)}{2}\,;$$

thus

$$g(y) \geqslant f(y) - \frac{f(x_0)}{2} \geqslant f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2} > 0\,.$$

Therefore, $g \in B$; thus $B(f, \varepsilon) \subseteq B$.

Example 7.50. Find the closure of B given in the previous example.

Proof. Claim: $\text{cl}(B) = \{f \in \mathscr{C}([0,1], \mathbb{R}) \mid f(x) \geqslant 0\}$.
Proof of claim: We show that for every $f \in \{f \in \mathscr{C}([0,1], \mathbb{R}) \mid f(x) \geqslant 0\}$, there exists $f_k \in B$ such that $\|f_k - f\|_\infty \to 0$ as $k \to \infty$. Take $f_k(x) = f(x) + \frac{1}{k}$, then $f_k \in B$ ($\because f_k(x) > 0$), and

$$\|f_k - f\|_\infty = \sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| \leqslant \sup_{x\in[0,1]} \frac{1}{k} = \frac{1}{k} \to 0 \text{ as } k \to \infty\,.$$

□

7.5 The Arzelà-Ascoli Theorem

在這一節中，我們將研究一般情況下，連續函數列的逐點收斂與均勻收斂之間的具體差異為何。更具體地說，我們希望能找到一個條件，使得逐點收斂的連續函數列，其均勻收斂性等價於該條件成立。這個條件，刻劃了均勻收斂與逐點收斂的真實差異，而這個特別的條件，也將提供額外（且有效）的判斷法，幫助我們判斷在連續函數空間裡面的集合是否緊緻。

7.5.1 Equi-continuous family of functions

The first part of this section is devoted to the investigation of the difference between the pointwise convergence and the uniform convergence of sequence of continuous functions.

Definition 7.51. Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. A subset $B \subseteq \mathscr{C}_b(A; \mathcal{V})$ is said to be ***equi-continuous***（等度連續）if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \|f(x_1) - f(x_2)\| < \varepsilon \quad \text{whenever } d(x_1, x_2) < \delta,\ x_1, x_2 \in A, \text{ and } f \in B\,.$$

Remark 7.52. 1. If $B \subseteq \mathscr{C}_b(A; \mathcal{V})$ is equi-continuous, and C is a subset of B, then C is also equi-continuous.

2. In an equi-continuous set of functions B, every $f \in B$ is uniformly continuous.

Remark 7.53. For a uniformly continuous function f, let $\delta_f(\varepsilon)$ (we have defined this number in Remark 4.47) denote the largest δ that can be used in the definition of the uniform continuity; that is, $\delta_f(\varepsilon)$ has the property that

$$\|f(x) - f(y)\| < \varepsilon \text{ whenever } d(x,y) < \delta, x, y \in A \quad \Leftrightarrow \quad 0 < \delta \leqslant \delta_f(\varepsilon)\,.$$

Suppose that every element in $B \subseteq \mathscr{C}_b(A;\mathcal{V})$ is uniformly continuous on A. Then B is equi-continuous if and only if $\inf\limits_{f\in B} \delta_f(\varepsilon) > 0$.

Example 7.54. Let $B = \big\{f \in \mathscr{C}_b((0,1);\mathcal{V}) \,\big|\, |f'(x)| \leqslant 1 \text{ for all } x \in (0,1)\big\}$. Then B is equi-continuous (by choosing $\delta = \epsilon$ for any given ϵ, and applying the mean value theorem).

Example 7.55. Let $f_k : [0,1] \to \mathbb{R}$ be a sequence of functions given by

$$f_k(x) = \begin{cases} kx & \text{if } 0 \leqslant x \leqslant \dfrac{1}{k}\,, \\ 2 - kx & \text{if } \dfrac{1}{k} \leqslant x \leqslant \dfrac{2}{k}\,, \\ 0 & \text{if } x \geqslant \dfrac{2}{k}\,, \end{cases}$$

and $B = \{f_k\}_{k=1}^{\infty}$. Then B is not equi-continuous since the largest δ for each k is $\dfrac{\varepsilon}{k}$ which converges to 0.

Lemma 7.56. *Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $K \subseteq M$ be a compact subset. If $B \subseteq \mathscr{C}(K;\mathcal{V})$ is pre-compact, then B is equi-continuous.*

Proof. Suppose the contrary that B is not equi-continuous. Then there exists $\varepsilon > 0$ such that

$$\forall\, k \in \mathbb{N}, \exists\, x_k, y_k \in K \text{ and } f_k \in B \ni d(x_k, y_k) < \frac{1}{k} \text{ but } \|f_k(x_k) - f_k(y_k)\| \geqslant \varepsilon\,.$$

Since B is pre-compact in $\big(\mathscr{C}(K;\mathcal{V}), \|\cdot\|_\infty\big)$ and K is compact in (M,d), there exists a subsequence $\big\{f_{k_j}\big\}_{j=1}^{\infty}$ and $\{x_{k_j}\}_{j=1}^{\infty}$ such that $\big\{f_{k_j}\big\}_{j=1}^{\infty}$ converges uniformly to some function $f \in \big(\mathscr{C}(K;\mathcal{V}), \|\cdot\|_\infty\big)$ and $\{x_{k_j}\}_{j=1}^{\infty}$ converges to some $a \in K$. We must also have $\{y_{k_j}\}_{j=1}^{\infty}$ converges to a since $d(x_{k_j}, y_{k_j}) < \dfrac{1}{k_j}$.

Since f is continuous at a,

$$\exists\, \delta > 0 \ni \|f(x) - f(a)\| < \frac{\varepsilon}{5} \qquad \text{if } x \in B(a,\delta) \cap K.$$

Moreover, since $\{f_{k_j}\}_{j=1}^{\infty}$ converges to f uniformly on K and $x_{k_j}, y_{k_j} \to a$ as $j \to \infty$, there exists $N > 0$ such that

$$\|f_{k_j}(x) - f(x)\| < \frac{\varepsilon}{5} \qquad \text{if } j \geqslant N \text{ and } x \in K$$

and

$$d(x_{k_j}, a) < \delta \quad \text{and} \quad d(y_{k_j}, a) < \delta \qquad \text{if } j \geqslant N\,.$$

As a consequence, for all $j \geqslant N$,

$$\begin{aligned}\varepsilon \leqslant \big\|f_{k_j}(x_{k_j}) - f_{k_j}(y_{k_j})\big\| &\leqslant \big\|f_{k_j}(x_{k_j}) - f(x_{k_j})\big\| + \big\|f(x_{k_j}) - f(a)\big\| \\ &\quad + \big\|f(y_{k_j}) - f(a)\big\| + \big\|f(y_{k_j}) - f_{k_j}(y_{k_j})\big\| < \frac{4\varepsilon}{5}\end{aligned}$$

which is a contradiction. □

Alternative proof of Lemma 7.56. Suppose the contrary that B is not equi-continuous. Then there exists $\varepsilon > 0$ such that

$$\forall\, k \in \mathbb{N}, \exists\, x_k, y_k \in K \text{ and } f_k \in B \ni d(x_k, y_k) < \frac{1}{k} \text{ but } \|f_k(x_k) - f_k(y_k)\| \geqslant \varepsilon\,.$$

Since B is pre-compact in $\big(\mathscr{C}(K; \mathcal{V}), \|\cdot\|_\infty\big)$, there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ converges to some function f in $\big(\mathscr{C}(K; \mathcal{V}), \|\cdot\|_\infty\big)$. By Proposition 7.46, $\{f_{k_j}\}_{j=1}^{\infty}$ converges uniformly to f on K; thus there exists $N_1 > 0$ such that

$$\big\|f_{k_j}(x) - f(x)\big\| < \frac{\varepsilon}{4} \quad \forall\, j \geqslant N_1 \text{ and } x \in K\,.$$

Since $f \in \mathscr{C}(K; \mathcal{V})$, by Theorem 4.49, f is uniformly continuous on K; thus

$$\exists\, \delta > 0 \ni \|f(x) - f(y)\| < \frac{\varepsilon}{4} \quad \text{if } d(x, y) < \delta \text{ and } x, y \in K\,.$$

Let $N = \max\big\{N_1, \big[\frac{1}{\delta}\big] + 1\big\}$, and $j \geqslant N$. Then $d(x_{k_j}, y_{k_j}) < \delta$ and this further implies that

$$\varepsilon \leqslant \big\|f_{k_j}(x_{k_j}) - f_{k_j}(y_{k_j})\big\| \leqslant \big\|f_{k_j}(x_{k_j}) - f(x_{k_j})\big\| + \big\|f(x_{k_j}) - f(y_{k_j})\big\| + \big\|f(y_{k_j}) - f_{k_j}(y_{k_j})\big\| < \frac{3\varepsilon}{4}\,,$$

a contradiction. □

Corollary 7.57. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $K \subseteq M$ be a compact subset. If $\{f_k\}_{k=1}^{\infty}$ converges uniformly on K, then $\{f_k\}_{k=1}^{\infty}$ is equi-continuous.*

Example 7.58. Corollary 7.57 fails to hold if the compactness of K is removed. For example, let $\{f_k\}_{k=1}^\infty$ be a sequence of identical functions $f_k(x) = \dfrac{1}{x}$ on $(0,1)$. Then $\{f_k\}_{k=1}^\infty$ converges uniformly on $(0,1)$ but $\{f_k\}_{k=1}^\infty$ is not equi-continuous since none of f_k is uniformly continuous on $(0,1)$ which violates Remark 7.52.

We have just shown that if $\{f_k\}_{k=1}^\infty$ converges uniformly on a compact set K, then $\{f_k\}_{k=1}^\infty$ must be equi-continuous. The inverse statement, on the other hand, cannot be true. For example, taking $\{f_k\}_{k=1}^\infty$ to be a sequence of constant functions $f_k(x) = k$. Then $\{f_k\}_{k=1}^\infty$ obviously does not converge, not even any subsequence. Therefore, we would like to study under what additional conditions, equi-continuity of a sequence of functions (defined on a compact set K) indeed converges uniformly. The following lemma is an answer to the question.

Lemma 7.59. *Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a Banach space, $K \subseteq M$ be a compact set, and $\{f_k\}_{k=1}^\infty \subseteq \mathscr{C}(K;\mathcal{V})$ be a equi-continuous sequence of functions. If $\{f_k\}_{k=1}^\infty$ converges pointwise on a dense subset E of K (that is, $E \subseteq K \subseteq \text{cl}(E)$), then $\{f_k\}_{k=1}^\infty$ converges uniformly on K.*

Proof. Let $\varepsilon > 0$ be given. By the equi-continuity of $\{f_k\}_{k=1}^\infty$,

$$\exists\, \delta > 0 \ni \|f_k(x) - f_k(y)\| < \frac{\varepsilon}{3} \quad \text{if } d(x,y) < \delta,\ x,y \in K \text{ and } k \in \mathbb{N}\,.$$

Since K is compact, K is totally bounded; thus

$$\exists \{y_1, \cdots, y_m\} \subseteq K \ni K \subseteq \bigcup_{j=1}^m B\big(y_j, \frac{\delta}{2}\big)\,.$$

By the denseness of E in K, for each $j = 1, \cdots, m$, there exists $z_j \in E$ such that $d(z_j, y_j) < \dfrac{\delta}{2}$. Moreover, $B\big(y_j, \frac{\delta}{2}\big) \subseteq B(z_j, \delta)$; thus $K \subseteq \bigcup\limits_{j=1}^m B(z_j,\delta)$. Since $\{f_k\}_{k=1}^\infty$ converges pointwise on E, $\{f_k(z_j)\}_{k=1}^\infty$ converges as $k \to \infty$ for all $j = 1, \cdots, m$. Therefore,

$$\exists\, N_j > 0 \ni \|f_k(z_j) - f_\ell(z_j)\| < \frac{\varepsilon}{3} \qquad \forall\, k, \ell \geqslant N_j\,.$$

Let $N = \max\{N_1, \cdots, N_m\}$, then

$$\|f_k(z_j) - f_\ell(z_j)\| < \frac{\varepsilon}{3} \qquad \forall\, k, \ell \geqslant N \text{ and } j = 1, \cdots, m\,.$$

Now we are in the position of concluding the lemma. If $x \in K$, there exists $z_j \in E$ such that $d(x, z_j) < \delta$; thus if we further assume that $k, \ell \geqslant N$,

$$\|f_k(x) - f_\ell(x)\| \leqslant \|f_k(x) - f_k(z_j)\| + \|f_k(z_j) - f_\ell(z_j)\| + \|f_\ell(z_j) - f_\ell(x)\| < \varepsilon\,.$$

By Proposition 7.7, $\{f_k\}_{k=1}^{\infty}$ converges uniformly on K. □

Remark 7.60. Corollary 7.57 and Lemma 7.59 imply that "a sequence $\{f_k\}_{k=1}^{\infty} \subseteq \mathscr{C}(K;\mathcal{V})$ converges uniformly on K if and only if $\{f_k\}_{k=1}^{\infty}$ is equi-continuous and pointwise convergent (on a dense subset of K)".

7.5.2 Compact sets in $\mathscr{C}(K;\mathcal{V})$

The next subject in this section is to obtain a (useful) criterion of determining the compactness (or pre-compactness) of a subset $B \subseteq \mathscr{C}(K;\mathcal{V})$ which guarantees the existence of a convergent subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ of a given sequence $\{f_k\}_{k=1}^{\infty} \subseteq B$ in $\big(\mathscr{C}(K;\mathcal{V}), \|\cdot\|_\infty\big)$.

Lemma 7.61 (Cantor's Diagonal Process)**.** *Let E be a countable set, $(\mathcal{V}, \|\cdot\|)$ be a Banach space, and $f_k : E \to \mathcal{V}$ be a sequence of functions. Suppose that for each $x \in E$, $\big\{f_k(x)\big\}_{k=1}^{\infty}$ is pre-compact in $\mathcal{V}$. Then there exists a subsequence of $\{f_k\}_{k=1}^{\infty}$ that converges pointwise on E.*

Proof. Since E is countable, $E = \{x_\ell\}_{\ell=1}^{\infty}$.

1. Since $\big\{f_k(x_1)\big\}_{k=1}^{\infty}$ is pre-compact in $(\mathcal{V}, \|\cdot\|)$, there exists a subsequence $\big\{f_{k_j}\big\}_{j=1}^{\infty}$ such that $\big\{f_{k_j}(x_1)\big\}_{j=1}^{\infty}$ converges in $(\mathcal{V}, \|\cdot\|)$.
2. Since $\big\{f_k(x_2)\big\}_{k=1}^{\infty}$ is pre-compact in $(\mathcal{V}, \|\cdot\|)$, the sequence $\big\{f_{k_j}(x_2)\big\}_{j=1}^{\infty} \subseteq \big\{f_k(x_2)\big\}_{k=1}^{\infty}$ has a convergent subsequence $\big\{f_{k_{j_\ell}}(x_2)\big\}_{\ell=1}^{\infty}$.

Continuing this process, we obtain a sequence of sequences S_1, S_2, $\cdots$ such that

1. S_k consists of a subsequence of $\{f_k\}_{k=1}^{\infty}$ which converges at x_k, and
2. $S_k \supseteq S_{k+1}$ for all $k \in \mathbb{N}$.

Let g_k be the k-th element of S_k. Then the sequence $\{g_k\}_{k=1}^{\infty}$ is a subsequence of $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ converges at each point of E. □

The condition that "$\{f_k(x)\}_{k=1}^{\infty}$ is pre-compact in $\mathcal{V}$ for each $x \in E$" in Lemma 7.61 motivates the following

Definition 7.62. Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. A subset $B \subseteq \mathscr{C}_b(A; \mathcal{V})$ is said to be ***pointwise*** $\begin{array}{c}\textbf{\textit{compact}}\\ \textbf{\textit{pre-compact}}\\ \textbf{\textit{bounded}}\end{array}$ if the set $B_x \equiv \{f(x) \,|\, f \in B\}$ is $\begin{array}{c}\text{compact}\\ \text{pre-compact}\\ \text{bounded}\end{array}$ in $(\mathcal{V}, \|\cdot\|)$ for all $x \in A$.

Example 7.63. Let $f_k : [0,1] \to \mathbb{R}$ be given in Example 7.55, and $B = \{f_k\}_{k=1}^{\infty}$. Then B is pointwise compact: for each $x \in [0,1]$, B_x is a finite set since if $f_k(0) = 0$ for all $k \in \mathbb{N}$, while if $x > 0$, $f_k(x) = 0$ for all k large enough which implies that $\# B_x < \infty$.

是時候可以來看 $\mathscr{C}(K; \mathcal{V})$ 裡面的 compact sets 有什麼等價條件了。首先我們先看何時 $B \subseteq \mathscr{C}(K; \mathcal{V})$ 是 compact set。給定一個函數列 $\{f_k\}_{k=1}^{\infty} \subseteq B$，我們想知道能不能找到一個在 sup-norm 下收斂的 subsequence $\{f_{k_j}\}_{j=1}^{\infty}$（即 sequentially compact）。由 Diagonal Process (Lemma 7.61) 知，我們得在 K 中找一個稠密的子集合 E 使得 $\{f_k\}_{k=1}^{\infty}$ 在 E 上是 pointwise pre-compact（這個部份只保證了可以找到 subsequence 逐點收斂），然後加上 Lemma 7.59 的幫助，馬上知道加上 equi-continuity 的條件之後，逐點收斂會變均勻收斂。因此，很自然地我們會要求 B 滿足 pointwise pre-compact 還有 equi-continuous 這兩個條件來證出 B 是 $\mathscr{C}(K; \mathcal{V})$ 中的 compact set。而在一個 compact set K 中能不能找到一個稠密子集合則是由下面這個 Lemma 所提供。

Lemma 7.64. *A compact set K in a metric space (M, d) is separable; that is, there exists a countable subset E of K such that* $\mathrm{cl}(E) = K$.

Proof. Since K is compact, K is totally bounded; thus $\forall\, n \in \mathbb{N}$, there exists $E_n \subseteq K$ such that

$$\# E_n < \infty \quad \text{and} \quad K \subseteq \bigcup_{y \in E_n} B\big(y, \frac{1}{n}\big)\,.$$

Let $E = \bigcup_{n=1}^{\infty} E_n$. Then E is countable by Theorem 0.20. We claim that $\mathrm{cl}(E) = K$.

To see this, first by the definition of the closure of a set, $\mathrm{cl}(E) \subseteq K$ (since K is closed). Let $x \in K$. Since $K \subseteq \bigcup_{y \in E_n} B\big(y, \frac{1}{n}\big)$, $x \in B\big(y, \frac{1}{n}\big)$ for some $y \in E_n$. Therefore, $B\big(x, \frac{1}{n}\big) \cap E \neq \varnothing$ for all $n \in \mathbb{N}$. This implies that $x \in \bar{E} = \mathrm{cl}(E)$. □

Theorem 7.65. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a Banach space, $K \subseteq M$ be a compact set, and $B \subseteq \mathscr{C}(K;\mathcal{V})$ be equi-continuous and pointwise pre-compact. Then B is pre-compact in $\big(\mathscr{C}(K;\mathcal{V}),\|\cdot\|_\infty\big)$.*

Proof. We show that every sequence $\{f_k\}_{k=1}^\infty$ in B has a convergent subsequence. Since K is compact, there is a countable dense subset E of K (Lemma 7.64), and the diagonal process (Lemma 7.61) implies that there exists $\{f_{k_j}\}_{j=1}^\infty$ that converges pointwise on E. Since E is dense in K, by Lemma 7.59 $\{f_{k_j}\}_{j=1}^\infty$ converges uniformly on K; thus $\{f_{k_j}\}_{j=1}^\infty$ converges in $\big(\mathscr{C}(K;\mathcal{V}),\|\cdot\|_\infty\big)$ by Proposition 7.46. □

Remark 7.66. Lemma 7.56 and Theorem 7.65 imply that "a set $B \subseteq \mathscr{C}(K;\mathcal{V})$ is pre-compact if and only if B is equi-continuous and pointwise pre-compact". (That B is pre-compact implies that B is pointwise pre-compact is left as an exercise).

Corollary 7.67. *Let (M,d) be a metric space, and $K \subseteq M$ be a compact set. Assume that $B \subseteq \mathscr{C}(K;\mathbb{R})$ is equi-continuous and pointwise bounded on K. Then every sequence in B has a uniformly convergent subsequence.*

Proof. By the Bolzano-Weierstrass theorem the boundedness of $\{f_k(x)\}_{k=1}^\infty$ implies that $\{f_k(x)\}_{k=1}^\infty$ is pre-compact for all $x \in E$. Therefore, we can apply Theorem 7.65 under the setting $(\mathcal{V},\|\cdot\|) = (\mathbb{R},|\cdot|)$ to conclude the corollary. □

The following theorem provides how compact sets look like in $\mathscr{C}(K;\mathcal{V})$.

Theorem 7.68 (The Arzelà-Ascoli Theorem)**.** *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a Banach space, $K \subseteq M$ be a compact set, and $B \subseteq \mathscr{C}(K;\mathcal{V})$. Then B is compact in $\big(\mathscr{C}(K;\mathcal{V}),\|\cdot\|_\infty\big)$ if and only if B is closed, equi-continuous, and pointwise compact.*

Proof. "$\Leftarrow$" This direction is conclude by Theorem 7.65 and the fact that B is closed.

"$\Rightarrow$" By Lemma 7.56 and the fact that compact sets are closed, it suffices to shows that B is pointwise compact. Let $x \in K$ and $\{f_k(x)\}_{k=1}^\infty$ be a sequence in B_x. Since B is compact, there exists a subsequence $\{f_{k_j}\}_{j=1}^\infty$ that converges uniformly to some function $f \in B$. In particular, $\{f_{k_j}(x)\}_{j=1}^\infty$ converges to $f(x) \in B_x$. In other words, we find a subsequence $\{f_{k_j}(x)\}_{j=1}^\infty$ of $\{f_k(x)\}_{k=1}^\infty$ that converges to a point in B_x. This implies that B_x is sequentially compact; thus B_x is compact. □

Example 7.69. Let $f_k : [0,1] \to \mathbb{R}$ be a sequence of functions such that

(1) $|f_k(x)| \leqslant M_1$ for all $k \in \mathbb{N}$ and $x \in [0,1]$; (2) $|f_k'(x)| \leqslant M_2$ for all $k \in \mathbb{N}$ and $x \in [0,1]$.

Then $\{f_k\}_{k=1}^\infty$ is clearly pointwise bounded. Moreover, by the mean value theorem

$$\big|f_k(x) - f_k(y)\big| \leqslant M_2|x-y| \qquad \forall\, x, y \in [0,1], k \in \mathbb{N}$$

which implies that $\{f_k\}_{k=1}^\infty$ is equi-continuous. Therefore, by Corollary 7.67 there exists a subsequence $\{f_{k_j}\}_{j=1}^\infty$ that converges uniformly on $[0,1]$.

Question: If assumption (1) of Example 7.69 is omitted, can $\{f_k\}_{k=1}^\infty$ still have a convergent subsequence?

Answer: No! Take $f_k(x) = k$, then $\{f_k\}_{k=1}^\infty$ does not have a convergent subsequence (note that f_k is continuous and $f_k'(x) = 0$; that is, Assumption (2) is fulfilled).

Example 7.70. We show that Assumption (1) of Example 7.69 can be replaced by $f_k(0) = 0$ for all $k \in \mathbb{N}$.

Proof. (a) If $f_n(0) = 0$, then by the mean value theorem we have for all $x \in (0,1]$ and $k \in \mathbb{N}$, $f_k(x) - f_k(0) = f_k'(c_k)(x-0)$. Then Assumption (2) of Example 7.69 implies that

$$\big|f_k(x) - f_k(0)\big| = \big|f_k'(c_k)\big|\big|x\big| \leqslant M_2|x| \leqslant M_2$$

which shows that $\{f_k\}_{k=1}^\infty$ is uniformly bounded by M_2.

(b) $\{f_k\}_{k=1}^\infty$ are equi-continuous (same proof as in Example 7.69). □

7.6 The Stone-Weierstrass Theorem

Theorem 7.71 (Weierstrass)**.** *Let $f : [0,1] \to \mathbb{R}$ be continuous. Then for every $\varepsilon > 0$, there exists a polynomial $p : [0,1] \to \mathbb{R}$ such that $\|f - p\|_\infty < \varepsilon$. In other words, the collection of all polynomials is dense in the space $\big(\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty\big)$.*

Proof. For a fixed $n \in \mathbb{N}$, let $r_k(x) = C_k^n x^k (1-x)^{n-k}$. By looking at the partial derivatives with respect to x of the identity $(x+y)^n = \sum\limits_{k=0}^n C_k^n x^k y^{n-k}$, we find that

1. $\sum\limits_{k=0}^n r_k(x) = 1$; 2. $\sum\limits_{k=0}^n k r_k(x) = nx$; 3. $\sum\limits_{k=0}^n k(k-1) r_k(x) = n(n-1)x^2$.

As a consequence,

$$\sum_{k=0}^n (k-nx)^2 r_k(x) = \sum_{k=0}^n \big[k(k-1) + (1-2nx)k + n^2x^2\big] r_k(x) = nx(1-x)\,.$$

Let $\varepsilon > 0$ be given. Since $f : [0,1] \to \mathbb{R}$ is continuous on a compact $[0,1]$, f is uniformly continuous on $[0,1]$ (by Theorem 4.49); thus there exists $\delta > 0$ such that

$$\big|f(x) - f(y)\big| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |x-y| < \delta,\ x, y \in [0,1]\,.$$

Choose $n \in \mathbb{N}$ such that $\dfrac{\|f\|_\infty}{n\delta^2} < \varepsilon$, and define the ***Bernstein polynomial*** $p(x) = \sum\limits_{k=0}^{n} f\big(\frac{k}{n}\big) r_k(x)$. Then p is a polynomial. Moreover, for $x \in [0,1]$ we have

$$\begin{aligned}
\big|f(x) - p(x)\big| &= \Big|\sum_{k=0}^{n}\Big(f(x) - f\big(\frac{k}{n}\big)\Big) r_k(x)\Big| \leqslant \sum_{k=0}^{n}\Big|f(x) - f\big(\frac{k}{n}\big)\Big| r_k(x) \\
&\leqslant \Big(\sum_{|k-nx|<\delta n} + \sum_{|k-nx|\geqslant\delta n}\Big)\Big|f(x) - f\big(\frac{k}{n}\big)\Big| r_k(x) \\
&< \frac{\varepsilon}{2} + 2\|f\|_\infty \sum_{|k-nx|\geqslant\delta n} \frac{(k-nx)^2}{(k-nx)^2} r_k(x) \\
&\leqslant \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{n^2\delta^2} \sum_{k=0}^{n} (k-nx)^2 r_k(x) \leqslant \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{n\delta^2} x(1-x)\,.
\end{aligned}$$

Since $\sup\limits_{x\in[0,1]} x(1-x) = \dfrac{1}{4}$, we find that

$$\|f - p\|_\infty = \sup_{x\in[0,1]} \big|f(x) - p(x)\big| \leqslant \frac{\varepsilon}{2} + \frac{\|f\|_\infty}{2n\delta^2} < \varepsilon\,. \qquad \square$$

Remark 7.72. A polynomial of the form $p_n(x) = \sum\limits_{k=0}^{n} \beta_k r_k(x)$ is called a ***Bernstein polynomial of degree*** n, and the coefficients β_k are called Bernstein coefficients.

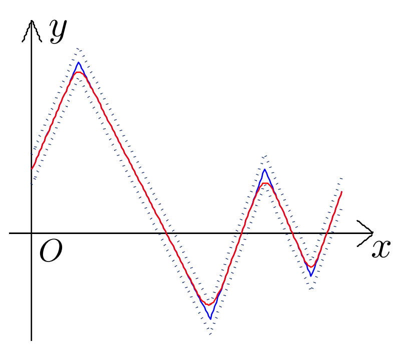


Figure 7.3: Using a Bernstein polynomial of degree 350 (the red curve) to approximate a "saw-tooth" function (the blue curve)

Corollary 7.73. *The collection of polynomials on* $[a,b]$ *is dense in* $\big(\mathscr{C}([a,b];\mathbb{R}), \|\cdot\|_\infty\big)$.

Proof. Let $g \in \mathscr{C}([a,b];\mathbb{R})$. Define $f(x) = g(x(b-a)+a)$. Then $f \in \mathscr{C}([0,1];\mathbb{R})$; thus there exists a sequence $p_n \in \mathscr{C}([0,1];\mathbb{R})$ such that

$$\lim_{n\to\infty} \sup_{y\in[0,1]} \big|f(y) - p_n(y)\big| = 0\,.$$

Therefore, with the change of variable $y = \dfrac{x-a}{b-a}$ (or $x = y(b-a)+a$),

$$\lim_{n\to\infty} \sup_{x\in[a,b]} \Big|g(x) - p_n\big(\frac{x-a}{b-a}\big)\Big| = \lim_{n\to\infty} \sup_{y\in[0,1]} \big|f(y) - p_n(y)\big| = 0\,;$$

thus by the fact $p_n\big(\dfrac{x-a}{b-a}\big)$ is a polynomial in x for all $n \in \mathbb{N}$ we conclude that there exists a sequence of polynomials converging to g uniformly on $[a,b]$. □

Definition 7.74. Let (M,d) be a metric space, and $E \subseteq M$ be a subset. A family $\mathcal{A}$ of real-valued functions defined on E is called an ***algebra*** if

1. $f + g \in \mathcal{A}$ for all $f, g \in \mathcal{A}$;
2. $f \cdot g \in \mathcal{A}$ for all $f, g \in \mathcal{A}$;
3. $\alpha f \in \mathcal{A}$ for all $f \in \mathcal{A}$ and $\alpha \in \mathbb{R}$.

In other words, $\mathcal{A}$ is an algebra if $\mathcal{A}$ is closed under addition, multiplication, and scalar multiplication.

Example 7.75. A function $g : [a,b] \to \mathbb{R}$ is called ***simple*** if we can divide up $[a,b]$ into sub-intervals on which g is constant except perhaps at the end-points. In other words, g is called simple if there is a partition $\mathcal{P} = \{x_0, x_1, \cdots, x_N\}$ of $[a,b]$ such that

$$g(x) = g\big(\frac{x_{i-1}+x_i}{2}\big) \qquad \text{if } \ x \in (x_{i-1}, x_i)\,.$$

Then the collection of all simple functions is an algebra.

Proposition 7.76. *Let (M,d) be a metric space, and $A \subseteq M$ be a subset. If $\mathcal{A} \subseteq \mathscr{C}_b(A;\mathbb{R})$ is an algebra, so is $\bar{\mathcal{A}}$.*

Proof. Let $f, g \in \bar{\mathcal{A}}$. Then there exists $\{f_k\}_{k=1}^\infty, \{g_k\}_{k=1}^\infty \subseteq \mathcal{A}$ such that $\{f_k\}_{k=1}^\infty$ converges uniformly to f on A, and $\{g_k\}_{k=1}^\infty$ converges uniformly to g on A. Since $\mathcal{A}$ is an algebra, $f_k + g_k, f_k \cdot g_k$ and αf_k belong to $\mathcal{A}$ for all $k \in \mathbb{N}$. By Theorem 2.48 and Proposition 7.46, the limit of $\{f_k + g_k\}_{k=1}^\infty$ and $\{\alpha f_k\}_{k=1}^\infty$ belong to $\bar{\mathcal{A}}$ which implies that $f+g$ and αf belong to $\bar{\mathcal{A}}$. Moreover,

$$\|f_k \cdot g_k - f \cdot g\|_\infty \leqslant \|f_k - f\|_\infty \|g_k\|_\infty + \|f\|_\infty \|g_k - g\|_\infty$$

which converges to 0 as $k \to \infty$; thus $f \cdot g$ is the limit of $\{f_k \cdot g_k\}_{k=1}^\infty$ so that $f \cdot g \in \bar{\mathcal{A}}$. Therefore, $\bar{\mathcal{A}}$ is an algebra. □

Corollary 7.77. *Let (M,d) be a metric space, $K \subseteq M$ be a compact set, and $\mathcal{A} \subseteq \mathscr{C}(K;\mathbb{R})$ be an algebra.*

1. *If $f \in \bar{\mathcal{A}}$, so is $|f|$.*
2. *If $f_1, \cdots, f_n \in \bar{\mathcal{A}}$, then* $\max\{f_1, \cdots, f_n\} \in \bar{\mathcal{A}}$ *and* $\min\{f_1, \cdots, f_n\} \in \bar{\mathcal{A}}$, *where*

$$\max\{f_1, \cdots, f_n\}(x) = \max\{f_1(x), \cdots, f_n(x)\},$$
$$\min\{f_1, \cdots, f_n\}(x) = \min\{f_1(x), \cdots, f_n(x)\}.$$

Proof. 1. Let $f \in \bar{\mathcal{A}}$. Then f is bounded so that $\mathrm{M} = \sup\limits_{x \in K} |f(x)| \in \mathbb{R}$. By Corollary 7.73, there exists a sequence of polynomial $\{p_n\}_{n=1}^\infty$ such that $\lim\limits_{n\to\infty} \sup\limits_{y \in [-M,M]} \big|p_n(y) - |y|\big| = 0$. Since $\mathcal{A}$ is an algebra, $\bar{\mathcal{A}}$ is also an algebra; thus $g_n \equiv p_n(f) \in \bar{\mathcal{A}}$. Moreover,

$$\sup_{x \in K} \big|g_n(x) - |f(x)|\big| = \sup_{x \in K} \big|p_n(f(x)) - |f(x)|\big| \leqslant \sup_{y \in [-M,M]} \big|p_n(y) - |y|\big|$$

which shows that $\{g_n\}_{n=1}^\infty$ converges uniformly to $|f|$ on K; thus $|f| \in \bar{\bar{\mathcal{A}}} = \bar{\mathcal{A}}$.

2. It suffices to show that $\max\{f,g\}$ and $\min\{f,g\}$ both belong to $\bar{\mathcal{A}}$ since

$$\max\{f_1, \cdots, f_n\} = \max\{\max\{f_1, \cdots, f_{n-1}\}, f_n\},$$
$$\min\{f_1, \cdots, f_n\} = \min\{\min\{f_1, \cdots, f_{n-1}\}, f_n\}.$$

Nevertheless, note that $\max\{f,g\} = \dfrac{f+g}{2} + \dfrac{|f-g|}{2}$ and $\min\{f,g\} = \dfrac{f+g}{2} - \dfrac{|f-g|}{2}$, we find that if $f, g \in \bar{\mathcal{A}}$ then $\max\{f,g\} \in \bar{\mathcal{A}}$ and $\min\{f,g\} \in \bar{\mathcal{A}}$. □

Definition 7.78. Let (M,d) be a metric space, and $E \subseteq M$ be a subset. A family $\mathscr{F}$ of real-valued functions defined on E is said to

1. ***separate points*** on E if for all $x, y \in E$ and $x \neq y$, there exists $f \in \mathscr{F}$ such that $f(x) \neq f(y)$.
2. ***vanish at no point*** of E if for each $x \in E$ there is $f \in \mathscr{F}$ such that $f(x) \neq 0$.

Example 7.79. Let $\mathscr{P}([a,b])$ denote the collection of polynomials defined on $[a,b]$ is an algebra. Moreover, $\mathscr{P}([a,b])$ separates points on $[a,b]$ since $p(x) = x$ does the separation, and $\mathscr{P}([a,b])$ vanishes at no point of $[a,b]$.

Example 7.80. Let $\mathscr{P}_{\text{even}}([a,b])$ denote the collection of all polynomials $p(x)$ of the form

$$p(x) = \sum_{k=0}^{n} a_k x^{2k} = a_n x^{2n} + a_{n-1} x^{2n-2} + \cdots + a_0 \,.$$

Then $\mathscr{P}_{\text{even}}([a,b])$ is an algebra. Moreover, $\mathscr{P}_{\text{even}}([a,b])$ vanishes at no point of $[a,b]$ since the constant functions are polynomials (since constant functions belongs to $\mathscr{P}([a,b])$. However, if $ab < 0$, $\mathscr{P}_{\text{even}}([a,b])$ does not separate points on $[a,b]$. On the other hand, if $ab \geqslant 0$, then $\mathscr{P}_{\text{even}}([a,b])$ separates points on $[a,b]$ since $p(x) = x^2$ does the job.

Lemma 7.81. *Let (M,d) be a metric space, and $E \subseteq M$ be a subset. Suppose that $\mathcal{A} \subseteq \mathscr{C}_b(E;\mathbb{R})$ is an algebra, $\mathcal{A}$ separates points on E, and $\mathcal{A}$ vanishes at no point of E. Then for all $x_1, x_2 \in E$, $x_1 \neq x_2$, and $c_1, c_2 \in \mathbb{R}$ (c_1, c_2 could be the same), there exists $f \in \mathcal{A}$ such that $f(x_1) = c_1$ and $f(x_2) = c_2$.*

Proof. Since $\mathcal{A}$ separates points on E, there exists $g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$, and since $\mathcal{A}$ vanishes at no point of E, there exists $h, k \in \mathcal{A}$ such that $h(x_1) \neq 0$ and $k(x_2) \neq 0$. Then

$$f(x) = c_1 \frac{\big[g(x) - g(x_2)\big] h(x)}{\big[g(x_1) - g(x_2)\big] h(x_1)} + c_2 \frac{\big[g(x) - g(x_1)\big] k(x)}{\big[g(x_2) - g(x_1)\big] k(x_2)}$$

has the desired property. □

Theorem 7.82 (Stone)**.** *Let (M,d) be a metric space, $K \subseteq M$ be a compact set, and $\mathcal{A} \subseteq \mathscr{C}(K;\mathbb{R})$ satisfying*

1. *$\mathcal{A}$ is an algebra.* 2. *$\mathcal{A}$ separates points on K.* 3. *$\mathcal{A}$ vanishes at no point of K.*

Then $\mathcal{A}$ is dense in $\mathscr{C}(K;\mathbb{R})$; that is, for every $f \in \mathscr{C}(K;\mathbb{R})$ and $\varepsilon > 0$, there exists $g \in \mathcal{A}$ such that $\|f - g\|_\infty < \varepsilon$.

Proof. We first show that for any given $f \in \mathscr{C}(K;\mathbb{R})$, $a \in K$ and $\varepsilon > 0$, there exists a function $g_a \in \bar{\mathcal{A}}$ such that

$$g_a(a) = f(a) \qquad \text{and} \qquad g_a(x) > f(x) - \varepsilon \quad \forall\, x \in K \,. \tag{7.6.1}$$

Let $f \in \mathscr{C}(K;\mathbb{R})$, $a \in K$ and $\varepsilon > 0$ be given. Since $\mathcal{A}$ is an algebra, so is $\bar{\mathcal{A}}$; thus Lemma 7.81 implies that there exists $h_b \in \bar{\mathcal{A}}$ such that $h_b(a) = f(a)$ and $h_b(b) = f(b)$. Note that every function in $\bar{\mathcal{A}}$ is continuous (by Theorem 7.8); thus the continuity of h_b provides $\delta = \delta_b > 0$ such that

$$h_b(x) > f(x) - \varepsilon \qquad \forall\, x \in \big[B\big(b, \delta_b\big) \cup B\big(a, \delta_b\big)\big] \cap K\,.$$

Let $U_b = B\big(b, \delta_b\big) \cup B\big(a, \delta_b\big)$. Then U_b is open. Since $K \subseteq \bigcup\limits_{\substack{b \in K \\ b \neq a}} U_b$ and K is compact, there exists a finite set $\{b_1, \cdots, b_m\} \subseteq K \backslash \{a\}$ such that $K \subseteq \bigcup\limits_{j=1}^{n} U_{b_j}$. Define $g_a = \max\big\{h_{b_1}, \cdots h_{b_m}\big\}$. Then $g_a(a) = f(a)$, and Corollary 7.77 implies that $g_a \in \bar{\mathcal{A}}$. Moreover, if $x \in K$, $x \in U_{b_j}$ for some j; thus

$$g_a(x) \geqslant h_{b_j}(x) > f(x) - \varepsilon$$

which implies that g satisfies (7.6.1).

Let $f \in \mathscr{C}(K;\mathbb{R})$ and $\varepsilon > 0$ be given. For any $a \in K$, let $g_a \in \bar{\mathcal{A}}$ be a function satisfying

$$g_a(a) = f(a) \quad \text{and} \quad g_a(x) > f(x) - \frac{\varepsilon}{2} \qquad \forall\, x \in K\,. \tag{7.6.2}$$

By the continuity of g_a, there exists $\delta = \delta_a > 0$ such that

$$g_a(x) < f(x) + \frac{\varepsilon}{2} \qquad \forall\, x \in B(a, \delta_a) \cap K\,. \tag{7.6.3}$$

By the compactness of K, there exists $\{a_1, \cdots, a_n\} \subseteq K$ such that

$$K \subseteq \bigcup_{j=1}^{m} B\big(a_j, \delta_{a_j}\big)\,.$$

Define $h = \min\big\{g_{a_1}, \cdots, g_{a_n}\big\}$. Corollary 7.77 implies that $h \in \bar{\mathcal{A}}$, and (7.6.2) shows that

$$h(x) > f(x) - \frac{\varepsilon}{2} \qquad \forall\, x \in K.$$

Moreover, if $x \in K$, there exists j such that $x \in B\big(a_j, \delta_{a_j}\big)$ and (7.6.3) further shows that

$$h(x) \leqslant g_{a_j}(x) < f(x) + \frac{\varepsilon}{2}\,;$$

thus

$$h(x) < f(x) + \frac{\varepsilon}{2} \qquad \forall\, x \in K.$$

Therefore, we establish the existence of $h \in \bar{\mathcal{A}}$ such that

$$\big|h(x) - f(x)\big| < \frac{\varepsilon}{2} \qquad \forall\, x \in K\,.$$

On the other hand, since $h \in \bar{\mathcal{A}}$, there exists $p \in \mathcal{A}$ such that

$$\left|p(x) - h(x)\right| < \frac{\varepsilon}{2} \qquad \forall\, x \in K\,;$$

thus

$$\left|p(x) - f(x)\right| \leqslant \left|p(x) - h(x)\right| + \left|h(x) - f(x)\right| < \varepsilon \qquad \forall\, x \in K$$

which concludes the theorem. □

Example 7.83. Let $K = [-1,1] \times [-1,1] \subseteq \mathbb{R}^2$. Consider the set $\mathscr{P}(K)$ of all polynomials $p(x,y)$ in two variables $(x,y) \in K$. Then $\mathscr{P}(K)$ is dense in $\mathscr{C}(K;\mathbb{R})$.

Reason: Since K is compact, and $\mathscr{P}(K)$ is definitely an algebra and the constant function $p(x,y) = 1 \in \mathscr{P}(K)$ vanishes at no point of K, it suffices to show that $\mathscr{P}(K)$ separates points. Let (a_1,b_1) and (a_2,b_2) be two different points in K. Then the polynomial

$$p(x,y) = (x-a_1)^2 + (y-b_1)^2$$

has the property that $p(a_1,b_1) \neq p(a_2,b_2)$. Therefore, $\mathscr{P}(K)$ separates points in K,

Example 7.84. Consider $\mathscr{P}_{\rm even}([0,1]) = \left\{p(x) = \sum\limits_{k=0}^{n} a_k x^{2k} \,\middle|\, a_k \in \mathbb{R}\right\}$ (see Example 7.80). Then $\mathcal{A} = \mathscr{P}_{\rm even}([0,1])$ satisfies all the conditions in the Stone theorem, so $\mathscr{P}_{\rm even}([0,1])$ is dense in $\mathscr{C}([0,1];\mathbb{R})$.

On the other hand, if $K = [-1,1]$, then $\mathscr{P}_{\rm even}([-1,1])$ does not separate points on K since if $p \in \mathscr{P}_{\rm even}([-1,1])$, $p(x) = p(-x)$; thus the Stone theorem cannot be applied to conclude the denseness of $\mathscr{P}_{\rm even}([-1,1])$ in $\mathscr{C}([-,1];\mathbb{R})$. In fact, $\mathscr{P}_{\rm even}([-1,1])$ is not dense in $\mathscr{C}([-1,1];\mathbb{R})$ since polynomials in $\mathscr{P}_{\rm even}([-1,1])$ are all even functions and $f(x) = x$ cannot be approximated by even functions.

Corollary 7.85. *Let $\mathscr{C}(\mathbb{T})$ be the collection of all 2π-periodic continuous real-valued functions, and $\mathscr{P}_n(\mathbb{T})$ be the collection of all real-valued trigonometric polynomials of degree n; that is,*

$$\mathscr{P}_n(\mathbb{T}) = \left\{\frac{c_0}{2} + \sum_{k=1}^{n} c_k \cos kx + s_k \sin kx \,\middle|\, \{c_k\}_{k=0}^n, \{s_k\}_{k=1}^n \subseteq \mathbb{R}\right\}.$$

Then $\mathscr{P}(\mathbb{T}) \equiv \bigcup\limits_{n=0}^{\infty} \mathscr{P}_n(\mathbb{T})$ is dense in $\mathscr{C}(\mathbb{T})$. In other words, if $f \in \mathscr{C}(\mathbb{T})$ and $\varepsilon > 0$ is given, there exists $p \in \mathscr{P}(\mathbb{T})$ such that

$$\left|f(x) - p(x)\right| < \varepsilon \qquad \forall\, x \in \mathbb{R}\,.$$

Proof. We note that $\mathscr{C}(\mathbb{T})$ can be viewed as the collection of all continuous functions defined on the unit circle $\mathbb{S}^1$ in the sense that every $f \in \mathscr{C}(\mathbb{T})$ corresponds to a unique $F \in \mathscr{C}(\mathbb{S}^1;\mathbb{R})$ such that $f(x) = F(\cos x, \sin x)$, and vice versa. Since $\mathbb{S}^1 \subseteq [-1,1] \times [-1,1]$ is compact, Example 7.83 provides that $\mathscr{P}(\mathbb{S}^1)$, the collection of all polynomials defined on $\mathbb{S}^1$, is an algebra that separates points of $\mathbb{S}^1$ and vanishes at no point on $\mathbb{S}^1$. The Stone-Weierstrass Theorem then implies that there exists $P \in \mathscr{P}(\mathbb{S}^1)$ such that

$$\big|F(x,y) - P(x,y)\big| < \varepsilon \qquad \forall\,(x,y) \in \mathbb{S}^1 \text{ (that is, } x^2+y^2=1).$$

Let $p(x) = P(\cos x, \sin x)$. Note that

$$\begin{aligned}\cos^n x &= \Big(\frac{e^{ix}+e^{-ix}}{2}\Big)^n = \sum_{k=0}^{n} \frac{1}{2^n} C^n_k e^{ikx} e^{-i(n-k)x} = \sum_{k=0}^{n} \frac{1}{2^n} C^n_k e^{i(2k-n)x} \\ &= \sum_{k=0}^{n} \frac{1}{2^n} C^n_k \big(\cos(2k-n)x + i\sin(2k-n)x\big) = \sum_{k=0}^{n} \frac{1}{2^n} C^n_k \cos(2k-n)x \in \mathscr{P}_n(\mathbb{T}),\end{aligned}$$

and similarly, $\sin^m x \in \mathscr{P}_m(\mathbb{T})$. Therefore, if $P(x,y) = \sum\limits_{k,\ell=0}^{n} a_{k,\ell} x^k y^\ell$, then $P(\cos x, \sin x) \in \mathscr{P}_{2n}(\mathbb{T})$ because of the product-to-sum formulas

$$\begin{aligned}\cos\theta\cos\varphi &= \frac{1}{2}\Big[\cos(\theta-\varphi) + \cos(\theta+\varphi)\Big],\\ \sin\theta\cos\varphi &= \frac{1}{2}\Big[\sin(\theta+\varphi) + \sin(\theta-\varphi)\Big],\\ \sin\theta\sin\varphi &= \frac{1}{2}\Big[\cos(\theta-\varphi) - \cos(\theta+\varphi)\Big].\end{aligned}$$

As a consequence, we conclude that

$$\big|f(x) - p(x)\big| = \big|F(\cos x, \sin x) - P(\cos x, \sin x)\big| < \varepsilon \qquad \forall\, x \in \mathbb{R}. \qquad \square$$

7.7 Exercises

§7.1 Pointwise and Uniform Convergence

Problem 7.1. Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f_k : A \to N$ be a sequence of functions such that for some function $f : A \to N$, we have that for all $x \in A$, if $\{x_k\}_{k=1}^{\infty} \subseteq A$ and $x_k \to x$ as $k \to \infty$, then

$$\lim_{k\to\infty} f_k(x_k) = f(x).$$

Show that

1. $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f.

2. If $\{f_{k_j}\}_{j=1}^{\infty}$ is a subsequence of $\{f_k\}_{k=1}^{\infty}$, and $\{x_j\}_{j=1}^{\infty} \subseteq A$ is a convergent sequence satisfying that $\lim\limits_{j\to\infty} x_j = x$, then

$$\lim_{j\to\infty} f_{k_j}(x_j) = f(x)\,.$$

3. Show that if in addition A is compact and f is continuous on A, then $\{f_k\}_{k=1}^{\infty}$ converges uniformly f on A.

Remark. Using the inequality

$$\rho\big(f_k(x_k), f(x)\big) \leqslant \rho\big(f(x_k), f(x)\big) + \sup_{x\in A} \rho\big(f_k(x), f(x)\big)\,,$$

we find that if $\{f_k\}_{k=1}^{\infty}$ converges uniformly to a continuous function f, then $\lim\limits_{k\to\infty} f_k(x_k) = f(x)$ as long as $\lim\limits_{k\to\infty} x_k = x$. Together with the conclusion in 3, we conclude that

> Let (M,d), (N,ρ) be metric spaces, $K \subseteq M$ be a compact set, $f_k : K \to N$ be a function for each $k \in \mathbb{N}$, and $f : K \to N$ be continuous. The sequence $\{f_k\}_{k=1}$ converges uniformly to f if and only if $\lim\limits_{k\to\infty} f_k(x_k) = f(x)$ whenever sequence $\{x_k\}_{k=1}^{\infty} \subseteq K$ converges to x.

Problem 7.2. Let (M,d) be a metric space, $A \subseteq M$, (N,ρ) be a complete metric space, and $f_k : A \to N$ be a sequence of functions (not necessary continuous) which converges uniformly on A. Suppose that $a \in \text{cl}(A)$ and

$$\lim_{x\to a} f_k(x) = L_k$$

exists for all $k \in \mathbb{N}$. Show that $\{L_k\}_{k=1}^{\infty}$ converges, and

$$\lim_{x\to a}\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty}\lim_{x\to a} f_k(x)\,.$$

Problem 7.3. Prove the Dini theorem:

> Let K be a compact set, and $f_k : K \to \mathbb{R}$ be continuous for all $k \in \mathbb{N}$ such that $\{f_k\}_{k=1}$ converges pointwise to a continuous function $f : K \to \mathbb{R}$. Suppose that $f_k \leqslant f_{k+1}$ for all $k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on K.

Hint: Mimic the proof of showing that $\{c_k\}_{k=1}^{\infty}$ converges to 0 in Lemma 6.64.

Problem 7.4. Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f_k : A \to N$ be uniformly continuous functions, and $\{f_k\}_{k=1}^{\infty}$ converges uniformly to $f : A \to N$ on A. Show that f is uniformly continuous on A.

Problem 7.5. Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a norm space, $B \subseteq A \subseteq M$, $f_k : A \to \mathcal{V}$ be bounded for each $k \in \mathbb{N}$, and $\{g_n\}_{n=1}^{\infty}$ be the Cesàro mean of $\{f_k\}_{k=1}^{\infty}$; that is, $g_n = \dfrac{1}{n}\sum\limits_{k=1}^{n} f_k$. Show that if $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on B, then $\{g_n\}_{n=1}^{\infty}$ converges uniformly to f on B.

Problem 7.6. Complete the following.

1. Suppose that $f_k, f, g : [0,\infty) \to \mathbb{R}$ are functions such that

    (a) $\forall\, R > 0$, f_k and g are Riemann integrable on $[0,R]$;

    (b) $|f_k(x)| \leqslant g(x)$ for all $k \in \mathbb{N}$ and $x \in [0,\infty)$;

    (c) $\forall\, R > 0$, $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on $[0,R]$;

    (d) $\displaystyle\int_0^{\infty} g(x)dx \equiv \lim_{R\to\infty}\int_0^R g(x)dx < \infty$.

    Show that $\displaystyle\lim_{k\to\infty}\int_0^{\infty} f_k(x)dx = \int_0^{\infty} f(x)dx$; that is,

    $$\lim_{k\to\infty}\lim_{R\to\infty}\int_0^R f_k(x)dx = \lim_{R\to\infty}\lim_{k\to\infty}\int_0^R f_k(x)dx\,.$$

2. Let $f_k(x)$ be given by $f_k(x) = \begin{cases} 1 & \text{if } k-1 \leqslant x < k\,, \\ 0 & \text{otherwise.}\end{cases}$ Find the (pointwise) limit f of the sequence $\{f_k\}_{k=1}^{\infty}$, and check whether $\displaystyle\lim_{k\to\infty}\int_0^{\infty} f_k(x)dx = \int_0^{\infty} f(x)dx$ or not. Briefly explain why one can or cannot apply 1.

3. Let $f_k : [0,\infty) \to \mathbb{R}$ be given by $f_k(x) = \dfrac{x}{1+kx^4}$. Find $\displaystyle\lim_{k\to\infty}\int_0^{\infty} f_k(x)dx$.

§7.2 Series of Functions and The Weierstrass M-Test

Problem 7.7. Show that the series

$$\sum_{k=1}^{\infty}(-1)^k\frac{x^2+k}{k^2}$$

converges uniformly on every bounded interval.

Problem 7.8. Consider the function

$$f(x)=\sum_{k=1}^{\infty}\frac{1}{1+k^2x}\,.$$

On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? If f bounded?

Problem 7.9. Determine which of the following real series $\sum\limits_{k=1}^{\infty} g_k$ converge (pointwise or uniformly). Check the continuity of the limit in each case.

1. $g_k(x)=\begin{cases} 0 & \text{if } x\leqslant k\,,\\ (-1)^k & \text{if } x>k\,.\end{cases}$

2. $g_k(x)=\begin{cases} \dfrac{1}{k^2} & \text{if } |x|\leqslant k\,,\\ \dfrac{1}{x^2} & \text{if } |x|>k\,.\end{cases}$

3. $g_k(x)=\dfrac{(-1)^k}{\sqrt{k}}\cos(kx)$ on $\mathbb{R}$.

§7.3 Integration and Differentiation of Series

Problem 7.10. In the following series of functions defined on $\mathbb{R}$, find its domain of convergence (classify it into domain of absolute and conditional convergence). If the series is a power series, find its radius of convergence. Also discuss whether the series is uniformly convergent in every compact subsets of its domain of convergence. Determine which series can be differentiated or integrated term by term in its domain of convergence.

(1) $\sum\limits_{k=1}^{\infty}\dfrac{x}{k^\alpha+k^\beta x^2}$, $\alpha\geqslant 0$, $\beta>0$;

(2) $\sum\limits_{k=1}^{\infty}\dfrac{1}{2^k}\sqrt{1-x^{2k}}$;

(3) $\sum\limits_{k=1}^{\infty}\dfrac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots(2k)}\Big(1+\dfrac{1}{2}+\cdots+\dfrac{1}{k}\Big)x^{2k}$;

(4) $\sum\limits_{k=1}^{\infty}\dfrac{(-1)^{k-1}}{k\log(k+1)}x^{k!}$;

(5) $\sum_{k=1}^{\infty} a_k x^k$, where $\{a_k\}_{k=1}^{\infty}$ is defined by the recursive relation $a_k = 3a_{k-1} - 2a_{k-2}$ for $k \geqslant 2$, and $a_0 = 1$, $a_1 = 2$.

Also find the sum of the series in (5).

Problem 7.11. In this problem we investigate the differentiability of a complex power series. This requires a new proof of $\frac{d}{dx}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=1}^{\infty} k a_k x^{k-1}$ instead of making use of Theorem 7.11.

Let $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ be a real sequence, and $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a (real) power series with radius of convergence $R > 0$. Let $s_n(x) = \sum_{k=0}^{n} a_k x^k$ be the n-th partial sum, $R_n(x) = f(x) - s_n(x)$, and $g(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$. For $x, x_0 \in (-\rho, \rho) \subsetneq (-R, R)$, where $x \neq x_0$, write

$$\frac{f(x)-f(x_0)}{x-x_0} - g(x) = \frac{s_n(x)-s_n(x_0)}{x-x_0} - s_n'(x_0) + \big(s_n'(x_0) - g(x_0)\big) + \frac{R_n(x)-R_n(x_0)}{x-x_0}\,. \tag{7.7.1}$$

1. Show that
$$\Big|\frac{R_n(x)-R_n(x_0)}{x-x_0}\Big| \leqslant \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1}\,,$$
and use the inequality above to show that $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = g(x_0)$.

2. Generalize the conclusion to complex power series; that is, show that if $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{C}$ and the power series $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > 0$, then
$$\frac{d}{dz}\sum_{k=0}^{\infty} a_k z^k = \sum_{k=1}^{\infty} k a_k z^{k-1} \qquad \forall\, |z| < R\,.$$
Assume that you have known $\frac{d}{dz}\sum_{k=0}^{n} a_k z^k = \sum_{k=1}^{n} k a_k z^{k-1}$ for all $n \in \mathbb{N} \cup \{0\}$ (this can be proved using the definition of differentiability of functions with values in normed vector spaces provided in Chapter 5).

Problem 7.12. Suppose that the series $\sum_{n=0}^{\infty} a_n = 0$, and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $-1 < x \leqslant 1$. Show that f is continuous at $x = 1$ by complete the following.

1. Write $s_n = a_0 + a_1 + \cdots + a_n$ and $S_n(x) = a_0 + a_1x + \cdots + a_nx^n$. Show that
$$S_n(x) = (1-x)(s_0 + s_1x + \cdots + s_{n-1}x^{n-1}) + s_nx^n$$
and $f(x) = (1-x)\sum\limits_{n=0}^{\infty} s_nx^n$.

2. Using the representation of f from above to conclude that $\lim\limits_{x\to 1^-} f(x) = 0$.

3. What if $\sum\limits_{n=0}^{\infty} a_n$ is convergent but not zero?

Problem 7.13. Construct the function $g(x)$ by letting $g(x) = |x|$ if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and extending g so that it becomes periodic (with period 1). Define
$$f(x) = \sum_{k=1}^{\infty} \frac{g(4^{k-1}x)}{4^{k-1}}\,.$$

1. Use the Weierstrass M-test to show that f is continuous on $\mathbb{R}$.

2. Prove that f is differentiable at no point.

(So there exists a continuous which is nowhere differentiable!)

Hint: Google Blancmange function!

§7.4 The Space of Continuous Functions

Problem 7.14. Let $\delta : (\mathscr{C}([-1,1];\mathbb{R}), \|\cdot\|_\infty) \to \mathbb{R}$ be defined by $\delta(f) = f(0)$. Show that δ is linear and uniformly continuous.

Problem 7.15. Let (M, d) be a metric space, and $K \subseteq M$ be a compact subset.

1. Show that the set $U = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a < f(x) < b \text{ for all } x \in K\big\}$ is open in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

2. Show that the set $F = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a \leqslant f(x) \leqslant b \text{ for all } x \in K\big\}$ is closed in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

3. Let $A \subseteq M$ be a subset, not necessarily compact. Prove or disprove that the set $B = \big\{f \in \mathscr{C}_b(A;\mathbb{R}) \,\big|\, f(x) > 0 \text{ for all } x \in A\big\}$ is open in $\big(\mathscr{C}_b(A;\mathbb{R}), \|\cdot\|_\infty\big)$.

§7.5 The Arzelà-Ascoli Theorem

Problem 7.16. Which of the following set B of continuous functions are equi-continuous in the metric space M? Are the closure $\bar{B}$ compact in M?

1. $B = \{ \sin kx \,|\, k = 0,1,2,\cdots \}$, $M = \mathscr{C}([0,\pi];\mathbb{R})$.
2. $B = \{ \sin\sqrt{x + 4k^2\pi^2} \,|\, k = 0,1,2,\cdots \}$, $M = \mathscr{C}_b([0,\infty);\mathbb{R})$.
3. $B = \Big\{ \dfrac{x^2}{x^2 + (1-kx)^2} \,\Big|\, k = 0,1,2,\cdots \Big\}$, $M = \mathscr{C}([0,1];\mathbb{R})$.
4. $B = \{ (k+1)x^k(1-x) \,|\, k \in \mathbb{N} \}$, $M = \mathscr{C}([0,1];\mathbb{R})$.

Problem 7.17. Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed space, and $A \subseteq M$ be a subset. Suppose that $B \subseteq \mathscr{C}_b(A;\mathcal{V})$ be equi-continuous. Prove or disprove that $\text{cl}(B)$ is equi-continuous.

Problem 7.18. Let $f_k : [a,b] \to \mathbb{R}$ be a sequence of differentiable functions such that $f_k(a)$ is bounded and $|f_k'(x)| \leqslant M$ for all $x \in [a,b]$ and $k \in \mathbb{N}$. Show that $\{f_k\}_{k=1}^{\infty}$ contains an uniformly convergent subsequence. Must the limit function differentiable?

Problem 7.19. Let $\mathscr{C}^{0,\alpha}([0,1];\mathbb{R})$ denote the "space"

$$\mathscr{C}^{0,\alpha}([0,1];\mathbb{R}) \equiv \Big\{ f \in \mathscr{C}([0,1];\mathbb{R}) \,\Big|\, \sup_{x,y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty \Big\},$$

where $\alpha \in (0,1]$. For each $f \in \mathscr{C}^{0,\alpha}([0,1];\mathbb{R})$, define

$$\|f\|_{\mathscr{C}^{0,\alpha}} = \sup_{x\in[0,1]} |f(x)| + \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

1. Show that $\big(\mathscr{C}^{0,\alpha}([0,1];\mathbb{R}), \|\cdot\|_{\mathscr{C}^{0,\alpha}}\big)$ is a complete normed space.
2. Show that the set $B = \{ f \in \mathscr{C}([0,1];\mathbb{R}) \,|\, \|f\|_{\mathscr{C}^{0,\alpha}} < 1 \}$ is equi-continuous.
3. Show that $\text{cl}(B)$ is compact in $\big(\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty\big)$.

Problem 7.20. Given $f : \mathbb{R} \to \mathbb{R}$ a continuous periodic function of period 1; that is, $f(x+1) = f(x)$ for all $x \in \mathbb{R}$, and $x_1,\cdots,x_m \in [0,1]$ arbitrary m points, define a new function

$$I(f;x_1,\cdots,x_m)(x) = \frac{1}{m}\big[f(x+x_1) + \cdots + f(x+x_m)\big] \qquad \forall\, x \in \mathbb{R}.$$

Prove that the set

$$B = \{I(f; x_1, \cdots, x_m) \,|\, x_1, \cdots, x_m \in [0,1], m \in \mathbb{N}\}$$

is uniformly bounded and equi-continuous in the space $\mathscr{C}([0,1];\mathbb{R})$. Moreover, show that the derived set $B' = \left\{\int_0^1 f(x)dx\right\}$; that is, the derived set of B consists of one single function which is a constant function $y = \int_0^1 f(x)dx$.

Problem 7.21. Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a Banach space, $K \subseteq M$ be compact, and $\{f_k\}_{k=1}^\infty \subseteq \mathscr{C}(K;\mathcal{V})$ be a sequence of continuous functions. Suppose that for all $x \in K$, if $\{x_k\}_{k=1}^\infty, \{y_k\}_{k=1}^\infty \subseteq K$ and $\lim\limits_{k\to\infty} x_k = \lim\limits_{k\to\infty} y_k = x$, the limits $\lim\limits_{k\to\infty} f_k(x_k)$ and $\lim\limits_{k\to\infty} f_k(y_k)$ exist and are identical. Show that $\{f_k\}_{k=1}^\infty$ converges uniformly on K. How about if K is not compact?

Problem 7.22. Assume that $\{f_k\}_{k=1}^\infty$ is a sequence of monotone increasing functions on $\mathbb{R}$ with $0 \leqslant f_k(x) \leqslant 1$ for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$.

1. Show that there is a subsequence $\{f_{k_j}\}_{j=1}^\infty$ which converges pointwise to a function f (This is usually called the ***Helly selection theorem***).
2. If the limit f is continuous, show that $\{f_{k_j}\}_{j=1}^\infty$ converges uniformly to f on any compact set of $\mathbb{R}$.

§7.6 The Stone-Weierstrass Theorem

Problem 7.23. Define B to be the set of all even functions in the space $\mathscr{C}([-1,1];\mathbb{R})$; that is, $f \in B$ if and only if f is continuous on $[-1,1]$ and $f(x) = f(-x)$ for all $x \in [-1,1]$. Prove that B is closed but not dense in $\mathscr{C}([-1,1];\mathbb{R})$. Hence show that even polynomials are dense in B, but not in $\mathscr{C}([-1,1];\mathbb{R})$.

Problem 7.24. Let $f : [0,1] \to \mathbb{R}$ be a continuous function.

1. Suppose that
$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \mathbb{N} \cup \{0\}\,.$$
Show that $f = 0$ on $[0,1]$.

2. Suppose that for some $m \in \mathbb{N}$,
$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \{0, 1, \cdots, m\}\,.$$
Show that $f(x) = 0$ has at least $(m+1)$ distinct real roots around which $f(x)$ change signs.

Problem 7.25. Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that
$$\lim_{n\to\infty} \int_0^1 f(x)\cos(nx)\, dx = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \int_0^1 f(x)\sin(nx)\, dx = 0\,.$$

Problem 7.26. Put $p_0 = 0$ and define
$$p_{k+1}(x) = p_k(x) + \frac{x^2 - p_k^2(x)}{2} \qquad \forall\, k \in \mathbb{N} \cup \{0\}\,.$$
Show that $\{p_k\}_{k=1}^\infty$ converges uniformly to $|x|$ on $[-1, 1]$.
Hint: Use the identity
$$|x| - p_{k+1}(x) = \big[|x| - p_k(x)\big]\Big[1 - \frac{|x| + p_k(x)}{2}\Big] \tag{7.7.2}$$
to prove that $0 \leqslant p_k(x) \leqslant p_{k+1}(x) \leqslant |x|$ if $|x| \leqslant 1$, and that
$$|x| - p_k(x) \leqslant |x|\Big(1 - \frac{|x|}{2}\Big)^k < \frac{2}{k+1}$$
if $|x| \leqslant 1$.

Problem 7.27. Let $f : [0,1] \to \mathbb{R}$ be continuous and $\varepsilon > 0$. Prove that there is a simple function g (defined in Example 7.75) such that $\|f - g\|_\infty < \varepsilon$.

Problem 7.28. Suppose that p_n is a sequence of polynomials converging uniformly to f on $[0, 1]$ and f is not a polynomial. Prove that the degrees of p_n are not bounded.
Hint: An Nth-degree polynomial p is uniquely determined by its values at $N + 1$ points $x_0, \cdots, x_N$ via Lagrange's interpolation formula
$$p(x) = \sum_{k=0}^N \pi_k(x)\frac{p(x_k)}{\pi_k(x_k)}\,,$$
where $\pi_k(x) = (x - x_0)(x - x_1)\cdots(x - x_N)/(x - x_k) = \prod\limits_{\substack{1\leqslant j\leqslant N\\ j\neq k}} (x - x_j)$.

Problem 7.29. Consider the set of all functions on $[0,1]$ of the form

$$h(x) = \sum_{j=1}^{n} a_j e^{b_j x},$$

where a_j, $b_j \in \mathbb{R}$. Is this set dense in $\mathscr{C}([0,1];\mathbb{R})$?

Problem 7.30 (True or False). Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1. Let $f_n : [a,b] \to \mathbb{R}$ be an uniformly convergent sequence of continuous functions. Then the sequence of the indefinite integrals $g_n(x)$ defined by

$$g_n(x) = \int_a^x f_n(t)\,dt \qquad \forall\, x \in [a,b]$$

   converges uniformly to a continuously differentiable function.

2. Let $f_n : [0,1] \to \mathbb{R}$ be a equi-continuous sequence of functions such that the sequence $\big\{f_n(\frac{1}{2})\big\}_{n=1}^{\infty}$ is bounded in $\mathbb{R}$. Then $\{f_n\}_{n=1}^{\infty}$ contains a convergent subsequence.

Chapter 8

Fourier Series

讓我們回顧一下之前已經有的一些結論。在 §7.6 中我們學到了 Stone-Weierstrass 定理，它告訴我們定義在 $[0,1]$ 上的連續函數 f 可以用多項式（例如 Bernstein 多項式）去逼近（在均勻收斂的意義下），而我們也注意到 Bernstein 多項式，在取不同次數 n 的多項式做逼近時，每一個單項式 x^k 前面的係數都跟 n 和 k 有關。但是從定理 7.23 中我們又發現，對某些擁有很好性質的函數 f（叫做解析函數 Analytic functions），即使取不同次數 n 的多項式做逼近時，每個單項式 x^k 前面的係數可以取成只跟函數 f 的 k 次導數有關（跟 n 無關）。這給了我們一個很粗略的概念，知道想用多項式去逼近連續函數時，多項式的係數有些時候會跟多項次的次數有關，有時則無關。

在這一章中，我們在前四節特別關注在週期為 2π 的連續函數。由 Corollary 7.85 我們知道這樣的函數可用形如

$$p_n(x) = \frac{c_0^{(n)}}{2} + \sum_{k=1}^{n}(c_k^{(n)} \cos kx + s_k^{(n)} \sin kx)$$

的三角多項式 (trigonometric polynomials) 所逼近（在均勻收斂的意義下），其中上標 (n) 代表的是係數可能與用來逼近的三角多項式的次數 n 有關係。跟前一段所述的經驗類似，在數學理論上我們想知道下面問題的答案：

1. 什麼樣的函數，可以用係數與逼近次數無關的三角多項式去逼近。對這樣的函數，三角多項式要怎麼挑？

2. 對於實在沒辦法用係數與逼近次數無關的三角多項式去逼近的連續週期函數，有什麼好的方法逼近？而上面所挑出來的那個係數跟逼近次數無關的三角多項式，在次數接近無窮大時出了什麼問題？

上述的問題解決之後，我們用變數變換，也可以得到對於週期為 $2L$ 的函數的相關理論。

另外，由於在進行的過程中，我們發現我們所關心用來逼近連續函數的三角多項式（叫富氏級數），其係數的定法只要求函數可積分即可，因此，一個自然衍生的問題則是：對不連續（但可積分）的函數來說，有沒有什麼收斂理論可以說明？這個部份的研究則是第四、五節的主要重點。在第六節中，我們則提供了一個快速傅利葉變換 (FFT) 的演算法可供電腦去計算富氏級數（的係數）。

8.1 Basic Properties of the Fourier Series

Let $f \in \mathscr{C}(\mathbb{T})$ be given. We first assume that the trigonometric polynomials used to approximate f can be chosen in such a way that the coefficients does not depend on the degree of approximation; that is, $c_k^{(n)} = c_k$ and $s_k^{(n)} = s_k$. In this case, if $p_n \to f$ uniformy on $[-\pi, \pi]$, by Theorem 7.17 we must have

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} p_n(x) \cos kx \, dx = \int_{-\pi}^{\pi} f(x) \cos kx \, dx \qquad \forall\, k \in \{0, 1, \cdots, n\}$$

and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} p_n(x) \sin kx \, dx = \int_{-\pi}^{\pi} f(x) \sin kx \, dx \qquad \forall\, k \in \{1, \cdots, n\}\,.$$

Since

$$\int_{-\pi}^{\pi} \cos kx \cos \ell x \, dx = \int_{-\pi}^{\pi} \sin kx \sin \ell x \, dx = \pi \delta_{k\ell} \qquad \forall\, k, \ell \in \mathbb{N}$$

and

$$\int_{-\pi}^{\pi} \sin kx \cos \ell x \, dx = 0 \qquad \forall\, k \in \mathbb{N}, \ell \in \mathbb{N} \cup \{0\}\,,$$

we find that

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx \qquad \text{and} \qquad s_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx\,. \tag{8.1.1}$$

This induces the following

Definition 8.1. For a Riemann integrable function $f : [-\pi, \pi] \to \mathbb{R}$, the ***Fourier series*** of f, denoted by $s(f, \cdot)$, is given by

$$s(f, x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} (c_k \cos kx + s_k \sin kx)$$

whenever the sum makes sense, where sequences $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ given by (8.1.1) are called the ***Fourier coefficients*** associated with f. The n-th partial sum of the Fourier series to f, denoted by $s_n(f,\cdot)$, is given by

$$s_n(f,x) = \frac{c_0}{2} + \sum_{k=1}^{n} (c_k \cos kx + s_k \sin kx)\,.$$

We note that for the Fourier series $s(f,x)$ to be defined, f is not necessary continuous. Our goal is to establish the convergence of Fourier series in various senses.

Remark 8.2. Because of the Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$, we can write

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)(e^{iky} + e^{-iky})dy \quad \text{and} \quad s_k = \frac{1}{2\pi i}\int_{-\pi}^{\pi} f(y)(e^{iky} - e^{-iky})dy$$

thus

$$\begin{aligned} s_n(f,x) &= \frac{c_0}{2} + \sum_{k=1}^{n} \big(c_k \frac{e^{ikx} + e^{-ikx}}{2} + s_k \frac{e^{ikx} - e^{-ikx}}{2i}\big) \\ &= \frac{1}{2}\Big[c_0 + \sum_{k=1}^{n} \big((c_k - is_k)e^{ikx} + (c_k + is_k)e^{-ikx}\big)\Big] \\ &= \frac{1}{2}\Big[c_0 + \sum_{k=1}^{n} (c_k - is_k)e^{ikx} + \sum_{k=-n}^{-1} (c_{-k} + is_{-k})e^{ikx}\Big] \\ &= \frac{1}{2}\Big[c_0 + \frac{1}{\pi}\sum_{k=1}^{n} \int_{-\pi}^{\pi} f(y)e^{-iky}dye^{ikx} + \frac{1}{\pi}\sum_{k=-n}^{-1} \int_{-\pi}^{\pi} f(y)e^{-iky}dye^{ikx}\Big]\,. \end{aligned}$$

Define $\widehat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy$. Then $\widehat{f}_k = \frac{c_{|k|} + is_{|k|}}{2}$ (here we treat $s_0 = 0$), and

$$s_n(f,x) = \sum_{k=-n}^{n} \widehat{f}_k e^{ikx}\,.$$

The sequence $\{\widehat{f}_k\}_{k=-\infty}^{\infty}$ is also called the Fourier coefficients associated with f, and one can write the Foruier series of f as $\sum\limits_{k=-\infty}^{\infty} \widehat{f}_k e^{ikx}$.

Remark 8.3. Given a continuous function g with period $2L$ (or a function g which is Riemann integrable on $[-L,L]$), let $f(x) = g\big(\frac{Lx}{\pi}\big)$. Then f is a continuous function with period 2π (or f is a Riemann integrable function on $[-\pi, \pi]$), and the Fourier series of f is given by

$$s(f, x) = \frac{c_0}{2} + \sum_{k=1}^{n} (c_k \cos kx + s_k \sin kx)\,,$$

where c_k and s_k are given by (8.1.1). Now, define the Fourier series of g by $s(g, x) = s\big(f, \frac{\pi x}{L}\big)$. Then the Fourier series of g is given by

$$s(g, x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} \big(c_k \cos \frac{k\pi x}{L} + s_k \sin \frac{k\pi x}{L}\big)\,,$$

where $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ is also called the Fourier coefficients associated with g and are given by

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} g\big(\frac{Lx}{\pi}\big) \cos kx\, dx = \frac{1}{L} \int_{-L}^{L} g(x) \cos \frac{k\pi x}{L}\, dx$$

and similarly, $s_k = \dfrac{1}{L} \displaystyle\int_{-L}^{L} g(x) \sin \frac{k\pi x}{L}\, dx$. Similar to Remark 8.2, the Fourier series of g can also be written as

$$\sum_{k=-\infty}^{\infty} \widehat{g}_k e^{\frac{i\pi kx}{L}}\,,$$

where $\widehat{g}_k = \dfrac{1}{2L} \displaystyle\int_{-L}^{L} g(y) e^{\frac{-i\pi ky}{L}}\, dy$.

Example 8.4. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \pi\,, \\ -x & \text{if } -\pi < x < 0\,, \end{cases}$$

and $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. To find the Fourier representation of f, we compute the Fourier coefficients by

$$s_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx = \frac{1}{\pi} \Big(\int_{0}^{\pi} x \sin kx\, dx - \int_{-\pi}^{0} x \sin kx\, dx \Big) = 0$$

and

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx = \frac{1}{\pi} \Big(\int_{0}^{\pi} x \cos kx\, dx - \int_{-\pi}^{0} x \cos kx\, dx \Big) = \frac{2}{\pi} \int_{0}^{\pi} x \cos kx\, dx\,.$$

If $k = 0$, then $c_0 = \dfrac{2}{\pi} \displaystyle\int_{0}^{\pi} x\, dx = \pi$, while if $k \in \mathbb{N}$,

$$c_k = \frac{2}{\pi} \Big(\frac{x \sin kx}{k} \Big|_0^{\pi} - \int_{0}^{\pi} \frac{\sin kx}{k}\, dx \Big) = \frac{2}{\pi} \frac{\cos kx}{k^2} \Big|_0^{\pi} = \frac{2((-1)^k - 1)}{\pi k^2}\,.$$

Therefore, $c_{2k} = 0$ and $c_{2k-1} = -\dfrac{4}{\pi(2k-1)^2}$ for all $k \in \mathbb{N}$. Therefore, the Fourier series of f is given by

$$s(f,x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}\,.$$

Example 8.5. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } -\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}\,, \\ 0 & \text{if } -\pi \leqslant x < -\dfrac{\pi}{2} \text{ or } \dfrac{\pi}{2} < x \leqslant \pi\,, \end{cases}$$

and $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$. We compute the Fourier coefficients of f and find that $s_k = 0$ for all $k \in \mathbb{N}$ and $c_0 = 1$, as well as

$$c_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos kx\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos kx\,dx = \frac{2\sin\frac{k\pi}{2}}{\pi k}\,.$$

Therefore, $c_{2k} = 0$ and $c_{2k-1} = \dfrac{2(-1)^{k+1}}{\pi(2k-1)}$ for all $k \in \mathbb{N}$; thus the Fourier series of f is given by

$$s(f,x) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1}\cos(2k-1)x\,.$$

Example 8.6. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x \qquad \text{if} \quad -\pi < x \leqslant \pi$$

and $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$. Then the Fourier coefficients of f are computed as follows: $c_k = 0$ for all $k \in \mathbb{N} \cup \{0\}$ since f is (more or less) an odd function, and

$$\begin{aligned} s_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin kx\,dx = \frac{2}{\pi}\Big(-\frac{x\cos kx}{k}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos kx}{k}\,dx\Big) \\ &= \frac{2(-1)^{k+1}}{k}\,. \end{aligned}$$

Therefore, the Fourier series of f is given by

$$s(f,x) = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kx\,.$$

8.2 Uniform Convergence of the Fourier Series

Before proceeding, we note that Remark 8.2 implies that

$$s_n(f,x) = \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{ik(x-y)}\, dy = \int_{-\pi}^{\pi} f(y) \Big(\frac{1}{2\pi} \sum_{k=-n}^{n} e^{ik(x-y)} \Big)\, dy\,.$$

Define $D_n(x) = \frac{1}{2\pi} \sum\limits_{k=-n}^{n} e^{ikx}$. Then D_n is 2π-periodic, and

$$s_n(f,x) = \int_{-\pi}^{\pi} f(y) D_n(x-y)\, dy\,.$$

For 2π-periodic Riemann integrable functions f and g, we define the convolution of f and g on the circle by

$$(f \star g)(x) = \int_{-\pi}^{\pi} f(y) g(x-y)\, dy\,.$$

Then $s_n(f,x) = (D_n \star f)(x)$.

Note that $D_n(0) = \frac{2n+1}{2\pi}$, and if $e^{ix} \neq 1$,

$$D_n(x) = \frac{1}{2\pi} \frac{e^{-inx}\big[e^{i(2n+1)x} - 1\big]}{e^{ix} - 1} = \frac{1}{2\pi} \frac{e^{i(n+1/2)x} - e^{-i(n+1/2)x}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin(n+\frac{1}{2})x}{2\pi \sin\frac{x}{2}}$$

so that we have the following

Definition 8.7. The function $D_n : \mathbb{R} \to \mathbb{R}$ defined by

$$D_n(x) = \begin{cases} \dfrac{\sin(n+\frac{1}{2})x}{2\pi \sin\frac{x}{2}} & \text{if } x \notin \{2k\pi \,|\, k \in \mathbb{Z}\}\,, \\ \dfrac{2n+1}{2\pi} & \text{if } x \in \{2k\pi \,|\, k \in \mathbb{Z}\}\,, \end{cases} \tag{8.2.1}$$

is called the ***Dirichlet kernel***.

By the fact that $D_n(x) = \frac{1}{2\pi} \sum\limits_{k=-n}^{n} e^{ikx}$, we immediately conclude the following

Lemma 8.8. *For each $n \in \mathbb{N}$ and $x \in \mathbb{R}$,* $\displaystyle\int_{-\pi}^{\pi} D_n(x-y)\, dy = 1$.

In the following, we first consider the uniform convergence of the Fourier series of 2π-periodic continuously differentiable functions.

Definition 8.9. The normed vector space $\big(\mathscr{C}^1(\mathbb{T}), \|\cdot\|_{\mathscr{C}^1(\mathbb{T})}\big)$ is a vector space over $\mathbb{R}$ consisting of all 2π-periodic real-valued continuously differentiable functions and is equipped with a norm

$$\|f\|_{\mathscr{C}^1(\mathbb{T})} = \|f\|_\infty + \|f'\|_\infty = \max_{x\in\mathbb{R}} \big|f(x)\big| + \max_{x\in\mathbb{R}} \big|f'(x)\big| \qquad \forall\, f \in \mathscr{C}^1(\mathbb{T})\,.$$

Theorem 8.10. *For any $f \in \mathscr{C}^1(\mathbb{T})$,the Fourier series of f converges uniformly to f on $\mathbb{R}$; that is, the sequence $\{s_n(f,\cdot)\}_{n=1}^\infty$ converges uniformly to f on $\mathbb{R}$.*

Proof. By Lemma 8.8, we find that for all $x \in \mathbb{R}$,

$$\begin{aligned} s_n(f,x) - f(x) = (D_n \star f - f)(x) &= \int_{-\pi}^{\pi} D_n(x-y)\big(f(y) - f(x)\big)dy \\ &= \int_{-\pi}^{\pi} D_n(y)\big(f(x-y) - f(x)\big)dy\,. \end{aligned}$$

We break the integral into two parts: one is the integral on $|y| \leqslant \delta$ and the other is the integral on $\delta < |y| \leqslant \pi$. Since $f \in \mathscr{C}^1(\mathbb{T})$,

$$|f(x-y) - f(x)| \leqslant \|f'\|_\infty |y|\,;$$

thus by the fact that $\dfrac{x}{\sin x} \leqslant \dfrac{\pi}{2}$ for $0 < x < \dfrac{\pi}{2}$, we obtain that

$$\begin{aligned} &\Big|\int_{|y|\leqslant\delta} D_n(y)\big(f(x-y) - f(x)\big)dy\Big| \\ &\qquad \leqslant \int_{-\delta}^{\delta} \frac{\big|f(x-y) - f(x)\big|}{2\pi\big|\sin\frac{y}{2}\big|}dy \leqslant \frac{\|f'\|_\infty}{2\pi}\int_{-\delta}^{\delta} \frac{y}{\sin\frac{y}{2}}dy \leqslant \|f'\|_\infty \delta\,. \end{aligned} \tag{8.2.2}$$

Now we take care of the integral on $\delta < |y| \leqslant \pi$ by first looking at the integral on $\delta < y < \pi$. Integrating by parts,

$$\begin{aligned} &\int_{\delta}^{\pi} D_n(y)\big(f(x-y) - f(y)\big)\,dy = \frac{1}{2\pi}\int_{\delta}^{\pi} \sin\big(n+\frac{1}{2}\big)y\frac{f(x-y) - f(x)}{\sin\frac{y}{2}}dy \\ &\quad = -\frac{1}{2\pi}\frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}\Big|_{y=\delta}^{y=\pi} + \frac{1}{2\pi}\int_{\delta}^{\pi}\frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\frac{d}{dy}\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}dy\,. \end{aligned}$$

For the first term on the right-hand side,

$$\Big|\frac{1}{2\pi}\frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}\Big|_{y=\delta}^{y=\pi}\Big| \leqslant \frac{2\|f\|_\infty}{2\pi n\sin\frac{\delta}{2}} \leqslant \frac{\|f\|_\infty}{n\sin\frac{\delta}{2}} \qquad \forall\, x \in \mathbb{R}\,.$$

For the second term on the right-hand side,

$$\begin{aligned}
&\Big|\frac{1}{2\pi}\int_\delta^\pi \frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\frac{d}{dy}\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}dy\Big| \\
&\qquad\leqslant \frac{1}{2\pi}\bigg[\Big|\int_\delta^\pi \frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\frac{f'(x-y)}{\sin\frac{y}{2}}dy\Big|+\Big|\int_\delta^\pi \frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\frac{\cos\frac{y}{2}\big(f(x-y)-f(x)\big)}{2\sin^2\frac{y}{2}}dy\Big|\bigg] \\
&\qquad\leqslant \frac{1}{2\pi}\Big[\|f'\|_\infty\frac{\pi-\delta}{\big(n+\frac{1}{2}\big)\sin\frac{\delta}{2}}+\|f\|_\infty\frac{\pi-\delta}{\big(n+\frac{1}{2}\big)\sin^2\frac{\delta}{2}}\Big] \leqslant \frac{\|f\|_{\mathscr{C}^1(\mathbb{T})}}{n\sin^2\frac{\delta}{2}}\,.
\end{aligned}$$

Similarly,

$$\Big|\int_{-\pi}^{-\delta} D_n(y)\big(f(x-y)-f(x)\big)dy\Big| \leqslant \frac{\|f\|_\infty}{n\sin\frac{\delta}{2}}+\frac{\|f\|_{\mathscr{C}^1(\mathbb{T})}}{n\sin^2\frac{\delta}{2}}\,;$$

thus for all $x\in\mathbb{R}$,

$$\begin{aligned}
\big|s_n(f,x)-f(x)\big| &\leqslant \Big|\Big(\int_{-\delta}^{\delta}+\int_\delta^\pi+\int_{-\pi}^{-\delta}\Big)D_n(y)\big(f(x-y)-f(x)\big)\,dy\Big| \\
&\leqslant \|f'\|_\infty\delta+\frac{2\|f\|_\infty}{n\sin\frac{\delta}{2}}+\frac{2\|f\|_{\mathscr{C}^1(\mathbb{T})}}{n\sin^2\frac{\delta}{2}} \leqslant \|f'\|_\infty\delta+\frac{4\|f\|_{\mathscr{C}^1(\mathbb{T})}}{n\sin^2\frac{\delta}{2}}\,.
\end{aligned}$$

Let $\varepsilon>0$ be given. Choose a fixed $\delta>0$ such that $\|f'\|_\infty\delta<\dfrac{\varepsilon}{2}$. For this fixed δ, choose $N>0$ such that

$$\frac{4\|f\|_{\mathscr{C}^1(\mathbb{T})}}{N\sin^2\frac{\delta}{2}}<\frac{\varepsilon}{2}\,.$$

Then if $n\geqslant N$ and $x\in\mathbb{R}$, we have

$$\big|s_n(f,x)-f(x)\big|<\frac{\varepsilon}{2}+\frac{4\|f\|_{\mathscr{C}^1(\mathbb{T})}}{n\sin^2\frac{\delta}{2}}\leqslant\frac{\varepsilon}{2}+\frac{4\|f\|_{\mathscr{C}^1(\mathbb{T})}}{N\sin^2\frac{\delta}{2}}<\varepsilon\,.$$ □

Next we consider the convergence of the Fourier series of less regular functions. The functions of which we prove the convergence of the Fourier series belong to the so-called Hölder class continuous functions.

Definition 8.11. A function $f\in\mathscr{C}(\mathbb{T})$ is said to be ***Hölder continuous with exponent*** $\alpha\in(0,1]$, denoted by $f\in\mathscr{C}^{0,\alpha}(\mathbb{T})$, if $\sup\limits_{\substack{x,y\in\mathbb{R}\\x\neq y}}\dfrac{|f(x)-f(y)|}{|x-y|^\alpha}<\infty$. Let $\|\cdot\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}$ be defined by

$$\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}=\sup_{x\in\mathbb{T}}\big|f(x)\big|+\sup_{\substack{x,y\in\mathbb{R}\\x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\alpha}\,.$$

Then $\|\cdot\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}$ is a norm on $\mathscr{C}^{0,\alpha}(\mathbb{T})$, and

$$\mathscr{C}^{0,\alpha}(\mathbb{T})=\big\{f\in\mathscr{C}(\mathbb{T})\,\big|\,\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}<\infty\big\}\,.$$

In particular, when $\alpha = 1$, a function in $\mathscr{C}^{0,1}(\mathbb{T})$ is said to be Lipschitz continuous on $\mathbb{T}$; thus $\mathscr{C}^{0,1}(\mathbb{T})$ consists of Lipschitz continuous functions on $\mathbb{T}$.

The uniform convergence of $s_n(f,\cdot)$ to f for $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ with $\alpha \in (0,1)$ requires a lot more work. The idea is to estimate $\|f - s_n(f,\cdot)\|_\infty$ in terms of the quantity $\inf\limits_{p\in\mathscr{P}_n(\mathbb{T})} \|f-p\|_\infty$. Since $s_n(f,\cdot) \in \mathscr{P}_n(\mathbb{T})$, it is obvious that

$$\inf_{p\in\mathscr{P}_n(\mathbb{T})} \|f-p\|_\infty \leqslant \|f - s_n(f,\cdot)\|_\infty .$$

The goal is to show the inverse inequality

$$\|f - s_n(f,\cdot)\|_\infty \leqslant C_n \inf_{p\in\mathscr{P}_n(\mathbb{T})} \|f-p\|_\infty \tag{8.2.3}$$

for some constant C_n, and pick a suitable $p \in \mathscr{P}_n(\mathbb{T})$ which gives a good upper bound for $\|f - s_n(f,\cdot)\|_\infty$. The inverse inequality is established via the following

Proposition 8.12. *The Dirichlet kernel D_n satisfies that for all $n \in \mathbb{N}$,*

$$\int_{-\pi}^{\pi} |D_n(x)| dx \leqslant 2 + \log n . \tag{8.2.4}$$

Proof. The validity of (8.2.4) for the case $n = 1$ is left to the reader, and we provide the proof for the case $n \geqslant 2$ here. Recall that $D_n(x) = \dfrac{\sin(n+\frac{1}{2})x}{2\pi \sin\frac{x}{2}}$ if $x \in (0,\pi]$. Therefore,

$$\int_{-\pi}^{\pi} |D_n(x)| dx = 2\int_0^{\pi} |D_n(x)| dx = \int_0^{\frac{1}{n}} 2|D_n(x)| dx + \int_{\frac{1}{n}}^{\pi} \Big|\frac{\sin(n+\frac{1}{2})x}{\pi \sin\frac{x}{2}}\Big| dx .$$

Since $|D_n(x)| \leqslant \dfrac{2n+1}{2\pi}$ for all $0 < x \leqslant \dfrac{1}{n}$, the first integral can be estimated by

$$\int_0^{\frac{1}{n}} 2|D_n(x)| dx \leqslant \frac{1}{\pi}\frac{2n+1}{n} . \tag{8.2.5}$$

Since $\dfrac{2x}{\pi} \leqslant \sin x$ for $0 \leqslant x \leqslant \dfrac{\pi}{2}$, the second integral can be estimated by

$$\int_{\frac{1}{n}}^{\pi} \Big|\frac{\sin(n+\frac{1}{2})x}{\pi \sin\frac{x}{2}}\Big| dx \leqslant \int_{\frac{1}{n}}^{\pi} \frac{1}{x}\, dx = \log\pi + \log n . \tag{8.2.6}$$

We then conclude (8.2.4) from (8.2.5) and (8.2.6) by noting that $\log\pi + \dfrac{2n+1}{n\pi} \leqslant 2$ for all $n \geqslant 2$. ◻

Remark 8.13. A more subtle estimate can be done to show that

$$\int_{-\pi}^{\pi} \big|D_n(x)\big| dx \geqslant c_1 + c_2 \log n \qquad \forall\, n \in \mathbb{N}$$

for some positive constants c_1 and c_2. Therefore, the integral of $|D_n|$ on $[-\pi, \pi]$ blows up as $n \to \infty$.

With the help of Proposition 8.12, we are able to prove the inverse inequality (8.2.3). The following theorem is a direct consequence of Proposition 8.12.

Theorem 8.14. *Let $f \in \mathscr{C}(\mathbb{T})$; that is, f is a continuous function with period 2π. Then*

$$\big\|f - s_n(f,\cdot)\big\|_\infty \leqslant (3 + \log n) \inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty \,. \tag{8.2.7}$$

Proof. For $n \in \mathbb{N}$ and $x \in \mathbb{T}$,

$$\big|s_n(f,x)\big| \leqslant \int_{-\pi}^{\pi} \big|D_n(y)\big|\big|f(x-y)\big| dy \leqslant (2 + \log n)\|f\|_\infty \,.$$

Given $\varepsilon > 0$, let $p \in \mathscr{P}_n(\mathbb{T})$ such that

$$\|f - p\|_\infty \leqslant \inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty + \varepsilon.$$

Then by the fact that $s_n(p,x) = p(x)$ if $p \in \mathscr{P}_n(\mathbb{T})$, we obtain that

$$\begin{aligned}
\big\|f - s_n(f,\cdot)\big\|_\infty &\leqslant \big\|f - p\big\|_\infty + \big\|p - s_n(f,\cdot)\big\|_\infty \leqslant \big\|f - p\big\|_\infty + \big\|s_n(f-p,\cdot)\big\|_\infty \\
&\leqslant \big\|f - p\big\|_\infty + (2 + \log n)\|f - p\|_\infty \\
&\leqslant (3 + \log n)\Big[\inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty + \varepsilon\Big],
\end{aligned}$$

and (8.2.7) is obtained by passing to the limit as $\varepsilon \to 0$. □

Having established Theorem 8.14, the study of the uniform convergence of $s_n(f,\cdot)$ to f then amounts to the study of the quantity $\inf\limits_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty$. The estimate of $\inf\limits_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty$ for $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, where $\alpha \in (0,1)$, is more difficult, and requires a clever choice of p. We begin with the following

Lemma 8.15. *If f is a continuous function on $[a,b]$, then for all $\delta_1, \delta_2 > 0$,*

$$\sup_{|x-y| \leqslant \delta_1} \big|f(x) - f(y)\big| \leqslant \Big(1 + \frac{\delta_1}{\delta_2}\Big) \sup_{|x-y| \leqslant \delta_2} \big|f(x) - f(y)\big| \,.$$

The proof of Lemma 8.15 is not very difficult, and is left to the readers.

Now we are in position to prove the theorem due to D. Jackson.

Theorem 8.16 (Jackson). *There exists a constant $C > 0$ such that*

$$\inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty \leqslant C \sup_{|x-y| \leqslant \frac{1}{n}} |f(x) - f(y)| \qquad \forall\, f \in \mathscr{C}(\mathbb{T}).$$

Proof. Let $p(x) = 1 + c_1 \cos x + \cdots + c_n \cos nx$ be a positive trigonometric function of degree n with coefficients $\{c_i\}_{i=1}^n$ determined later. Define an operator K on $\mathscr{C}(\mathbb{T})$ by

$$(\mathrm{K}f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(y) f(x-y)\, dy.$$

Then $\mathrm{K}f \in \mathscr{P}_n(\mathbb{T})$. Lemma 8.15 then implies

$$\begin{aligned}
\big|(\mathrm{K}f)(x) - f(x)\big| &\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} p(y) \big|f(x-y) - f(x)\big| dy \\
&\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} p(y) \big(1 + n|y|\big) \sup_{|x-y| \leqslant \frac{1}{n}} \big|f(x) - f(y)\big| dy \\
&= \Big[1 + \frac{n}{2\pi} \int_{-\pi}^{\pi} |y| p(y)\, dy\Big] \sup_{|x-y| \leqslant \frac{1}{n}} \big|f(x) - f(y)\big|.
\end{aligned}$$

Since $y^2 \leqslant \dfrac{\pi^2}{2}(1 - \cos y)$ for $y \in [-\pi, \pi]$, by the Cauchy-Schwarz inequality (Corollary 2.27) we find that

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} |y| p(y)\, dy &\leqslant \Big[\frac{1}{2\pi} \int_{-\pi}^{\pi} y^2 p(y)\, dy\Big]^{\frac{1}{2}} \Big[\frac{1}{2\pi} \int_{-\pi}^{\pi} p(y)\, dy\Big]^{\frac{1}{2}} \\
&\leqslant \Big[\frac{\pi}{4} \int_{-\pi}^{\pi} (1 - \cos y) p(y)\, dy\Big]^{\frac{1}{2}} = \frac{\pi}{2} \sqrt{2 - c_1}.
\end{aligned}$$

Therefore,

$$\|\mathrm{K}f - f\|_\infty \leqslant \Big(1 + \frac{n\pi}{2} \sqrt{2 - c_1}\Big) \sup_{|x-y| \leqslant \frac{1}{n}} \big|f(x) - f(y)\big|.$$

To conclude the theorem, we need to show that the number $n\sqrt{2 - c_1}$ can be made bounded by choosing p properly. Nevertheless, let

$$\begin{aligned}
p(x) &= c \Big| \sum_{k=0}^{n} \sin \frac{(k+1)\pi}{n+2} e^{ikx} \Big|^2 = c \sum_{k=0}^{n} \sum_{\ell=0}^{n} \sin \frac{(k+1)\pi}{n+2} \sin \frac{(\ell+1)\pi}{n+2} e^{i(k-\ell)x} \\
&= c \sum_{k=0}^{n} \sin^2 \frac{(k+1)\pi}{n+2} + 2c \sum_{\substack{k,\ell=0 \\ k > \ell}}^{n} \sin \frac{(k+1)\pi}{n+2} \sin \frac{(\ell+1)\pi}{n+2} \cos(k-\ell)x
\end{aligned}$$

and choose c so that $p(x) = 1 + c_1 \cos x + \cdots + c_n \cos nx$. Then

$$\begin{aligned} c^{-1} &= \sum_{k=0}^{n} \sin^2 \frac{(k+1)\pi}{n+2} = \frac{1}{2} \sum_{k=0}^{n} \Big[1 - \cos \frac{2(k+1)\pi}{n+2}\Big] \\ &= \frac{n+1}{2} - \frac{\sin \frac{(2n+3)\pi}{n+2} - \sin \frac{\pi}{n+2}}{4 \sin \frac{\pi}{n+2}} = \frac{n+2}{2}\,, \end{aligned}$$

and

$$\begin{aligned} c_1 &= 2c \sum_{k=1}^{n} \sin \frac{(k+1)\pi}{n+2} \sin \frac{k\pi}{n+2} = c \sum_{k=1}^{n} \Big[\cos \frac{\pi}{n+2} - \cos \frac{(2k+1)\pi}{n+2}\Big] \\ &= c\Big[n \cos \frac{\pi}{n+2} - \frac{\sin \frac{(2n+2)\pi}{n+2} - \sin \frac{2\pi}{n+2}}{2 \sin \frac{\pi}{n+2}}\Big] \\ &= c\Big[n \cos \frac{\pi}{n+2} + \frac{\sin \frac{2\pi}{n+2}}{\sin \frac{\pi}{n+2}}\Big] = c(n+2) \cos \frac{\pi}{n+2} = 2 \cos \frac{\pi}{n+2}. \end{aligned}$$

As a consequence,

$$\begin{aligned} n\sqrt{2 - c_1} &= n\Big(2 - 2\cos \frac{\pi}{n+2}\Big)^{\frac{1}{2}} = 2n \sin \frac{\pi}{2(n+2)} \\ &= 2(n+2) \sin \frac{\pi}{2(n+2)} - 4 \sin \frac{\pi}{2(n+2)} \\ &= \pi \frac{2(n+2)}{\pi} \sin \frac{\pi}{2(n+2)} - 4 \sin \frac{\pi}{2(n+2)} \end{aligned}$$

which is bounded by π; thus

$$\inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty \leqslant \|\mathrm{K}f - f\|_\infty \leqslant \Big(1 + \frac{\pi^2}{2}\Big) \sup_{|x-y| \leqslant \frac{1}{n}} \big|f(x) - f(y)\big|\,. \qquad \square$$

Finally, since $\lim\limits_{n\to\infty} n^{-\alpha} \log n = 0$ for all $\alpha \in (0, 1]$, we conclude the following

Theorem 8.17. *For any $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ with $\alpha \in (0, 1]$, the Fourier series of f converges uniformly to f on $\mathbb{R}$.*

Remark 8.18. The converse of Theorem 8.16 is the Bernstein theorem which states that if f is a 2π-periodic function with the property that there exist a constant C (independent of n) and $\alpha \in (0, 1)$ such that

$$\inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty \leqslant Cn^{-\alpha} \qquad \forall\, n \in \mathbb{N}\,, \tag{8.2.8}$$

then $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$. In other words, (8.2.8) is an equivalent condition to the Hölder continuity with exponent α of 2π-periodic continuous functions.

8.3 Cesàro Mean of Fourier Series

While Corollary 7.85 shows that the collection of trigonometric polynomials

$$\Big\{\frac{c_0}{2}+\sum_{k=1}^{n}(c_k\cos kx+s_k\sin kx)\,\Big|\,\{c_k\}_{k=0}^{n},\{s_k\}_{k=1}^{n}\subseteq\mathbb{R}\Big\}$$

is dense in $\mathscr{C}(\mathbb{T})$, Theorem 8.17 only implies the uniform convergence of the Fourier series of Hölder continuous functions. To approximate continuous functions uniformly, the coefficients of the trigonometric polynomials should depend on the order of approximation.

The motivation of the discussion below is due to the following observation. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence. Define a new sequence $\{b_n\}_{n=1}^{\infty}$, called the ***Cesàro mean*** of the sequence $\{a_k\}_{k=1}^{\infty}$, by

$$b_n=\frac{a_1+\cdots+a_n}{n}=\frac{1}{n}\sum_{k=1}^{n}a_k\,.$$

If $\{a_k\}_{k=1}^{\infty}$ converges to a, then $\{b_n\}_{n=1}^{\infty}$ converges to a as well. Even though the convergence of a sequence cannot be guaranteed by the convergence of its Cesàro mean, it is worthwhile investigating the convergence behavior of the Cesàro mean.

Let $\sigma_n(f,\cdot)$ denote the Cesàro mean of the Fourier series of f given by

$$\sigma_n(f,\cdot)\equiv\frac{1}{n+1}\sum_{k=0}^{n}s_k(f,\cdot)=\frac{1}{n+1}\sum_{k=0}^{n}(D_k\star f)=\Big(\frac{1}{n+1}\sum_{k=0}^{n}D_k\Big)\star f\,.$$

We note that the coefficients of the Cesàro mean $\sigma_n(f,\cdot)$ depend on the order of approximation n since

$$\sigma_n(f,x)=\frac{c_0}{2}+\sum_{k=1}^{n}\Big(\underbrace{\frac{n+1-k}{n+1}c_k}_{\equiv c_k^{(n)}}\cos kx+\underbrace{\frac{n+1-k}{n+1}s_k}_{\equiv s_k^{(n)}}\sin kx\Big)\,.$$

Recall that $D_k(x)=\dfrac{\sin(k+\frac{1}{2})x}{2\pi\sin\frac{x}{2}}$. By the product-to-sum formula, we find that if $x\in(0,\pi)$,

$$\begin{aligned}\sum_{k=0}^{n}D_k(x)&=\frac{1}{2\pi\sin\frac{x}{2}}\sum_{k=0}^{n}\sin(k+\frac{1}{2})x=\frac{1}{4\pi\sin^2\frac{x}{2}}\sum_{k=0}^{n}2\sin\frac{x}{2}\sin(k+\frac{1}{2})x\\&=\frac{1}{4\pi\sin^2\frac{x}{2}}\sum_{k=0}^{n}\big(\cos kx-\cos(k+1)x\big)\\&=\frac{1}{4\pi\sin^2\frac{x}{2}}\big(1-\cos(n+1)x\big)=\frac{\sin^2\frac{n+1}{2}x}{2\pi\sin^2\frac{x}{2}}\,.\end{aligned}$$

This induces the following

Definition 8.19. The ***Fejér kernel*** is the Cesàro mean of the Dirichlet kernel given by

$$F_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(x) = \frac{1}{2\pi(n+1)}\frac{\sin^2\frac{(n+1)x}{2}}{\sin^2\frac{x}{2}}\,.$$

We note that $\sigma_n(f,\cdot) = F_n \star f$, where $F_n \geqslant 0$ and has the property that $\displaystyle\int_{-\pi}^{\pi} F_n(x)\,dx = 1$ (since the integral of the Dirichlet kernel is 1). Moreover, for any $\delta > 0$,

$$\lim_{n\to\infty}\int_{\delta\leqslant|x|\leqslant\pi} F_n(x)\,dx = 0 \tag{8.3.1}$$

since $\big|F_n(x)\big| \leqslant \dfrac{1}{2\pi(n+1)\sin^2\frac{\delta}{2}}$ if $\delta \leqslant |x| \leqslant \pi$. Inequality (8.3.1) allows us to show that $\big\{\sigma_n(f,\cdot)\big\}_{n=1}^{\infty}$ converges uniformly to f.

Theorem 8.20. *For any $f \in \mathscr{C}(\mathbb{T})$, the Cesàro mean $\big\{\sigma_n(f,\cdot)\big\}_{n=1}^{\infty}$ of the Fourier series of f converges uniformly to f.*

Proof. Let $\varepsilon > 0$ be given. Since $f \in \mathscr{C}(\mathbb{T})$, f is uniformly continuous on $\mathbb{R}$; thus there exists $\delta > 0$ such that

$$\big|f(x) - f(y)\big| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |x-y| < \delta\,.$$

Therefore, by the fact that $\displaystyle\int_{-\pi}^{\pi} F_n(x)dx = 1$ and $F_n \geqslant 0$,

$$\begin{aligned}
\big|\sigma_n(f,x) - f(x)\big| &= \Big|\int_{-\pi}^{\pi} F_n(y)f(x-y)\,dy - \int_{-\pi}^{\pi} F_n(y)f(x)\,dy\Big| \\
&\leqslant \int_{-\pi}^{\pi} F_n(y)\big|f(x-y) - f(x)\big|\,dy \\
&= \int_{|y|<\delta} F_n(y)\big|f(x-y) - f(x)\big|\,dy + \int_{\delta\leqslant|y|\leqslant\pi} F_n(y)\big|f(x-y) - f(x)\big|\,dy \\
&\leqslant \varepsilon\int_{|y|<\delta} F_n(y)\,dy + 2\|f\|_\infty\int_{\delta\leqslant|y|\leqslant\pi} F_n(y)\,dy \\
&\leqslant \frac{\varepsilon}{2} + 2\|f\|_\infty\int_{\delta\leqslant|y|\leqslant\pi} F_n(y)\,dy\,.
\end{aligned}$$

Using (8.3.1), there exists $N > 0$ such that

$$2\|f\|_\infty\int_{\delta\leqslant|y|\leqslant\pi} F_n(y)\,dy < \frac{\varepsilon}{2} \quad \textit{whenever} \quad n \geqslant N\,.$$

Therefore, $\big|\sigma_n(f,x) - f(x)\big| < \varepsilon$ whenever $n \geqslant N$ and $x \in \mathbb{R}$; thus we conclude that the Cesàro mean $\big\{\sigma_n(f,\cdot)\big\}_{n=1}^{\infty}$ converges uniformly to f. □

8.4 Convergence of Fourier Series for Functions with Jump Discontinuity

In previous sections we discussed the convergence of the Fourier series of continuous functions. However, since the Fourier series can be defined for bounded Riemann integrable functions, it is natural to ask what happen if the function under consideration is not continuous. We note that in this case we cannot apply Corollary 7.85 at all so no uniform convergence is expected.

In this section, we focus on the convergence behavior of Fourier series of functions with only jump discontinuities.

Definition 8.21. A function $f : [-\pi, \pi] \to \mathbb{R}$ is said to have jump discontinuity at $a \in (-\pi, \pi)$ if

1. $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to a^-} f(x)$ both exist.
2. $\lim\limits_{x \to a^+} f(x) \neq \lim\limits_{x \to a^-} f(x)$.

Now suppose that $f : [-\pi, \pi] \to \mathbb{R}$ is piecewise Hölder continuous with exponent $\alpha \in (0, 1]$; that is, there exists $\{a_1, \cdots, a_m\} \subseteq (-\pi, \pi)$ such that $f \in \mathscr{C}^{0,\alpha}((a_j, a_{j+1}); \mathbb{R})$ for all $j \in \{0, \cdots, m\}$, where $a_0 = -\pi$ and $a_{m+1} = \pi$, and $f \in \mathscr{C}^{0,\alpha}(I; \mathbb{R})$ if and only if

$$\sup_{x,y \in I, x \neq y} \frac{\big|f(x) - f(y)\big|}{|x - y|^\alpha} < \infty \,.$$

Then for all $a \in (-\pi, \pi)$, the limits $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to a^-} f(x)$ exist since if $\{x_k\}_{k=1}^\infty$ is a sequence in $(-\pi, \pi)$ which approaches to a from the right/left, then for some $0 \leqslant j \leqslant m$ we must have $x_k \in (a_j, a_{j+1})$ for all large k so that the Hölder continuity implies that

$$\big|f(x_k) - f(x_\ell)\big| \leqslant M|x_k - x_\ell|^\alpha \qquad \forall\, k, \ell \text{ large}$$

which shows that $\{f(x_k\}_{k=1}^\infty$ is a Cauchy sequence $\big($converging to $\lim\limits_{x \to a^\pm} f(x)\big)$. In other words, if $f : [-\pi, \pi] \to \mathbb{R}$ is piecewise Hölder continuous and $a \in (-\pi, \pi)$ is a discontinuity of f, then f has either removable discontinuity at a $\big($which means $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) \neq f(a)\big)$ or jump discontinuity at a. In the following, we always assume that f is piecewise Hölder continuous with exponent $\alpha \in (0, 1]$ and has only jump discontinuities at $\{a_1, \cdots, a_m\}$ in $(-\pi, \pi)$.

Let $f(a_j^+) = \lim\limits_{x\to a_j^+} f(x)$, $f(a_j^-) = \lim\limits_{x\to a_j^-} f(x)$, and define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \frac{1}{2\pi}(x-\pi) \qquad \forall\, x \in [0, 2\pi) \tag{8.4.1}$$

and $\phi(x+2\pi) = \phi(x)$ for all $x \in \mathbb{R}$. Since f has jump discontinuities at $\{a_1, \cdots, a_m\}$, with a_0^- denoting a_{m+1}^- the function $g : [-\pi, \pi] \to \mathbb{R}$ defined by

$$g(x) \equiv \begin{cases} f(x) + \displaystyle\sum_{j=0}^{m} \big(f(a_j^+) - f(a_j^-)\big)\phi(x-a_j) & \text{if } x \neq a_k \text{ for all } k, \\ \dfrac{f(a_k^+) + f(a_k^-)}{2} + \displaystyle\sum_{\substack{0\leqslant j\leqslant m \\ j\neq k}} \big(f(a_j^+) - f(a_j^-)\big)\phi(a_k - a_j) & \text{if } x = a_k \text{ for some } k, \end{cases} \tag{8.4.2}$$

is Hölder continuous with exponent α and $g(a_0^+) = g(a_0^-) = g(-\pi)$. Let G be the 2π-periodic extension of g; that is, $G = g$ on $[-\pi, \pi]$ and $G(x+2\pi) = G(x)$ for all $x \in \mathbb{R}$. Then $G \in \mathscr{C}^{0,\alpha}(\mathbb{T})$; thus Theorem 8.17 implies that $s_n(G, \cdot) \to G$ uniformly on $\mathbb{R}$. In particular, $s_n(g, \cdot) \to g$ uniformly on $[-\pi, \pi]$.

Using the identity

$$\int_{-\pi}^{\pi} \phi(x-a)e^{-ikx}\,dx = e^{-ika}\int_{-\pi}^{\pi} \phi(x)e^{-ikx}\,dx = \widehat{\phi}_k e^{-ika}\,,$$

we obtain that

$$s_n(\phi(\cdot - a), x) = \sum_{k=-n}^{n} \widehat{\phi}_k e^{ik(x-a)} = s_n(\phi, x-a)\,; \tag{8.4.3}$$

thus (8.4.2) implies that the Fourier series of f is given by

$$\begin{aligned} s_n(f, x) &= s_n(g, x) - \sum_{j=0}^{m} \big(f(a_j^+) - f(a_j^-)\big) s_n(\phi(\cdot - a_j), x) \\ &= s_n(g, x) - \sum_{j=0}^{m} \big(f(a_j^+) - f(a_j^-)\big) s_n(\phi, x - a_j)\,. \end{aligned} \tag{8.4.4}$$

Therefore, to understand the convergence of the Fourier series of f, without loss of generality it suffices to consider the convergence of $s_n(\phi, \cdot)$.

8.4.1 Uniform convergence on compact subsets

In this sub-section, we show that the Fourier series of a piecewise Hölder continuous function whose discontinuities are all jump discontinuities converges uniformly on each compact subset containing no jump discontinuities.

Based on the discussion above, we first study the convergence of $s_n(\phi,\cdot)$. Since ϕ is an odd function, for $k \in \mathbb{N}$,

$$\begin{aligned} s_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\sin kx\,dx = \frac{1}{\pi^2}\int_0^{\pi}(x-\pi)\sin kx\,dx \\ &= \frac{1}{\pi^2}\Big[\frac{-(x-\pi)\cos kx}{k}\Big|_{x=0}^{x=\pi} + \int_0^{\pi}\frac{\cos kx}{k}\,dx\Big] = -\frac{1}{\pi k}\,. \end{aligned}$$

Therefore, the n-th partial sum of the Fourier series of ϕ is given by

$$s_n(\phi,x) = -\frac{1}{\pi}\sum_{k=1}^{n}\frac{\sin kx}{k}\,. \tag{8.4.5}$$

Lemma 8.22. *The series* $\sum\limits_{k=1}^{\infty}\frac{\sin kx}{k}$ *converges uniformly on* $[-\pi,-\delta]\cup[\delta,\pi]$ *for all* $0<\delta<\pi$.

Proof. Let $0<\delta<\pi$ be given, and $S_n(x)$ denote the sum $\sum\limits_{k=1}^{n}\sin kx$. Using the identity

$$\sum_{k=1}^{n}\sin kx = \frac{\cos(n+\frac{1}{2})x - \cos\frac{x}{2}}{2\sin\frac{x}{2}} \qquad \forall\, x \in [-\pi,-\delta]\cup[\delta,\pi]\,,$$

we find that $|S_n| \leqslant M < \infty$ for some fixed constant M. For $m>n$,

$$\begin{aligned} \sum_{k=n+1}^{m}\frac{1}{k}\sin kx &= \frac{1}{m}(S_m - S_{m-1}) + \frac{1}{m-1}(S_{m-1}-S_{m-2}) + \cdots + \frac{1}{n+1}(S_{n+1}-S_n) \\ &= \frac{S_m}{m} - \frac{S_n}{n+1} + \frac{1}{m(m-1)}S_{m-1} + \frac{1}{(m-1)(m-2)}S_{m-2} + \cdots \frac{1}{(n+2)(n+1)}S_{n+1}\,; \end{aligned}$$

thus

$$\Big|\sum_{k=n+1}^{m}\frac{1}{k}\sin kx\Big| \leqslant M\Big(\frac{1}{m} + \frac{1}{n+1} + \sum_{k=n+2}^{m}\frac{1}{k(k-1)}\Big) \leqslant 2M\Big(\frac{1}{m}+\frac{1}{n}\Big)\,.$$

Since the right-hand side converges to 0 as $n,m\to\infty$, the Cauchy criterion (for the convergence of series of functions) implies that the series

$$\sum_{k=1}^{\infty}\frac{\sin kx}{k}$$

converges uniformly on $[-\pi,-\delta]\cup[\delta,\pi]$. ◻

Lemma 8.22 provides the uniform convergence of $s_n(\phi,\cdot)$ in $[-\pi,-\delta]\cup[\delta,\pi]$. To see the limit is exactly ϕ, we consider an anti-derivative Φ of ϕ and establish that $\Phi' = s(\phi,\cdot)$.

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and $\Psi(x) = \dfrac{x^2}{4\pi}$ for $x \in [-\pi, \pi]$. Then $\Psi \in \mathscr{C}^{0,1}(\mathbb{T})$ is an even function and the Fourier coefficients of Ψ is

$$\widehat{\Psi}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{x^2}{4\pi}\,dx = \frac{\pi}{12}$$

and for $k \neq 0$,

$$\widehat{\Psi}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{x^2}{4\pi}e^{-ikx}\,dx = \frac{1}{8\pi^2}\int_{-\pi}^{\pi} x^2(\cos kx + i\sin kx)\,dx = \frac{(-1)^k}{2k^2\pi}\,.$$

Therefore, using (8.4.3) we find that the Fourier series of $\Phi \equiv \Psi(\cdot - \pi)$ is

$$\begin{aligned} s(\Phi, x) = s(\Psi, x-\pi) &= \frac{\pi}{12} + \sum_{k\in\mathbb{Z}, k\neq 0} \widehat{\Psi}_k e^{ik(x-\pi)} = \frac{\pi}{12} + \frac{1}{2\pi}\sum_{k\in\mathbb{Z}, k\neq 0}\frac{e^{ikx}}{k^2} \\ &= \frac{\pi}{12} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\cos kx}{k^2}\,. \end{aligned}$$

Since $\Phi \in \mathscr{C}^{0,1}(\mathbb{T})$, $s_n(\Phi,\cdot)$ converges uniformly to Φ on $\mathbb{R}$. Moreover, $s_n(\Phi,\cdot)' = s_n(\phi,\cdot)$ which converges uniformly on $[-\pi,-\delta] \cup [\delta,\pi]$. Therefore, Theorem 7.11 implies that $s(\phi,\cdot)$, the uniform limit of $s_n(\phi,\cdot)$, must equal Φ' on $[-\pi,-\delta]\cup[\delta,\pi]$. Finally, we note that $\phi = \Phi'$ on $[-\pi,-\delta]\cup[\delta,\pi]$, so we establish that $s_n(\phi,\cdot) \to \phi$ uniformly on $[-\pi,-\delta]\cup[\delta,\pi]$.

Since a discontinuity of a piecewise Hölder continuous function f is either removable or a jump discontinuity, and the value of the function at removable discontinuities does not change the value of the Fourier series of f, the uniform convergence of $s_n(\phi,\cdot)$ to ϕ on $[-\pi,-\delta]\cup[\delta,\pi]$ for all $0 < \delta < \pi$ implies the following

Theorem 8.23. *Let $f : (-\pi,\pi) \to \mathbb{R}$ be piecewise Hölder continuous with exponent $\alpha \in (0,1]$. If f is continuous on (a,b), then the Fourier series of f converges uniformly to f on any compact subsets of (a,b).*

By Remark 8.3, we can also conclude the following

Corollary 8.24. *Let $f : (-L,L) \to \mathbb{R}$ be piecewise Hölder continuous with exponent $\alpha \in (0,1]$. If f is continuous on (a,b), then the Fourier series of f converges uniformly to f on any compact subsets of (a,b) (where the Fourier series of f is given in Remark 8.3). In particular, $\lim\limits_{n\to\infty} s_n(f,x_0) = f(x_0)$ if f is continuous at x_0. In other words, the Fourier series of f converges pointwise to f except the discontinuities.*

8.4.2 Gibbs phenomenon

In this sub-section, we show that the Fourier series evaluated at the jump discontinuity converges to the average of the limits from the left and the right. Moreover, the convergence of the Fourier series is never uniform in the domain including these jump discontinuities due to the famous Gibbs phenomenon: near the jump discontinuity the maximum difference between the limit of the Fourier series and the function itself is at least 8% of the jump. To be more precise, we have the following

Theorem 8.25. *Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic piecewise Hölder continuous with exponent $\alpha \in (0, 1]$. Then*

$$\lim_{n\to\infty} s_n(f, x_0) = \frac{f(x_0^+) + f(x_0^-)}{2} \qquad \forall\, x_0 \in \mathbb{R}\,. \tag{8.4.6}$$

Moreover, if x_0 is a jump discontinuity of f so that

$$f(x_0^+) - f(x_0^-) = a \neq 0\,,$$

then there exists a constant $c > 0$, independent of f, x_0 and L (in fact, $c = \dfrac{1}{\pi}\displaystyle\int_0^\pi \frac{\sin x}{x}\,dx - \frac{1}{2} \approx$ 0.089490), such that

$$\lim_{n\to\infty} s_n\big(f, x_0 + \frac{L}{n}\big) = f(x_0^+) + ca\,, \tag{8.4.7a}$$

$$\lim_{n\to\infty} s_n\big(f, x_0 - \frac{L}{n}\big) = f(x_0^-) - ca\,. \tag{8.4.7b}$$

Proof. By Remark 8.3, W.L.O.G. we can assume that $L = \pi$. Let $\{a_1, \cdots, a_m\} \subseteq (-\pi, \pi)$ be the collection of jump discontinuities of f in $(-\pi, \pi)$, $a_0 = -\pi$, $a_{m+1} = \pi$ (so by periodicity $f(a_0^-) = f(a_{m+1}^-)$ automatically), and define g by (8.4.2). Then $g \in \mathscr{C}^{0,\alpha}(\mathbb{T})$. Suppose that x_0 is a jump discontinuity of f in $[-\pi, \pi)$ (so a_0 could be a possible jump discontinuity of f). Then $x_0 = a_k$ for some $k \in \{0, 1, \cdots, m\}$. Therefore, by the fact that ϕ is continuous at $x_0 - a_j$ if $j \neq k$ and $s_n(\phi, 0) = 0$ for all $n \in \mathbb{N}$, Corollary 8.24 implies that

$$\begin{aligned}&\sum_{j=0}^{m} \big(f(a_j^+) - f(a_j^-)\big) \lim_{n\to\infty} s_n(\phi, x_0 - a_j)\\ &\quad = \sum_{\substack{0\leqslant j\leqslant m\\ j\neq k}} \big(f(a_j^+) - f(a_j^-)\big) \lim_{n\to\infty} s_n(\phi, x_0 - a_j) = \sum_{\substack{0\leqslant j\leqslant m\\ j\neq k}} \big(f(a_j^+) - f(a_j^-)\big)\phi(x_0 - a_j)\,.\end{aligned}$$

On the other hand,

$$\lim_{n\to\infty} s_n(g, x_0) = g(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2} + \sum_{\substack{0\leqslant j\leqslant m\\ j\neq k}} \big(f(a_j^+) - f(a_j^-)\big)\phi(x_0 - a_j)\,.$$

Identity (8.4.6) is then concluded using (8.4.4).

Now we focus on (8.4.7a). Since $g \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, $s_n(g,\cdot) \to g$ uniformly on $\mathbb{R}$; thus

$$\lim_{n\to\infty} s_n\big(g, x_0 + \frac{\pi}{n}\big) = g(x_0)\,.$$

Similarly, since $s_n(\phi,\cdot) \to \phi$ uniformly on $[-\pi,-\delta] \cup [\delta,\pi]$ for all $\delta > 0$, if $j \neq k$,

$$\lim_{n\to\infty} s_n\big(\phi, x_0 + \frac{\pi}{n} - a_j\big) = \phi(x_0 - a_j)\,.$$

On the other hand,

$$s_n\big(\phi, \frac{\pi}{n}\big) = -\sum_{k=1}^{n} \frac{1}{\pi k} \sin\frac{k\pi}{n} = -\frac{1}{\pi}\sum_{k=1}^{n} \frac{n}{k\pi} \sin\frac{k\pi}{n}\frac{\pi}{n} \to -\frac{1}{\pi}\int_0^\pi \frac{\sin x}{x} dx \equiv -\big(c + \frac{1}{2}\big)\,.$$

As a consequence,

$$\begin{aligned}
\lim_{n\to\infty} s_n\big(f, x_0 + \frac{\pi}{n}\big) &= \lim_{n\to\infty}\Big[s_n\big(g, x_0 + \frac{\pi}{n}\big) - \sum_{j=0}^{m}\big(f(a_j^+) - f(a_j^-)\big)s_n\big(\phi, x_0 + \frac{\pi}{n} - a_j\big)\Big] \\
&= g(x_0) - \sum_{\substack{0\leqslant j\leqslant m\\ j\neq k}} \big(f(a_j^+) - f(a_j^-)\big)\phi(x_0 - a_j) + \big(c + \frac{1}{2}\big)\big(f(x_0^+) - f(x_0^-)\big) \\
&= f(x_0^+) + c\big(f(x_0^+) - f(x_0^-)\big)\,.
\end{aligned}$$

Identity (8.4.7b) can be proved in the same fashion, and is left as an exercise. □

Remark 8.26. Let f be a function given in Theorem 8.25, x_0 be a jump discontinuity of f, and $I = (x_0, x_0 + r)$ for some $r > 0$ so that f is continuous on I. By the definition of the right limit, there exists $0 < \delta < r$ such that

$$\big|f(x) - f(x_0^+)\big| < \frac{c|a|}{2} \qquad \forall\, x \in (x_0, x_0 + \delta)\,.$$

Choose $N > 0$ such that $\frac{L}{N} < \delta$. Then $x_0 + \frac{L}{N} \in (x_0, x_0 + \delta)$ for all $n \geqslant N$; thus if $n \geqslant N$,

$$\begin{aligned}
\sup_{x\in I}\big|s_n(f,x) - f(x)\big| &\geqslant \big|s_n(f, x_0 + \frac{L}{N}) - f(x_0 + \frac{L}{N})\big| \\
&\geqslant \big|s_n(f, x_0 + \frac{L}{N}) - f(x_0^+)\big| - \big|f(x_0 + \frac{L}{N}) - f(x_0^+)\big| \\
&\geqslant \big|s_n(f, x_0 + \frac{L}{N}) - f(x_0^+)\big| - \frac{c|a|}{2}
\end{aligned}$$

which implies that

$$\liminf_{n\to\infty}\sup_{x\in I}\big|s_n(f,x)-f(x)\big|\geqslant c|a|-\frac{c|a|}{2}=\frac{c|a|}{2}\,.$$

Therefore, $\{s_n(f,\cdot)\}_{n=1}^\infty$ does not converge uniformly (to f) on I, while Corollary 8.24 shows that $\{s_n(f,\cdot)\}_{n=1}^\infty$ converges pointwise to f on I. Similarly, if x_0 is a jump discontinuity of f and f is continuous on (x_0-r,x_0) for some $r>0$, then $\{s_n(f,\cdot)\}_{n=1}^\infty$ converge pointwise but not uniformly on (x_0-r,x_0).

For a function f given in Theorem 8.25, let $\widetilde{f}$ be defined by

$$\widetilde{f}(x)=\left\{\begin{array}{cl} f(x) & \text{if } f \text{ is continuous at } x\,,\\ \dfrac{f(x^+)+f(x^-)}{2} & \text{if } x \text{ is a discontinuity of } f\,.\end{array}\right.$$

Then $s_n(\widetilde{f},\cdot)=s_n(f,\cdot)$ for all $n\in\mathbb{N}$, and Corollary 8.24 and Theorem 8.25 together imply that $\{s_n(f,\cdot)\}_{n=1}^\infty$ converges pointwise to $\widetilde{f}$. However, the discussion above shows that $\{s_n(f,\cdot)\}_{n=1}^\infty$ cannot converge uniformly on I as long as I contains jump discontinuities of f.

8.5 The Inner-Product Point of View

除了逐點收斂或均勻收斂的觀點之外，還有一個更自然（就數學而言）的觀點可以用來看 Fourier series。我們可以把定義在 $[-\pi,\pi]$ 的所有 square integrable 函數（定義在下）所形成的集合看成一個向量空間，然後在上面定義一個內積的結構。一個可積分函數（也可視為一個向量）的 Fourier series 可以看成這個向量在一組正交基底向量的線性組合。

Let $L^2(\mathbb{T})$ denote the collection of Riemann measurable, square integrable function on $[-\pi,\pi]$ modulo the relation that $f\sim g$ if $f-g=0$ except on a set of measure zero (or $f=g$ almost everywhere). In other words,

$$L^2(\mathbb{T})=\Big\{f:[-\pi,\pi]\to\mathbb{C}\,\Big|\,f \text{ is Riemann measurable and } \int_{-\pi}^{\pi}|f(x)|^2\,dx<\infty\Big\}\Big/\sim\,.$$

Here we **abuse** the use of notation $L^2(\mathbb{T})$ for that it indeed denotes a more general space. We also note that the domain $[-\pi,\pi]$ can be replaced by any intervals with $-\pi$, π as end-points for we can easily modify functions defined on those domains to functions defined on $[-\pi,\pi]$ without changing the square integrability.

Define a bilinear function $\langle\cdot,\cdot\rangle$ on $L^2(\mathbb{T})\times L^2(\mathbb{T})$ by

$$\langle f,g\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\overline{g(x)}\,dx\,.$$

Then $\langle\cdot,\cdot\rangle$ is an inner product on $L^2(\mathbb{T})$. Indeed, if f,g belong to $L^2(\mathbb{T})$, then the product $f\bar{g}$ is also Riemann measurable, and the Cauchy-Schwartz inequality as well as the monotone convergence theorem imply that

$$\begin{aligned}\big|\langle f,g\rangle\big|&=\lim_{k\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|(f\wedge k)(x)\big|\big|(g\wedge k)(x)\big|\,dx\\&\leqslant\lim_{k\to\infty}\frac{1}{2\pi}\Big(\int_{-\pi}^{\pi}\big|(f\wedge k)(x)\big|^2dx\Big)^{\frac{1}{2}}\Big(\int_{-\pi}^{\pi}\big|(g\wedge k)(x)\big|^2dx\Big)^{\frac{1}{2}}\\&=\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|f(x)\big|^2dx\Big)^{\frac{1}{2}}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|g(x)\big|^2dx\Big)^{\frac{1}{2}}=\|f\|_{L^2(\mathbb{T})}\|g\|_{L^2(\mathbb{T})}<\infty\,;\end{aligned}$$

thus the definition of the inner product $\langle\cdot,\cdot\rangle$ given above is well-defined. The norm induced by the inner product above is denoted by $\|\cdot\|_{L^2(\mathbb{T})}$.

For $k\in\mathbb{Z}$, define $\mathbf{e}_k:[-\pi,\pi]\to\mathbb{C}$ by $\mathbf{e}_k(x)=e^{ikx}$. Then $\{\mathbf{e}_k\}_{k=-\infty}^{\infty}$ is an orthonormal set in $L^2(\mathbb{T})$ since

$$\langle\mathbf{e}_k,\mathbf{e}_\ell\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ikx}e^{-i\ell x}\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(k-\ell)x}\,dx=\begin{cases}1&\text{if }k=\ell\,,\\0&\text{if }k\neq\ell\,.\end{cases}$$

Let $\mathcal{V}_n=\operatorname{span}(\mathbf{e}_{-n},\mathbf{e}_{-n+1},\cdots,\mathbf{e}_0,\mathbf{e}_1,\cdots,\mathbf{e}_n)=\Big\{\sum\limits_{k=-n}^{n}a_k\mathbf{e}_k\,\Big|\,\{a_k\}_{k=-n}^{n}\subseteq\mathbb{C}\Big\}$. For each vector $f\in L^2(\mathbb{T})$, the orthogonal projection of f onto $\mathcal{V}_n$ is, conceptually, given by

$$\sum_{k=-n}^{n}\langle f,\mathbf{e}_k\rangle\mathbf{e}_k=\sum_{k=-n}^{n}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-ikx}\,dx\Big)\mathbf{e}_k=\sum_{k=-n}^{n}\widehat{f}_k\mathbf{e}_k\,.$$

By the definition of $\mathbf{e}_k$, we obtain that the orthogonal projection of f on $\mathcal{V}_n$ is exactly $s_n(f,\cdot)$. We also note that $\mathcal{V}_n=\mathscr{P}_n(\mathbb{T})$.

Now we prove that $s_n(f,\cdot)$ is exactly the orthogonal projection of f onto $\mathcal{V}_n=\mathscr{P}_n(\mathbb{T})$.

Proposition 8.27. *Let $f\in L^2(\mathbb{T})$. Then*

$$\langle f-s_n(f,\cdot),p\rangle=0\qquad\forall\,p\in\mathscr{P}_n(\mathbb{T})\,.$$

Proof. Let $p\in\mathscr{P}_n(\mathbb{T})$. Then $p=s_n(p,\cdot)$; thus

$$\begin{aligned}\langle f-s_n(f,\cdot),p\rangle=\langle f,p\rangle-\langle s_n(f,\cdot),p\rangle&=\Big\langle f,\sum_{k=-n}^{n}\widehat{p}_k\mathbf{e}_k\Big\rangle-\Big\langle\sum_{k=-n}^{n}\widehat{f}_k\mathbf{e}_k,p\Big\rangle\\&=\sum_{k=-n}^{n}\overline{\widehat{p}_k}\langle f,\mathbf{e}_k\rangle-\sum_{k=-n}^{n}\widehat{f}_k\overline{\langle p,\mathbf{e}_k\rangle}=\sum_{k=-n}^{n}\overline{\widehat{p}_k}\widehat{f}_k-\sum_{k=-n}^{n}\widehat{f}_k\overline{\widehat{p}_k}=0\,.\end{aligned}$$

□

Theorem 8.28. *Let $f \in L^2(\mathbb{T})$. Then*

$$\|f-p\|^2_{L^2(\mathbb{T})} = \|f - s_n(f,\cdot)\|^2_{L^2(\mathbb{T})} + \|s_n(f,\cdot) - p\|^2_{L^2(\mathbb{T})} \qquad \forall\, p \in \mathscr{P}_n(\mathbb{T})\,. \tag{8.5.1}$$

Proof. By Proposition 8.27, if $p \in \mathscr{P}_n(\mathbb{T})$, $s_n(f,\cdot) - p = s_n(f-p,\cdot) \in \mathscr{P}_n(\mathbb{T})$; thus

$$\begin{aligned}\|f-p\|^2_{L^2(\mathbb{T})} &= \langle f-p, f-p\rangle = \langle f - s_n(f,\cdot) + s_n(f,\cdot) - p, f - s_n(f,\cdot) + s_n(f,\cdot) - p\rangle \\ &= \big\|f - s_n(f,\cdot)\big\|^2_{L^2(\mathbb{T})} + 2\mathrm{Re}\big(\langle f - s_n(f,\cdot), s_n(f,\cdot) - p\rangle\big) + \big\|s_n(f,\cdot) - p\big\|^2_{L^2(\mathbb{T})} \\ &= \big\|f - s_n(f,\cdot)\big\|^2_{L^2(\mathbb{T})} + \big\|s_n(f,\cdot) - p\big\|^2_{L^2(\mathbb{T})}\end{aligned}$$

which concludes the proposition. □

We note that (8.5.1) implies that

$$\|f - s_n(f,\cdot)\|_{L^2(\mathbb{T})} \leqslant \|f-p\|_{L^2(\mathbb{T})} \qquad \forall\, p \in \mathscr{P}_n(\mathbb{T})\,. \tag{8.5.2}$$

Since $s_n(f,\cdot) \in \mathscr{P}_n(\mathbb{T})$, we conclude that

$$\|f - s_n(f,\cdot)\|_{L^2(\mathbb{T})} = \inf_{p\in\mathscr{P}_n(\mathbb{T})} \|f-p\|_{L^2(\mathbb{T})}\,.$$

Moreover, letting $p = 0$ in (8.5.1) we establish the famous Bessel's inequality.

Corollary 8.29. *Let $f \in L^2(\mathbb{T})$. Then for all $n \in \mathbb{N}$,*

$$\|s_n(f,\cdot)\|_{L^2(\mathbb{T})} \leqslant \|f\|_{L^2(\mathbb{T})}\,. \tag{8.5.3}$$

In particular,

$$\sum_{k=-\infty}^{\infty} |\widehat{f}_k|^2 \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(x)\big|^2\,dx\,. \qquad \text{(Bessel's inequality)}$$

Remark 8.30. When $f \in L^2(\mathbb{T})$ and f is real-valued, then

$$\sum_{k=-\infty}^{\infty} |\widehat{f}_k|^2 = \frac{c_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}(c_k^2 + s_k^2)\,;$$

thus in this case the Bessel inequality can also be written as

$$\frac{c_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}(c_k^2 + s_k^2) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(x)\big|^2\,dx\,.$$

Next, we prove that the Bessel inequality is in fact an equality, called the Parseval identity. Using (8.5.1), it is equivalent to that $\big\{s_n(f,\cdot)\big\}_{n=1}^\infty$ converges to f in the sense of L^2-norm; that is,

$$\lim_{n\to\infty}\big\|s_n(f,\cdot)-f\big\|_{L^2(\mathbb{T})}=0 \qquad \forall\, f\in L^2(\mathbb{T})\,.$$

Before proceeding, we first prove that every $f\in L^2(\mathbb{T})$ can be approximated by a sequence $\{g_n\}_{n=1}^\infty\subseteq\mathscr{C}(\mathbb{T})$ in the sense of L^2-norm.

Proposition 8.31. *Let $f\in L^2(\mathbb{T})$. Then for all $\varepsilon>0$ there exists $g\in\mathscr{C}(\mathbb{T})$ (here $\mathscr{C}(\mathbb{T})$ denotes the collections of 2π-periodic complex-valued continuous functions on $\mathbb{R}$) such that*

$$\|f-g\|_{L^2(\mathbb{T})}<\varepsilon\,.$$

In other words, $\mathscr{C}(\mathbb{T})$ is dense in $\big(L^2(\mathbb{T}),\|\cdot\|_{L^2(\mathbb{T})}\big)$.

Proof. W.L.O.G., we can assume that f is real-valued and non-zero. Let $\varepsilon>0$ be given. Since $f\in L^2(\mathbb{T})$, the monotone convergence theorem (Corollary 6.105) implies that

$$\lim_{k\to\infty}\|f-(-k)\vee(f\wedge k)\|^2_{L^2(\mathbb{T})}=\lim_{k\to\infty}\int_{-\pi}^{\pi}\mathbf{1}_{\{|f(x)|>k\}}(x)|f(x)|^2\,dx=0\,;$$

thus there exists $N>0$ such that

$$\|f-(-k)\vee(f\wedge k)\|_{L^2(\mathbb{T})}<\frac{\varepsilon}{2}\qquad\forall\, k\geqslant N\,.$$

Let $h=(-N)\vee(f\wedge N)$. Then h is bounded and Riemann measurable; thus h is Riemann integrable on $[-\pi,\pi]$. Therefore, there exists a partition $\mathcal{P}=\{-\pi=x_0<x_1<\cdots<x_n=\pi\}$ of $[-\pi,\pi]$ such that $U(h,\mathcal{P})-L(h,\mathcal{P})<\dfrac{\pi\varepsilon^2}{8N}$. Define

$$S(x)=\sum_{k=0}^{n-1}\sup_{\xi\in[x_k,x_{k+1}]}h(\xi)\mathbf{1}_{[x_k,x_{k+1}]}(x)\quad\text{ and }\quad s(x)=\sum_{k=0}^{n-1}\inf_{\xi\in[x_k,x_{k+1}]}h(\xi)\mathbf{1}_{[x_k,x_{k+1}]}(x)\,,$$

where $\mathbf{1}_A$ denotes the characteristic/indicator function of set A. Then

1. $-N\leqslant s\leqslant h\leqslant S\leqslant N$ on $[-\pi,\pi]\backslash\{x_1,x_2,\cdots,x_{n-1}\}$;
2. $\displaystyle\int_{-\pi}^{\pi}S(x)\,dx=U(h,\mathcal{P})$; 3. $\displaystyle\int_{-\pi}^{\pi}s(x)\,dx=L(h,\mathcal{P})$.

The properties above show that

$$\begin{aligned}\int_{-\pi}^{\pi}\big|h(x)-s(x)\big|\,dx&=\int_{-\pi}^{\pi}h(x)-s(x)\,dx\leqslant\int_{-\pi}^{\pi}\big(S(x)-s(x)\big)\,dx\\&=U(h,\mathcal{P})-L(h,\mathcal{P})<\frac{\pi\varepsilon^2}{8N}\,.\end{aligned}$$

Now, similar to the construction of g and h in the proof of Lemma 6.63, for the step function s defined on $[-\pi,\pi]$ we can always find a continuous function $g\in\mathscr{C}(\mathbb{T})$ such that

1. $\|g\|_{L^\infty(\mathbb{T})}\leqslant N$. 2. $\displaystyle\int_{-\pi}^{\pi}\big|s(x)-g(x)\big|\,dx<\frac{\pi\varepsilon^2}{8N}$.

Therefore,

$$\int_{-\pi}^{\pi}\big|h(x)-g(x)\big|\,dx\leqslant\int_{-\pi}^{\pi}\big|h(x)-s(x)\big|\,dx+\int_{-\pi}^{\pi}\big|s(x)-g(x)\big|\,dx<\frac{\pi\varepsilon^2}{4N}$$

which implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|h(x)-g(x)\big|^2\,dx\leqslant\frac{N}{\pi}\int_{[-\pi,\pi]}\big|h(x)-g(x)\big|\,dx<\frac{\varepsilon^2}{4}\,;$$

thus $\|h-g\|_{L^2(\mathbb{T})}<\dfrac{\varepsilon}{2}$. The proposition is then concluded by the triangle inequality. □

Theorem 8.32. *Let $f\in L^2(\mathbb{T})$. Then*

$$\lim_{n\to\infty}\big\|f-s_n(f,\cdot)\big\|_{L^2(\mathbb{T})}=0 \tag{8.5.4}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|f(x)\big|^2\,dx=\sum_{k=-\infty}^{\infty}|\widehat{f}_k|^2\,. \qquad \text{(Parseval's identity)}$$

Proof. Let $\varepsilon>0$ be given. By Proposition 8.31 there exists $g\in\mathscr{C}(\mathbb{T})$ such that

$$\|f-g\|_{L^2(\mathbb{T})}<\frac{\varepsilon}{3}\,.$$

By the denseness of the trigonometric polynomials in $\mathscr{C}(\mathbb{T})$, there exists $h\in\mathscr{P}(\mathbb{T})$ such that $\sup\limits_{x\in\mathbb{R}}\big|g(x)-h(x)\big|<\dfrac{\varepsilon}{3}$. Suppose that $h\in\mathscr{P}_N(\mathbb{T})$. Using (8.5.2),

$$\big\|g-s_N(g,\cdot)\big\|^2_{L^2(\mathbb{T})}\leqslant\|g-h\|^2_{L^2(\mathbb{T})}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|g(x)-h(x)\big|^2\,dx\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\varepsilon^2}{9}\,dx=\frac{\varepsilon^2}{9}\,.$$

Since $s_N(g,\cdot)\in\mathscr{P}_n(\mathbb{T})$ if $n\geqslant N$, using (8.5.2) again we must have

$$\big\|g-s_n(g,\cdot)\big\|_{L^2(\mathbb{T})}\leqslant\big\|g-s_N(g,\cdot)\big\|_{L^2(\mathbb{T})}\leqslant\frac{\varepsilon}{3}\qquad\forall\,n\geqslant N\,.$$

Therefore, for $n\geqslant N$, inequality (8.5.3) and the triangle inequality yield that

$$\begin{aligned}\big\|f-s_n(f,\cdot)\big\|_{L^2(\mathbb{T})}&\leqslant\|f-g\|_{L^2(\mathbb{T})}+\|g-s_n(g,\cdot)\|_{L^2(\mathbb{T})}+\|s_n(g-f,\cdot)\|_{L^2(\mathbb{T})}\\&\leqslant 2\|f-g\|_{L^2(\mathbb{T})}+\|g-s_n(g,\cdot)\|_{L^2(\mathbb{T})}<\varepsilon\,;\end{aligned}$$

thus (8.5.4) is concluded. Finally, using (8.5.1) with $p=0$ we obtain that

$$\int_{-\pi}^{\pi} \big|f(x)\big|^2\,dx = \int_{-\pi}^{\pi} \big|s_n(f,x)\big|^2\,dx + \int_{-\pi}^{\pi} \big|f(x)-s_n(f,x)\big|^2\,dx\,.$$

Using the fact that $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \big|s_n(f,x)\big|^2\,dx = \sum_{k=-n}^{n} \big|\widehat{f}_k\big|^2$ and passing to the limit as $n\to\infty$, we conclude the Parseval identity. ◻

Example 8.33. Example 8.6 provides that $\displaystyle\int_{-\pi}^{\pi} x^2\,dx = \pi\sum_{k=1}^{\infty}\frac{4}{k^2}$; thus $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$.

Remark 8.34. The Parseval identity implies that

$$\langle f,g\rangle_{L^2(\mathbb{T})} = \sum_{k=-\infty}^{\infty} \widehat{f}_k\overline{\widehat{g}_k} \qquad \forall\, f,g\in L^2(\mathbb{T}) \tag{8.5.5}$$

since the polarization identity shows that

$$\begin{aligned}
\langle f,g\rangle_{L^2(\mathbb{T})} &= \frac{1}{4}\Big[\|f+g\|^2_{L^2(\mathbb{T})} - \|f-g\|^2_{L^2(\mathbb{T})} + i\|f+ig\|^2_{L^2(\mathbb{T})} - i\|f-ig\|^2_{L^2(\mathbb{T})}\Big]\\
&= \frac{1}{4}\sum_{k=-\infty}^{\infty}\Big[|\widehat{f}_k+\widehat{g}_k|^2 - |\widehat{f}_k-\widehat{g}_k|^2 + i|\widehat{f}_k+i\widehat{g}_k|^2 - i|\widehat{f}_k-i\widehat{g}_k|^2\Big]\\
&= \frac{1}{4}\sum_{k=-\infty}^{\infty}\Big[\big(|\widehat{f}_k|^2 + 2\mathrm{Re}(\widehat{f}_k\overline{\widehat{g}_k}) + |\widehat{g}_k|^2\big) - \big(|\widehat{f}_k|^2 - 2\mathrm{Re}(\widehat{f}_k\overline{\widehat{g}_k} + |\widehat{g}_k|^2\big)\\
&\qquad\qquad + i\big(|\widehat{f}_k|^2 + 2\mathrm{Im}(\widehat{f}_k\overline{\widehat{g}_k}) + |\widehat{g}_k|^2\big) - i\big(|\widehat{f}_k|^2 - 2\mathrm{Im}(\widehat{f}_k\overline{\widehat{g}_k}) + |\widehat{g}_k|^2\big)\Big]\\
&= \sum_{k=-\infty}^{\infty}\big[\mathrm{Re}(\widehat{f}_k\overline{\widehat{g}_k}) + i\mathrm{Im}(\widehat{f}_k\overline{\widehat{g}_k})\big] = \sum_{k=-\infty}^{\infty}\widehat{f}_k\overline{\widehat{g}_k}\,.
\end{aligned}$$

8.6 The Discrete Fourier "Transform" and the Fast Fourier "Transform"

Let $f:\mathbb{R}\to\mathbb{R}$ be a periodic function with period L and f is bounded Riemann integrable on $[0,L)$. Similar to Remark 8.2, the Fourier series of f, defined in Remark 8.3, can be written as

$$s(f,x) = \sum_{k=-\infty}^{\infty}\widehat{f}_k e^{\frac{2\pi ikx}{L}}\,,$$

where $\widehat{f}_k = \dfrac{1}{L}\displaystyle\int_0^L f(y)e^{\frac{-2\pi iky}{L}}\,dy$; thus $\widehat{f}_k$ can be approximated by the Riemann sum

$$\frac{1}{L}\sum_{\ell=0}^{N-1} f\Big(\frac{L\ell}{N}\Big)e^{\frac{-2\pi ik\ell}{N}}\frac{L}{N} = \frac{1}{N}\sum_{\ell=0}^{N-1} f\Big(\frac{L\ell}{N}\Big)e^{\frac{-2\pi ik\ell}{N}}\,.$$

In other words, the values of f at N evenly distributed points can be used to determine an approximation of the Fourier coefficients of f.

There is another point of view of finding the sum $\dfrac{1}{N}\displaystyle\sum_{\ell=0}^{N-1} f\Big(\frac{L\ell}{N}\Big)e^{\frac{-2\pi ik\ell}{N}}$. Even though $s_n(f,x)$ will be a good approximation of $s(f,x)$ for large n, the computation of the exact Fourier coefficients will be expensive (and probably impossible). Therefore, instead of compute the exact Fourier coefficients, we look for a Fourier-like series of the form

$$\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{\frac{2\pi ikx}{L}}\,.$$

so that it agrees with the value of f at points $\Big\{\dfrac{Lj}{N}\Big\}_{j=0}^{N-1}$. Therefore, we look for $\{X_k\}_{k=0}^{N-1}$ satisfying that

$$\frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{\frac{2\pi i}{N}} & e^{\frac{4\pi i}{N}} & \cdots & e^{\frac{2\pi(N-1)i}{N}} \\ 1 & e^{\frac{4\pi i}{N}} & e^{\frac{8\pi i}{N}} & \cdots & e^{\frac{4\pi(N-1)i}{N}} \\ \vdots & & & \ddots & \vdots \\ 1 & e^{\frac{2\pi(N-1)i}{N}} & e^{\frac{4\pi(N-1)i}{N}} & \cdots & e^{\frac{2\pi(N-1)^2 i}{N}} \end{bmatrix}\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{N-1}\end{bmatrix} = \begin{bmatrix} f(0) \\ f\big(\frac{L}{N}\big) \\ \vdots \\ f\Big(\dfrac{(N-1)L}{N}\Big)\end{bmatrix}.$$

Let $\boldsymbol{v}_k = \big[v_k^{(1)}, v_k^{(2)}, \cdots, v_k^{(N)}\big]^{\mathrm{T}}$ denote the k-th column of the $N\times N$ matrix F on the left-hand side of the equation above. Then $v_k^{(j)} = e^{\frac{2\pi(k-1)(j-1)i}{N}}$ so that

$$\begin{aligned}\boldsymbol{v}_\ell\cdot\boldsymbol{v}_k = \boldsymbol{v}_k^*\boldsymbol{v}_\ell &= \sum_{j=1}^{N} e^{\frac{-2\pi(j-1)(k-1)i}{N}} e^{\frac{2\pi(j-1)(\ell-1)i}{N}} = \sum_{j=1}^{N} e^{\frac{2\pi(j-1)(\ell-k)i}{N}} \\ &= \sum_{j=0}^{N-1}\cos\frac{2\pi j(\ell-k)}{N} + i\sum_{j=0}^{N-1}\sin\frac{2\pi j(\ell-k)i}{N}\end{aligned}$$

which shows that

$$\boldsymbol{v}_\ell\cdot\boldsymbol{v}_k = \begin{cases} N & \text{if } k=\ell\,, \\ 0 & \text{if } k\neq\ell\,. \end{cases}$$

Therefore, $F^*F = N\mathrm{I}_{N\times N}$; thus

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-\frac{2\pi i}{N}} & e^{-\frac{4\pi i}{N}} & \cdots & e^{-\frac{2\pi(N-1)i}{N}} \\ 1 & e^{-\frac{4\pi i}{N}} & e^{-\frac{8\pi i}{N}} & \cdots & e^{-\frac{4\pi(N-1)i}{N}} \\ \vdots & & & \ddots & \vdots \\ 1 & e^{-\frac{2\pi(N-1)i}{N}} & e^{-\frac{4\pi(N-1)i}{N}} & \cdots & e^{-\frac{2\pi(N-1)^2 i}{N}} \end{bmatrix} \begin{bmatrix} f(0) \\ f\big(\frac{L}{N}\big) \\ \vdots \\ f\big(\frac{(N-1)L}{N}\big) \end{bmatrix}.$$

The discussions above induce the following

Definition 8.35. The ***discrete Fourier transform***, symbolized by DFT, of a sequence of N complex numbers $\{x_0, x_1, \cdots, x_{N-1}\}$ is a sequence $\{X_k\}_{k\in\mathbb{Z}}$ defined by

$$X_k = \sum_{\ell=0}^{N-1} x_\ell e^{\frac{-2\pi i k\ell}{N}} \qquad \forall\, k \in \mathbb{Z}\,.$$

We note that the sequence $\{X_k\}_{k\in\mathbb{Z}}$ is N-periodic; that is, $X_{k+N} = X_k$ for all $k \in \mathbb{Z}$. Therefore, often time we only focus on one of the following N consecutive terms $\{X_0, X_1, \cdots, X_{N-1}\}$ of the DFT.

Example 8.36. The DFT of the sequence $\{x_0, x_1\}$ is $\{x_0 + x_1, x_0 - x_1\}$.

8.6.1 The inversion formula

Let $\{X_k\}_{k=0}^{N-1}$ be the discrete Fourier transform of the sequence $\{x_\ell\}_{\ell=0}^{N-1}$. Then $\{x_\ell\}_{\ell=0}^{N-1}$ can be recovered given $\{X_k\}_{k=0}^{N-1}$ by the inversion formula

$$x_\ell = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{\frac{2\pi i k\ell}{N}}\,. \tag{8.6.1}$$

To see this, we compute $\sum\limits_{k=0}^{N-1}\Big(\sum\limits_{j=0}^{N-1} x_j e^{\frac{-2\pi i kj}{N}}\Big)e^{\frac{2\pi i k\ell}{N}}$ and obtain that

$$\begin{aligned}\sum_{k=0}^{N-1}\Big(\sum_{j=0}^{N-1} x_j e^{\frac{-2\pi i kj}{N}}\Big)e^{\frac{2\pi i k\ell}{N}} &= \sum_{j=0}^{N-1}\Big(x_j \sum_{k=0}^{N-1} e^{\frac{2\pi i k(\ell-j)}{N}}\Big) = Nx_\ell + \sum_{\substack{j=0\\ j\neq\ell}}^{N-1}\Big(x_j\sum_{k=0}^{N-1} e^{\frac{2\pi i k(\ell-j)}{N}}\Big) \\ &= Nx_\ell + \sum_{\substack{j=0\\ j\neq\ell}}^{N-1}\Big(x_j \frac{e^{2\pi i(\ell-j)}-1}{e^{\frac{2\pi i(\ell-j)}{N}}-1}\Big) = Nx_\ell\,.\end{aligned}$$

The map from $\{X_k\}_{k=0}^{N-1}$ to $\{x_\ell\}_{\ell=0}^{N-1}$ is called the ***discrete inverse Fourier transform***.

We note that the inversion formula (8.6.1) is an analogy of

$$f(x) = \sum_{k=-\infty}^{\infty} \widehat{f}_k e^{ikx}$$

for all piecewise constant function f and $x \in \mathbb{R}$ at which f is continuous.

Remark 8.37. Given a sample data $[x_0, x_1, \cdots, x_{N-1}]$ which is the values of a function f on N evenly distributed points on $[0, L)$ (for some unknown $L > 0$), the DFT $[X_0, X_1, \cdots, X_{N-1}]$ can be thought as Fourier coefficients which provides the approximation

$$f(x) \approx \sum_{k=0}^{N-1} X_k e^{\frac{2\pi ikx}{L}} = \sum_{k=-[\frac{N}{2}]}^{-1} X_{k+N} e^{\frac{2\pi ikx}{L}} + \sum_{k=0}^{[\frac{N-1}{2}]} X_k e^{\frac{2\pi ikx}{L}},$$

where $\approx$ becomes $=$ if $x = \frac{L\ell}{N}$, $0 \leqslant \ell \leqslant N-1$. Therefore, for $0 \leqslant k \leqslant \big[\frac{N-1}{2}\big]$ each X_k is the coefficient associated with the wave with frequency $\frac{k}{L}$. To determine L, we introduce the ***sampling frequency*** F_s which is the number of samples per unit time/length. Then $F_s = \frac{N}{L}$ so that X_k is the coefficient associated with the wave with frequency $\frac{F_s}{N}k$.

8.6.2 The fast Fourier transform

Let $M = [m_{k\ell}]$ be an $N \times N$ matrix with entry $m_{k\ell}$ defined by

$$m_{k\ell} = e^{\frac{-2\pi ik\ell}{N}} \qquad 0 \leqslant k, \ell \leqslant N-1,$$

and write $\boldsymbol{x} = [x_0, x_1, \cdots, x_{N-1}]^{\mathrm{T}}$ and $\boldsymbol{X} = [X_0, \cdots, X_{N-1}]^{\mathrm{T}}$. Then $\boldsymbol{X} = M\boldsymbol{x}$ and it requires N^2 multiplications to compute $\boldsymbol{X}$. The ***fast Fourier transform***, symbolized by FFT, is a much faster way to compute $\boldsymbol{X}$. In the following, we show that when $N = 2^\gamma$ for some $\gamma \in \mathbb{N}$, then there is a way to compute the DFT with at most $N \log_2 N$ multiplications.

With $N = 2^\gamma$, suppose that $(x_0, \cdots, x_{N-1})$ is a given sequence, and $\{X_k\}_{k=0}^{N-1}$ is the DFT of $\{x_k\}_{k=0}^{N-1}$. Let $\omega = e^{-\frac{2\pi i}{N}}$, and

$$\boldsymbol{x}_{\text{even}} = \begin{bmatrix} x_0 & x_2 & x_4 & \cdots & x_{N-2} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{x}_{\text{odd}} = \begin{bmatrix} x_1 & x_3 & x_5 & \cdots & x_{N-1} \end{bmatrix}$$

Then

$$\begin{aligned} X_j = \sum_{\ell=0}^{N-1} x_\ell\, \omega^{j\ell} &= \sum_{\substack{0\leqslant \ell \leqslant N-1 \\ \ell \text{ is even}}} x_\ell \omega^{j\ell} + \omega^j \sum_{\substack{0\leqslant \ell \leqslant N-1 \\ \ell \text{ is odd}}} x_\ell\, \omega^{j(\ell-1)} \\ &= \boldsymbol{x}_{\text{even}} \cdot \begin{bmatrix} \omega^0 & \omega^{2j} & \omega^{4j} & \cdots & \omega^{j(N-2)} \end{bmatrix} + \omega^j \boldsymbol{x}_{\text{odd}} \cdot \begin{bmatrix} \omega^0 & \omega^{2j} & \omega^{4j} & \cdots & \omega^{j(N-2)} \end{bmatrix}. \end{aligned}$$

In particular, for $0 \leqslant j \leqslant \dfrac{N}{2} - 1$,

$$
\begin{aligned}
X_{\frac{N}{2}+j} &= \boldsymbol{x}_{\text{even}} \cdot \left[\omega^0 \ \omega^{2(\frac{N}{2}+j)} \ \omega^{4(\frac{N}{2}+j)} \ \cdots \ \omega^{(\frac{N}{2}+j)(N-2)}\right] \\
&\quad + \omega^{\frac{N}{2}+j} \boldsymbol{x}_{\text{odd}} \cdot \left[\omega^0 \ \omega^{2(\frac{N}{2}+j)} \ \omega^{4(\frac{N}{2}+j)} \ \cdots \ \omega^{(\frac{N}{2}+j)(N-2)}\right] \\
&= \boldsymbol{x}_{\text{even}} \cdot \left[\omega^0 \ \omega^{2j} \ \omega^{4j} \ \cdots \ \omega^{j(N-2)}\right] - \omega^j \boldsymbol{x}_{\text{odd}} \cdot \left[\omega^0 \ \omega^{2j} \ \omega^{4j} \ \cdots \ \omega^{j(N-2)}\right],
\end{aligned}
$$

where we have used the fact that $\omega^{\frac{N}{2}} = -1$ to conclude the equality. We note that

$$
\left\{\boldsymbol{x}_{\text{even}} \cdot \left[\omega^0 \ \omega^{2j} \ \omega^{4j} \ \cdots \ \omega^{j(N-2)}\right]\right\}_{j=0}^{N/2}
$$

is exactly the DFT of the sequence $\{x_0, x_2, \cdots, x_{N-2}\}$ and

$$
\left\{\boldsymbol{x}_{\text{odd}} \cdot \left[\omega^0 \ \omega^{2j} \ \omega^{4j} \ \cdots \ \omega^{j(N-1)}\right]\right\}_{j=0}^{N/2}
$$

is exactly the DFT of the sequence $\{x_1, x_3, \cdots, x_{N-1}\}$. In other words, to compute the DFT of $\{x_0, \cdots, x_{N-1}\}$, where $N = 2^\gamma$, it suffices to compute the DFTs of the sequence $\{x_0, x_2, \cdots, x_{N-2}\}$ and $\{x_1, x_3, \cdots, x_{N-1}\}$. If the DFTs of the sequences $\{x_0, x_2, \cdots, x_{N-2}\}$ and $\{x_1, x_3, \cdots, x_{N-1}\}$ are known, it requires another $\dfrac{N}{2}$ multiplications to compute the DFT of $\{x_0, x_1, \cdots, x_{N-1}\}$.

Now we compute the total multiplications it requires to compute the DFT of the sequence $\{x_k\}_{k=0}^{2^\gamma-1}$ using the procedure above. Suppose that to compute the DFT of $\{x_k\}_{k=0}^{2^\gamma-1}$ requires $f(\gamma)$ multiplications. Then

$$
f(\gamma) = 2f(\gamma-1) + 2^{\gamma-1}.
$$

It is easy to see that it requires no multiplication to compute the DFT of $\{x_0, x_1\}$ since it is simply $\{x_0 + x_1, x_0 - x_1\}$; thus $f(1) = 0$. Solving the iteration relation above, we obtain that $f(\gamma) = 2^{\gamma-1}(\gamma - 1)$ which implies the total multiplications requires to compute the DFT of $\{x_k\}_{k=0}^{N-1}$, where $N = 2^\gamma$, is $\dfrac{N}{2}(\log_2 N - 1)$.

Example 8.38. To compute the DFT of $\{x_0, x_1, \cdots, x_7\}$, we first compute the DFT of $\{x_0, x_2, x_4, x_6\}$ and $\{x_1, x_3, x_5, x_7\}$, and it requires another 4 multiplications (to compute the multiplication of ω^j and the j-th term of the DFT of $\{x_1, x_3, x_5, x_7\}$ for $0 \leqslant j \leqslant 3$). Nevertheless, instead of computing the DFT of $\{x_0, x_2, x_4, x_6\}$ and $\{x_1, x_3, x_5, x_7\}$ directly using matrix multiplication $\boldsymbol{X} = M\boldsymbol{x}$, we again divide the sequence of length 4 into further shorter sequence $\{x_0, x_4\}$, $\{x_2, x_6\}$, $\{x_1, x_5\}$ and $\{x_3, x_7\}$. Once the DFT of those sequence of length 2 are computed, it requires another $2 \times 2 = 4$ multiplications to compute the DFT of $\{x_0, x_2, x_4, x_6\}$ and $\{x_1, x_3, x_5, x_7\}$. By Example 8.36, it does not require any multiplications to compute the DFT of sequences of length 2; thus the total multiplications required to compute the DFT of $\{x_0, x_1, \cdots, x_7\}$ is $4 + 4 = 8$.

8.7 Fourier Series for Functions of Two Variables

In this section we briefly introduce the Fourier series of complex-valued functions defined on $\Omega \equiv [-L_1, L_1] \times [-L_2, L_2]$. Let

$$L^2(\Omega) = \Big\{ f : \Omega \to \mathbb{C} \,\Big|\, \int_\Omega \big|f(x_1, x_2)\big|^2 \, d(x_2, x_2) < \infty \Big\} \Big/ \sim$$

equipped with the inner product

$$\langle f, g \rangle \equiv \frac{1}{\nu(\Omega)} \int_\Omega f(x_1, x_2) \overline{g(x_1, x_2)} \, d(x_1, x_2) \,,$$

where $\nu(\Omega)$ denotes the area of Ω and $\sim$ again denotes the equivalence relation defined by $f \sim g$ if and only if $f - g = 0$ except on a set of measure zero. Let $\mathbf{e}_{k\ell}(\boldsymbol{x}) = e^{i\pi(\frac{k}{L_1}, \frac{\ell}{L_2}) \cdot \boldsymbol{x}}$, here $\boldsymbol{x} = (x_1, x_2)$. Then $\{\mathbf{e}_{k\ell}\}_{k,\ell \in \mathbb{Z}}$ is a complete orthonormal set in $L^2(\Omega)$; that is, for each $f \in L^2(\Omega)$, by defining the partial sum

$$s_{n,m}(f, \boldsymbol{x}) = \sum_{k=-n}^{n} \sum_{\ell=-m}^{m} \langle f, \mathbf{e}_{k\ell} \rangle \, \mathbf{e}_{k\ell}(\boldsymbol{x})$$

we have

$$\lim_{n,m \to \infty} \big\| f - s_{n,m}(f, \cdot) \big\|_{L^2(\Omega)} = 0 \,,$$

where $\| \cdot \|_{L^2(\Omega)}$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle$. The limit of $s_{n,m}(f, \cdot)$, as $n, m \to \infty$, in the inner product space $\big(L^2(\Omega), \langle \cdot, \cdot \rangle\big)$ is denoted by

$$s(f, \cdot) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \langle f, \mathbf{e}_{k\ell} \rangle \, \mathbf{e}_{k\ell}$$

and is called the Fourier series of f.

Given a collection of data $\{x_{mn}\}_{0 \leqslant n \leqslant M-1, 0 \leqslant n \leqslant N-1}$, the discrete Fourier transform (or simply DFT) of $\{x_{mn}\}_{0 \leqslant n \leqslant M-1, 0 \leqslant n \leqslant N-1}$ is a double sequence $\{X_{k\ell}\}_{k,\ell \in \mathbb{Z}}$ defined by

$$X_{k\ell} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x_{mn} \, \omega_M^{mk} \omega_N^{n\ell} \,,$$

where $\omega_{M} = e^{-\frac{2\pi i}{M}}$ and $\omega_{N} = e^{-\frac{2\pi i}{N}}$. The double sequence $\{X_{k\ell}\}_{k,\ell\in\mathbb{Z}}$ is doubly periodic satisfying $X_{k+M,\ell+N}$ for all $k, \ell \in \mathbb{Z}$; thus we usually only focus on the terms $\{X_{k\ell}\}_{0\leqslant k\leqslant M-1, 0\leqslant \ell\leqslant N-1}$. The discrete inverse Fourier transform of a double sequence $\{X_{k\ell}\}_{0\leqslant k\leqslant M-1, 0\leqslant \ell\leqslant N-1}$ is a double sequence $\{x_{mn}\}_{m,n\in\mathbb{Z}}$ defined by

$$x_{mn} = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{\ell=0}^{N-1} X_{k\ell}\,\overline{\omega_{M}}^{mk}\,\overline{\omega_{N}}^{n\ell},$$

where $\overline{\omega_{M}}$ and $\overline{\omega_{N}}$ are complex conjugate of ω_{M} and ω_{N} defined above.

Chapter 9

Fourier Transforms

Before introducing the Fourier transform, let us "motivate" the idea a little bit. In Section 8.5 we show that $\{\mathbf{e}_n\}_{n=-\infty}^{\infty}$, where $\mathbf{e}_n(x) = e^{inx}$, is a complete orthonormal set in $L^2(\mathbb{T})$. Similarly, let $L^2([-K,K])$ denote the inner-product space

$$L^2([-K,K]) = \big\{f : [-K,K] \to \mathbb{C} \,\big|\, f \text{ is square integrable}\big\}/\sim$$

equipped with the inner product

$$\langle f, g\rangle = \frac{1}{2K}\int_{-K}^{K} f(x)\overline{g(x)}\,dx\,,$$

where $\sim$ denotes the equivalence relation $f \sim g$ if and only if $f - g = 0$ except on a set of measure zero. Then the set $\Big\{\exp\big(\frac{in\pi x}{K}\big)\Big\}_{n=-\infty}^{\infty}$ is a complete orthonormal set in $L^2([-K,K])$; that is, any functions $f \in L^2([-K,K])$ can be expressed as

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{\frac{in\pi x}{K}}\,, \quad \text{where} \quad \widehat{f}(n) = \frac{1}{2K}\int_{-K}^{K} f(y)e^{-\frac{in\pi y}{K}}\,dy\,. \tag{9.0.1}$$

Moreover, $\sum\limits_{n=-\infty}^{\infty}|\widehat{f}(n)|^2 = \dfrac{1}{2K}\displaystyle\int_{-K}^{K}|f(x)|^2\,dx$. In other words, there is a one-to-one correspondence between $f \in L^2([-K,K])$ and $\widehat{f} \in \ell_2$, where ℓ^2 is the collection of square summable sequences; that is,

$$\ell^2 = \Big\{\{a_n\}_{n=-\infty}^{\infty}\,\Big|\, \sum_{n=-\infty}^{\infty}|a_n|^2 < \infty\Big\}\,.$$

We look for a space X so that there is also a one-to-one correspondence between the square integrable functions on $\mathbb{R}$ and X. Intuitively, we can check what "might" happen by letting $K \to \infty$ in (9.0.1).

Suppose that $K \gg 1$ and $K \in \mathbb{N}$. Making use of the Riemann sum to approximate the integral (by partition $[-K\pi, K\pi]$ into $2K^2$ intervals), we find that

$$\begin{aligned} f(x) &= \frac{1}{2K}\sum_{n=-\infty}^{\infty}\int_{-K}^{K} f(y)e^{\frac{in\pi(x-y)}{K}}\,dy \approx \frac{1}{2K}\sum_{n=-K^2}^{K^2-1}\int_{-K}^{K} f(y)\exp\Big[i\frac{n\pi}{K}(x-y)\Big]dy \\ &= \frac{1}{2\pi}\int_{-K}^{K}\Big(\sum_{n=-K^2}^{K^2-1} f(y)\exp\Big[i\frac{n\pi}{K}(x-y)\Big]\frac{\pi}{K}\Big)\,dy \\ &\approx \frac{1}{2\pi}\int_{-K}^{K}\Big(\int_{-K\pi}^{K\pi} f(y)\exp\big(i\xi(x-y)\big)\,d\xi\Big)\,dy = \frac{1}{2\pi}\int_{-K\pi}^{K\pi}\Big(\int_{-K}^{K} f(y)e^{i\xi(x-y)}\,dy\Big)\,d\xi \\ &\approx \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i\xi y}dy\Big]e^{i\xi x}d\xi\,. \end{aligned}$$

Therefore, if we define $\widehat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}} f(y)e^{-iy\xi}\,dy$, then the formal computation above suggests that

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{f}(\xi)e^{i\xi x}d\xi\,. \tag{9.0.2}$$

In the rest of this chapter, we are going to verify the identity above rigorously (for functions f with certain properties).

9.1 The Definition of the Fourier Transform

For notational convenience, we **abuse** the following notion from real analysis.

Definition 9.1. The space $L^1(\mathbb{R}^n)$ consists of all complex-valued functions that are integrable on $\mathbb{R}^n$ and whose integrals are absolutely convergent. In other words,

$$L^1(\mathbb{R}^n) = \Big\{f:\mathbb{R}^n \to \mathbb{C}\,\Big|\,\int_{\mathbb{R}^n}|f(x)|\,dx < \infty\Big\}\,;$$

that is, $f \in L^1(\mathbb{R}^n)$ if the limit $\displaystyle\lim_{R\to\infty}\int_{B(0,R)}\big|f(x)\big|\,dx = \|f\|_{L^1(\mathbb{R}^n)}$ exists.

Remark 9.2. Even though we have not defined the integral for complex-valued function, the definition of $L^1(\mathbb{R}^n)$ should be clear: when f is complex-valued function, the absolute integrability of f is equivalent to that the real part and the imaginary part of f are both absolutely integrable, and

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\,dx &= \int_{\mathbb{R}^n} \mathrm{Re}(f)(x)\,dx + i\int_{\mathbb{R}^n} \mathrm{Im}(f)(x)\,dx \\ &= \int_{\mathbb{R}^n} \frac{f(x)+\overline{f(x)}}{2}\,dx + \int_{\mathbb{R}^n} \frac{f(x)-\overline{f(x)}}{2}\,dx\,,\end{aligned}$$

where $\overline{f(x)}$ is the complex conjugate of $f(x)$.

Definition 9.3. For all $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f, denoted by $\mathscr{F}f$ or $\widehat{f}$, is a function defined by

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx \qquad \forall\, \xi \in \mathbb{R}^n\,,$$

where $x\cdot\xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$.

9.2 Some Properties of the Fourier Transform

Proposition 9.4. *$\mathscr{F} : L^1(\mathbb{R}^n) \to \mathscr{C}_b(\mathbb{R}^n;\mathbb{C})$, and*

$$\|\mathscr{F}f\|_\infty \equiv \sup_{\xi\in\mathbb{R}^n}\big|(\mathscr{F}f)(\xi)\big| \leqslant \|f\|_{L^1(\mathbb{R}^n)}\,. \tag{9.2.1}$$

Proof. Let $\xi \in \mathbb{R}^n$, and $\{\xi_k\}_{k=1}^\infty$ be a sequence converging to ξ. Define

$$g_k(x) = \frac{1}{\sqrt{2\pi}^n} f(x)e^{ix\cdot\xi_k} \quad \text{and} \quad g(x) = \frac{1}{\sqrt{2\pi}^n} f(x)e^{ix\cdot\xi}\,.$$

Then $\{g_k(x)\}_{k=1}^\infty$ converges to $g(x)$ for all $\big\{x \in \mathbb{R}^n \,\big|\, |g(x)| < \infty\big\}$, and for each $k \in \mathbb{N}$, g_k is integrable and $|g_k| \leqslant |f|$; thus the Dominated Convergence Theorem (Theorem 6.102) implies that

$$(\mathscr{F}f)(\xi) = \int_{\mathbb{R}^n} g(x)\,dx = \lim_{k\to\infty}\int_{\mathbb{R}^n} g_k(x)\,dx = \lim_{k\to\infty}(\mathscr{F}f)(\xi_k)\,.$$

Therefore, $(\mathscr{F}f)$ is continuous on $\mathbb{R}^n$. The validity of (9.2.1) should be clear, and is left as an exercise. □

Definition 9.5. A function f on $\mathbb{R}^n$ is said to have rapid decrease/decay if for all integers $N \geqslant 0$, there exists a_N such that

$$|x|^N|f(x)| \leqslant a_N \quad \text{as} \quad x \to \infty\,.$$

Definition 9.6. The *Schwartz space* $\mathscr{S}(\mathbb{R}^n)$ is the collection of all (complex-valued) smooth functions f on $\mathbb{R}^n$ such that f and all of its derivatives have rapid decrease. In other words,

$$\mathscr{S}(\mathbb{R}^n) = \big\{ u \in \mathscr{C}^\infty(\mathbb{R}^n) \,\big|\, |\cdot|^N D^k u \text{ is bounded for all } k, N \in \mathbb{N} \cup \{0\} \big\}.$$

Elements in $\mathscr{S}(\mathbb{R}^n)$ are called Schwartz functions.

The reader is encouraged to verify the following basic properties of $\mathscr{S}(\mathbb{R}^n)$:

1. $\mathscr{S}(\mathbb{R}^n)$ is a vector space.
2. $\mathscr{S}(\mathbb{R}^n)$ is an algebra under the pointwise product of functions.
3. $pu \in \mathscr{S}(\mathbb{R}^n)$ for all $u \in \mathscr{S}(\mathbb{R}^n)$ and all polynomial functions p.
4. $\mathscr{S}(\mathbb{R}^n)$ is closed under differentiation.
5. $\mathscr{S}(\mathbb{R}^n)$ is closed under translations and multiplication by complex exponentials $e^{ix\cdot\xi}$.

Remark 9.7. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and $\mathscr{C}_c^\infty(\Omega)$ denote the collection of all smooth functions with compact support in Ω (or equivalently, compactly supported in Ω); that is,

$$\mathscr{C}_c^\infty(\Omega) \equiv \left\{ u \in \mathscr{C}^\infty(\Omega) \,\middle|\, \{x \in \Omega \,|\, f(x) \neq 0\} \subset\subset \Omega \right\},$$

then $\mathscr{C}_c^\infty(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n)$. The set $\text{cl}\big(\big\{x \in \Omega \,\big|\, f(x) \neq 0\big\}\big)$, where the closure is taken in the metric space $(\Omega, |\cdot|)$, is called the ***support*** of f and is denoted by $\text{supp}(f)$.

The prototype element of $\mathscr{S}(\mathbb{R}^n)$ is $e^{-|x|^2}$ which is not compactly supported, but has rapidly decreasing derivatives.

The following lemma allows us to take the Fourier transform of Schwartz functions.

Lemma 9.8. *If $f \in \mathscr{S}(\mathbb{R}^n)$, then $f \in L^1(\mathbb{R}^n)$.*

Proof. If $f \in \mathscr{S}(\mathbb{R}^n)$, then $(1+|x|)^{n+1}|f(x)| \leqslant C$ for some $C > 0$. Therefore, with ω_{n-1} denoting the the surface area of the $(n-1)$-dimensional unit sphere,

$$\begin{aligned} \int_{\mathbb{R}^n} \big|f(x)\big|\,dx &\leqslant \int_{\mathbb{R}^n} \frac{C}{(1+|x|)^{n+1}}dx = \int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{C}{(1+r)^{n+1}} r^{n-1} dr dS \\ &\leqslant C\omega_{n-1} \int_0^\infty (1+r)^{-2} dr = C\omega_{n-1} \end{aligned}$$

which is a finite number. □

Next we show that $\widehat{f}$ is differentiable if $f \in \mathscr{S}(\mathbb{R}^n)$. Note that if $f \in \mathscr{S}(\mathbb{R}^n)$, then the function $y_j = x_j f(x)$ belongs to $\mathscr{S}(\mathbb{R}^n)$ for all $1 \leqslant j \leqslant n$.

Lemma 9.9. *If $f \in \mathscr{S}(\mathbb{R}^n)$, then $\widehat{f}$ is differentiable, and for each $j \in \{1, \cdots, n\}$, $\dfrac{\partial \widehat{f}}{\partial \xi_j}$ is given by*

$$\frac{\partial \widehat{f}}{\partial \xi_j}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} (-ix_j) f(x) e^{-ix\cdot\xi} dx = \mathscr{F}_x\Big[\frac{1}{i} x_j f(x)\Big](\xi) \,. \tag{9.2.2}$$

Proof. Let $\xi \in \mathbb{R}^n$ and $1 \leqslant j \leqslant n$ be given, and $\{h_k\}_{k=1}^\infty$ be a non-zero sequence converging to 0. Define

$$g_k(x) = \frac{1}{\sqrt{2\pi}^n} f(x) \frac{e^{ix\cdot(\xi + h_k \mathbf{e}_j)} - e^{ix\cdot\xi}}{h_k} = \frac{1}{\sqrt{2\pi}^n} f(x) e^{-ix\cdot\xi} \frac{e^{-ix_j h_k} - 1}{h_k} \,.$$

Note that the mean value theorem implies that

$$\Big|\frac{e^{-ix_j h_k} - 1}{h_k}\Big| = \Big|\frac{\cos(x_j h_k) - \cos 0}{h_k} - i \frac{\sin(x_j h_k)}{h_k}\Big| \leqslant 2|x_j| \,;$$

thus

$$\big|g_k(x)\big| \leqslant \frac{2}{\sqrt{2\pi}^n} \big|x_j f(x)\big| \qquad \forall\, x \in \mathbb{R}^n \text{ and } k \in \mathbb{N} \,.$$

Moreover, the fact that $f \in \mathscr{S}(\mathbb{R}^n)$ implies that the function $y = \dfrac{2}{\sqrt{2\pi}^n} \big|x_j f(x)\big|$ is integrable on $\mathbb{R}^n$. Therefore, by the fact that

$$\int_{\mathbb{R}^n} g_k(x)\, dx = \frac{\widehat{f}(\xi + h_j \mathbf{e}_k) - \widehat{f}(\xi)}{h_j} \qquad \text{and} \qquad \lim_{k\to\infty} g_k(x) = \frac{1}{\sqrt{2\pi}^n} (-ix_j) f(x) e^{-ix\cdot\xi} \,,$$

we conclude from the Dominated Convergence Theorem (Theorem 6.102) that

$$\lim_{k\to\infty} \frac{\widehat{f}(\xi + h_j \mathbf{e}_k) - \widehat{f}(\xi)}{h_j} = \lim_{k\to\infty} \int_{\mathbb{R}^n} g_k(x)\, dx = \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi}^n} (-ix_j) f(x) e^{-ix\cdot\xi}\, dx \,. \qquad \square$$

Corollary 9.10. *If $f \in \mathscr{S}(\mathbb{R}^n)$, then $\widehat{f} \in \mathscr{C}^\infty(\mathbb{R}^n)$. Moreover, if $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index,*

$$D_\xi^\alpha \widehat{f}(\xi) = \frac{1}{i^{|\alpha|}} \mathscr{F}_x\Big[x_1^{\alpha_1} \cdots x_n^{\alpha_n} f(x)\Big](\xi) \,.$$

Lemma 9.11. *If $f \in \mathscr{S}(\mathbb{R}^n)$, then for $j \in \{1, 2, \cdots, n\}$, $\mathscr{F}_x\Big[\dfrac{1}{i}\dfrac{\partial f}{\partial x_j}(x)\Big](\xi) = \xi_j \widehat{f}(\xi)$.*

Proof. W.L.O.G., we assume that $j = n$. Write $x = (x', x_n)$. Since $f \in \mathscr{S}(\mathbb{R}^n)$, there exists $C > 0$ such that

$$(1 + |x'|)^n |x_n| \big|f(x', x_n)\big| \leqslant C \qquad \forall\, x = (x', x_n) \in \mathbb{R}^n\,.$$

Then

1. For each $x' \in \mathbb{R}^{n-1}$, $f(x', \pm R) \to 0$ as $R \to \infty$.
2. The function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ defined by $g(x') = \dfrac{1}{(1 + |x'|)^n}$ is integrable on $\mathbb{R}^{n-1}$ (see the proof of Lemma 9.8), and $\big|f(x', \pm R)\big| \leqslant g(x')$ for each $x' \in \mathbb{R}^{n-1}$ and $R > 1$.

Therefore, the Dominated Convergence Theorem (Theorem 6.102) implies that

$$\lim_{R\to\infty} \int_{[-R,R]^{n-1}} f(x', \pm R) e^{-i(x',R)\cdot\xi}\, dx' = 0\,;$$

thus Fubini's Theorem and integrating by parts formula imply that

$$\begin{aligned}
\mathscr{F}\Big[\frac{1}{i}\frac{\partial f}{\partial x_n}(x)\Big](\xi) &= \frac{1}{i}\frac{1}{\sqrt{2\pi}^n} \lim_{R\to\infty} \int_{[-R,R]^n} \frac{\partial f}{\partial x_n}(x) e^{-ix\cdot\xi} dx \\
&= \frac{1}{i}\frac{1}{\sqrt{2\pi}^n} \lim_{R\to\infty} \int_{[-R,R]^{n-1}} \Big(\int_{-R}^{R} \frac{\partial f}{\partial x_n}(x) e^{-ix\cdot\xi} dx_n\Big) dx' \\
&= \frac{1}{i}\frac{1}{\sqrt{2\pi}^n} \lim_{R\to\infty} \Big[\Big(\int_{[-R,R]^{n-1}} f(x', x_n) e^{-i(x',x_n)\cdot\xi}\, dx'\Big)\Big|_{x_n=-R}^{x_n=R} + i\xi_n \int_{[-R,R]^n} f(x) e^{-ix\cdot\xi} dx\Big] \\
&= \xi_n \frac{1}{\sqrt{2\pi}^n} \lim_{R\to\infty} \int_{[-R,R]^n} f(x) e^{-ix\cdot\xi} dx = \xi_k \widehat{f}(\xi)\,.
\end{aligned}$$

□

Corollary 9.12. $\mathcal{P}(\xi_1, \cdots, \xi_n)\widehat{f}(\xi) = \mathscr{F}_x\Big[\mathcal{P}\Big(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_n}\Big) f(x)\Big](\xi)$ *for all* $f \in \mathscr{S}(\mathbb{R}^n)$ *and polynomial* $\mathcal{P}$.

Corollary 9.13. *The Fourier transform of a Schwartz function is a Schwartz function; that is,* $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$.

Proof. Let $\mathcal{P}$ be a polynomial and $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index. By Corollary 9.10 and 9.12,

$$\begin{aligned}
\mathcal{P}(\xi) D^\alpha \widehat{f}(\xi) &\equiv \mathcal{P}(\xi_1, \cdots, \xi_n) \frac{\partial^{|\alpha|} \widehat{f}}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}}(\xi) \\
&= \frac{1}{i^{|\alpha|}} \mathscr{F}_x\Big[\mathcal{P}\big(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_n}\big)\big[x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} f(x)\big]\Big](\xi)\,;
\end{aligned}$$

thus $\mathcal{P}D^\alpha \widehat{f}$ is the Fourier transform of a Schwartz function g defined by

$$g(x) = \frac{1}{i^{|\alpha|}}\mathcal{P}\Big(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_n}\Big)\big[x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} f(x)\big]\,.$$

By Proposition 9.4 and Lemma 9.8, $\mathcal{P}D^\alpha \widehat{f}$ is bounded. □

Remark 9.14. There exists a duality under $\wedge$ between differentiability and rapid decrease: the more differentiability f possesses, the more rapid decrease $\widehat{f}$ has and vice versa.

Definition 9.15. For all $f \in L^1(\mathbb{R}^n)$, we define operator $\mathscr{F}^*$ by

$$(\mathscr{F}^* f)(x) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}d\xi\,.$$

The function $\mathscr{F}^* f$ sometimes is also denoted by $\check{f}$.

The operator $\mathscr{F}^*$, indicated implicitly by the way it is written, is the formal adjoint of $\mathscr{F}$. To be more precise, we have the following

Lemma 9.16. $\langle \mathscr{F}u, v\rangle_{L^2(\mathbb{R}^n)} = \langle u, \mathscr{F}^* v\rangle_{L^2(\mathbb{R}^n)}$ *for all* $u, v \in \mathscr{S}(\mathbb{R}^n)$*, where* $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n)}$ *is an inner product on* $\mathscr{S}(\mathbb{R}^n)$ *given by*

$$\langle u, v\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)\overline{v(x)}\,dx\,.$$

Proof. Let $u, v \in \mathscr{S}(\mathbb{R}^n)$ be given, and define $f(x,y) = u(x)\overline{v(y)}e^{-ix\cdot y}$. By Tonelli's Theorem (Theorem 6.106), f is absolutely integrable on $\mathbb{R}^n \times \mathbb{R}^n$; thus Fubini's Theorem (Theorem 6.107) implies that

$$\begin{aligned}\langle \mathscr{F}u, v\rangle_{L^2(\mathbb{R}^n)} &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} u(x)e^{-ix\cdot\xi}dx\Big)\overline{v(\xi)}\,d\xi \\ &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x)\overline{e^{ix\cdot\xi}v(\xi)}\,d\xi\,dx \\ &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} u(x)\int_{\mathbb{R}^n}\overline{e^{ix\cdot\xi}v(\xi)}\,d\xi\,dx = \langle u, \mathscr{F}^* v\rangle_{L^2(\mathbb{R}^n)}\,.\end{aligned}$$

□

9.3 The Fourier Inversion Formula

We remind the readers that our goal is to prove (9.0.2), while having introduced operators $\mathscr{F}$ and $\mathscr{F}^*$, it is the same as showing that $\mathscr{F}$ and $\mathscr{F}^*$ are inverse to each other; that is, we want to show that

$$\mathscr{F}\mathscr{F}^* = \mathscr{F}^*\mathscr{F} = \mathrm{Id} \qquad \text{on} \quad \mathscr{S}(\mathbb{R}^n)\,.$$

For a given $t > 0$, let $P_t : \mathbb{R} \to \mathbb{R}$ be defined by $P_t(x) = \dfrac{1}{\sqrt{t}} \exp\big(-\dfrac{x^2}{2t}\big)$. Note that $P_t \in \mathscr{S}(\mathbb{R})$ for all $t > 0$, and P_t is normalized so that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P_t(x)\, dx = 1\,.$$

Now we compute the Fourier transform of P_t. By Lemma 9.9, we find that

$$\begin{aligned}\frac{d\widehat{P}_t}{d\xi}(\xi) &= \frac{-i}{\sqrt{2\pi t}} \int_{\mathbb{R}} x P_t(x) e^{-ix\xi}\, dx \\ &= \frac{-i}{\sqrt{2\pi t}} \int_{\mathbb{R}} x P_t(x) \cos(\xi x)\, dx - \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x P_t(x) \sin(\xi x)\, dx\,.\end{aligned}$$

Since the functions $y = xP_t(x)$ is absolutely integrable on $\mathbb{R}$ for each fixed $t > 0$, the integral $\displaystyle\int_{\mathbb{R}} x P_t(x) \cos(\xi x)\, dx$ converges absolutely; thus by the fact that $x\cos(\xi x)$ are odd functions in x, we have

$$\int_{\mathbb{R}} x P_t(x) \cos(\xi x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} x P_t(x) \cos(\xi x)\, dx = 0\,.$$

As a consequence,

$$\frac{d\widehat{P}_t}{d\xi}(\xi) = -\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x e^{-\frac{x^2}{2t}} \sin(x\xi) dx\,.$$

Integrating by parts,

$$\begin{aligned}\frac{d\widehat{P}_t}{d\xi}(\xi) &= -\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x e^{-\frac{x^2}{2t}} \sin(x\xi) dx = -\frac{1}{\sqrt{2\pi t}} \lim_{R\to\infty} \int_{-R}^{R} x e^{-\frac{x^2}{2t}} \sin(x\xi) dx \\ &= -\frac{1}{\sqrt{2\pi t}} \lim_{R\to\infty} \Big[- t e^{-\frac{x^2}{2t}} \sin(x\xi)\Big|_{x=-R}^{x=R} + \int_{-R}^{R} \xi t e^{-\frac{x^2}{2t}} \cos(x\xi)\, dx\Big] \\ &= -\frac{\xi t}{\sqrt{2\pi t}} \lim_{R\to\infty} \int_{-R}^{R} e^{-\frac{x^2}{2t}} \cos(x\xi)\, dx = -\frac{\xi t}{\sqrt{2\pi t}} \lim_{R\to\infty} \int_{-R}^{R} e^{-\frac{x^2}{2t}} \big[\cos(x\xi) - i\sin(x\xi)\big]\, dx \\ &= -\frac{\xi t}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{x^2}{2t}} e^{-ix\xi} dx = -\xi t \widehat{P}_t(\xi)\,,\end{aligned}$$

thus $\widehat{P}_t(\xi) = Ce^{-\frac{t\xi^2}{2}}$. By the fact that $\widehat{P}_t(0) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}} P_t(x) dx = 1$, we must have

$$\widehat{P}_t(\xi) = e^{-\frac{1}{2}t\xi^2}\,. \tag{9.3.1}$$

For $x \in \mathbb{R}^n$, if we define $\mathrm{P}_t(x) = \displaystyle\prod_{k=1}^{n} P_t(x_k) = \Big(\frac{1}{\sqrt{t}}\Big)^n e^{-\frac{|x|^2}{2t}}$, then (9.3.1) and Fubini's

Theorem imply that $\widehat{\mathrm{P}}_t(\xi) = e^{-\frac{1}{2}t|\xi|^2}$. Therefore,

$$\widehat{\mathrm{P}}_t(\xi) = \Big(\frac{1}{\sqrt{t}}\Big)^n \mathrm{P}_{\frac{1}{t}}(\xi)$$

which, together with the fact that $\check{f}(x) = \widehat{f}(-x)$, further shows that

$$\check{\widehat{\mathrm{P}}}_t(x) = \Big(\frac{1}{\sqrt{t}}\Big)^n \widehat{\mathrm{P}_{\frac{1}{t}}}(-x) = \Big(\frac{1}{\sqrt{t}}\Big)^n \Big(\frac{1}{\sqrt{t^{-1}}}\Big)^n \mathrm{P}_t(-x) = \mathrm{P}_t(x)\,.$$

Similarly, $\widehat{\check{\mathrm{P}}}_t(\xi) = \mathrm{P}_t(\xi)$, so we establish that

$$\mathscr{F}^*\mathscr{F}(\mathrm{P}_t) = \mathscr{F}\mathscr{F}^*(\mathrm{P}_t) = \mathrm{P}_t\,. \tag{9.3.2}$$

The proof of the following lemma is similar to that of Theorem 8.20.

Lemma 9.17. *If $g \in \mathscr{S}(\mathbb{R}^n)$, then $\mathrm{P}_t \star g \to g$ uniformly on $\mathbb{R}^n$ as $t \to 0^+$, where the convolution operator $\star$ is given by*

$$(\mathrm{P}_t \star g)(x) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} \mathrm{P}_t(x-y)g(y)\,dy = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} \mathrm{P}_t(y)g(x-y)\,dy\,. \tag{9.3.3}$$

Proof. Let $\varepsilon > 0$ be given. Since $g \in \mathscr{S}(\mathbb{R}^n)$, g is uniformly continuous; thus there exists $\delta > 0$ such that

$$\big|g(x) - g(y)\big| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |x-y| < \delta\,.$$

Since $\displaystyle\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} \mathrm{P}_t(x)\,dx = 1$, for all $x \in \mathbb{R}^n$ we have

$$\begin{aligned}\big|(\mathrm{P}_t \star g)(x) - g(x)\big| &= \frac{1}{\sqrt{2\pi}^n}\Big|\int_{\mathbb{R}^n} g(x-y)\mathrm{P}_t(y)\,dy - \int_{\mathbb{R}^n} g(x)\mathrm{P}_t(y)\,dy\Big| \\ &= \frac{1}{\sqrt{2\pi}^n}\Big|\int_{\mathbb{R}^n} \big[(g(x-y) - g(x)\big]\mathrm{P}_t(y)\,dy\Big| \\ &\leqslant \frac{\varepsilon}{2}\frac{1}{\sqrt{2\pi}^n}\int_{|y|<\delta} \mathrm{P}_t(y)\,dy + \frac{2\|g\|_\infty}{\sqrt{2\pi}^n}\int_{|y|\geqslant\delta} \mathrm{P}_t(y)\,dy\,,\end{aligned}$$

so we obtain that

$$\sup_{x\in\mathbb{R}^n}\big|(\mathrm{P}_t \star g)(x) - g(x)\big| \leqslant \frac{\varepsilon}{2} + \frac{2\|g\|_\infty}{\sqrt{2\pi}^n}\int_{|y|\geqslant\delta} \mathrm{P}_t(y)\,dy\,.$$

Note that

$$\int_{|y|>\delta} \mathrm{P}_t(y)\,dy = \frac{1}{\sqrt{t}^n}\int_{|y|>\delta} e^{-\frac{|y|^2}{2t}}\,dy = \int_{|z|>\frac{\delta}{\sqrt{t}}} e^{-\frac{|z|^2}{2}}\,dz = \int_{\mathbb{R}^n} \mathbf{1}_{B[0,\frac{\delta}{\sqrt{t}}]^\complement}(z)e^{-\frac{|z|^2}{2}}\,dz$$

which, by the Dominated Convergence Theorem, approaches 0 as $t \to 0^+$; thus there exists $h > 0$ such that if $0 < t < h$,

$$\frac{2\|g\|_\infty}{\sqrt{2\pi}^n} \int_{|y| \geqslant \delta} \mathrm{P}_t(y)\, dy < \frac{\varepsilon}{2}\,.$$

Therefore, we conclude that

$$\sup_{x \in \mathbb{R}^n} \big|(\mathrm{P}_t \star g)(x) - g(x)\big| < \varepsilon \quad \text{whenever} \quad 0 < t < h$$

which shows that $\mathrm{P}_t \star g \to g$ uniformly as $t \to 0^+$. □

Lemma 9.18. *If f and $g \in \mathscr{S}(\mathbb{R}^n)$, then*

$$(\check{f} \star g)(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi} \widehat{g}(\xi)\, d\xi\,.$$

Proof. By definition of $\check{f}$ and the convolution,

$$(\check{f} \star g)(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \check{f}(x-y) g(y)\, dy = \Big(\frac{1}{2\pi}\Big)^n \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} f(\xi) e^{i(x-y)\cdot\xi} g(y)\, d\xi \Big) dy\,.$$

By Tonelli's Theorem (Theorem 6.106), the function $h(\xi, y) = f(\xi) g(y) e^{i(x-y)\cdot\xi}$ is absolutely integrable on $\mathbb{R}^n \times \mathbb{R}^n$; thus Fubini's Theorem (Theorem 6.107) implies that

$$\begin{aligned}
(\check{f} \star g)(x) &= \Big(\frac{1}{2\pi}\Big)^n \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi} e^{-iy\cdot\xi} g(y)\, dy \Big) d\xi \\
&= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi} \Big(\frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} g(y)\, dy \Big) d\xi = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi} \widehat{g}(\xi)\, d\xi\,.
\end{aligned}$$

□

Theorem 9.19 (Fourier Inversion Formula)**.** *If $g \in \mathscr{S}(\mathbb{R}^n)$, then $\check{\widehat{g}}(\xi) = \widehat{\check{g}}(\xi) = g(\xi)$. In other words, $\mathscr{F}\mathscr{F}^* = \mathscr{F}^*\mathscr{F} = \mathrm{Id}$.*

Proof. Applying Lemma 9.18 with $f(\xi) = \widehat{\mathrm{P}}_t(\xi) = e^{-\frac{1}{2}t|\xi|^2}$ and using (9.3.2), we find that

$$(\mathrm{P}_t \star g)(x) = (\check{f} \star g)(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|\xi|^2} e^{ix\cdot\xi} \widehat{g}(\xi)\, d\xi\,.$$

Passing to the limit as $t \to 0^+$, by Lemma 9.17 and Dominated Convergence Theorem (Theorem 6.102) we obtain that

$$g(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \widehat{g}(\xi) e^{ix\cdot\xi} d\xi = \check{\widehat{g}}(x)\,.$$

Let $\widetilde{\ }$ denote the reflection operator given by $\widetilde{f}(x) = f(-x)$. Then the change of variable formula implies that

$$\begin{aligned}\check{g}(\xi) &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(x)e^{ix\cdot\xi}dx = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(x)e^{-i(-x)\cdot\xi}dx \\ &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(-x)e^{-ix\cdot\xi}dx = \widehat{\widetilde{g}}(\xi)\,.\end{aligned}$$

On the other hand,

$$\check{g}(\xi) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(x)e^{-ix\cdot(-\xi)}dx = \widehat{g}(-\xi) = \widetilde{\widehat{g}}(\xi)\,;$$

thus $\widehat{\widetilde{\widehat{g}}}(\xi) = \widehat{\widehat{\widetilde{g}}}(\xi) = \check{\widetilde{g}}(\xi) = g(\xi)$. □

Corollary 9.20. $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ *is a bijection.*

Remark 9.21. In view of the Fourier Inversion Formula (Theorem 9.19), $\mathscr{F}^*$ sometimes is written as $\mathscr{F}^{-1}$, and is called the ***inverse Fourier transform***.

Remark 9.22. In most of the engineering applications the Fourier transform of a function f is defined by

$$\mathscr{F}[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx\,.$$

In this case, the corresponding inverse Fourier transform $\mathscr{F}^{-1}$ and the adjoint Fourier transform $\mathscr{F}^*$ are given by

$$\mathscr{F}^{-1}[f](\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}\,dx \qquad \text{and} \qquad \mathscr{F}^*[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}\,dx$$

so that $\mathscr{F}^{-1} \neq \mathscr{F}^*$. In some applied fields such as the signal processing the Fourier transform of a function f is defined by

$$\mathscr{F}[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi ix\cdot\xi}\,dx\,.$$

In such a case, the inverse Fourier transform and the adjoint Fourier transform are identical and are given by

$$\mathscr{F}^{-1}[f](\xi) = \mathscr{F}^*[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{2\pi ix\cdot\xi}\,dx\,.$$

The proof of this fact is left as an exercise.

Theorem 9.23 (Plancherel formula for $\mathscr{S}(\mathbb{R}^n)$)**.** *If $f,\ g \in \mathscr{S}(\mathbb{R}^n)$, then*

$$\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathbb{R}^n)}\,.$$

Proof. Recall that $\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx$. By Fubini's theorem,

$$\begin{aligned}\langle \check{f}, g\rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \check{f}(x)\overline{g(x)}\,dx = \int_{\mathbb{R}^n}\Big[\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}d\xi\Big]\overline{g(x)}\,dx\\ &= \int_{\mathbb{R}^n} f(\xi)\Big[\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\overline{g(x)e^{-ix\cdot\xi}dx}\Big]d\xi = \langle f, \widehat{g}\rangle_{L^2(\mathbb{R}^n)}\,.\end{aligned}$$

Therefore, $\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \check{\widehat{f}}, g\rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathbb{R}^n)}$. □

Remark 9.24. The Plancherel formula is a "generalization" of the Parseval identity in the following sense. Define the ℓ^2 space as the collection of all square summable (complex) sequences; that is,

$$\ell^2 = \Big\{\{a_k\}_{k=-\infty}^{\infty} \subseteq \mathbb{C}\,\Big|\,\sum_{k=-\infty}^{\infty}|a_k|^2 < \infty\Big\}$$

with inner product

$$\Big\langle\{a_k\}_{k=-\infty}^{\infty}, \{b_k\}_{k=-\infty}^{\infty}\Big\rangle_{\ell^2} = \sum_{k=-\infty}^{\infty} a_k\overline{b_k}\,.$$

Define $\mathcal{F}: L^2(\mathbb{T}) \to \ell^2$ by $\mathcal{F}(f) = \{\widehat{f}_k\}_{k=-\infty}^{\infty}$. Then (8.5.5) shows that

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \langle \mathcal{F}(f), \mathcal{F}(g)\rangle_{\ell^2} \qquad \forall\, f, g \in L^2(\mathbb{T})$$

so that we obtain an identity similar to the Plancherel formula.

Remark 9.25. Even though in general an square integrable function might not be integrable, using the Plancherel formula the Fourier transform of L^2-functions can still be defined. Note that the Plancherel formula provides that

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\widehat{f}\|_{L^2(\mathbb{R}^n)} \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^n)\,. \tag{9.3.4}$$

If $f \in L^2(\mathbb{R}^n)$; that is, $|f|$ is square integrable, by the fact that $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, there exists a sequence $\{f_k\}_{k=1}^{\infty} \subseteq \mathscr{S}(\mathbb{R}^n)$ such that $\lim\limits_{k\to\infty}\|f_k - f\|_{L^2(\mathbb{R}^n)} = 0$. Then $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$; thus (9.3.4) implies that $\{\widehat{f}_k\}_{k=1}^{\infty}$ is also a Cauchy sequence in $L^2(\mathbb{R}^n)$. By the completeness of $L^2(\mathbb{R}^n)$ (which we did not cover in this lecture), there exists $g \in L^2(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \|\widehat{f}_k - g\|_{L^2(\mathbb{R}^n)} = 0 .$$

We note that such a limit g is independent of the choice of sequence $\{f_k\}_{k=1}^\infty$ used to approximate f; thus we can denote this limit g as $\widehat{f}$. In other words, $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Moreover, by that $f_k \to f$ and $\widehat{f}_k \to \widehat{f}$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$, we find that

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\widehat{f}\|_{L^2(\mathbb{R}^n)} \qquad \forall\, f \in L^2(\mathbb{R}^n)\,,$$

and the parallelogram law further implies that $\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathbb{R}^n)}$ for all $f, g \in L^2(\mathbb{R}^n)$. Similar argument applies to the case of inverse transform of L^2-functions; thus we conclude that

$$\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathbb{R}^n)} = \langle \check{f}, \check{g}\rangle_{L^2(\mathbb{R}^n)} \qquad \forall\, f, g \in L^2(\mathbb{R}^n)\,. \tag{9.3.5}$$

We will talk about how to define the Fourier transform of L^2-functions in another way in Section 9.4.

Theorem 9.26. *If $f, g \in \mathscr{S}(\mathbb{R}^n)$, then $\mathscr{F}(f \star g) = \widehat{f}\,\widehat{g}$. In particular, $f \star g \in \mathscr{S}(\mathbb{R}^n)$ if $f, g \in \mathscr{S}(\mathbb{R}^n)$.*

Proof. By the definition of the Fourier transform and the convolution,

$$\begin{aligned}
\widehat{f \star g}(\xi) &= \frac{1}{\sqrt{2\pi}^n}\mathscr{F}\Big(\int_{\mathbb{R}^n} f(\cdot - y)g(y)\,dy\Big)(\xi) \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\Big]e^{-ix\cdot\xi}dx \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} g(y)\Big(\int_{\mathbb{R}^n} f(x)e^{-i(x+y)\cdot\xi}dx\Big)dy \\
&= \Big(\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx\Big)\Big(\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(y)e^{-iy\cdot\xi}dy\Big)
\end{aligned}$$

which concludes the theorem. □

Corollary 9.27. *$\mathscr{F}^*(f \star g) = \check{f}\,\check{g}$, $\mathscr{F}(fg) = \widehat{f} \star \widehat{g}$ and $\mathscr{F}^*(fg) = \check{f} \star \check{g}$ for all $f, g \in \mathscr{S}(\mathbb{R}^n)$.*

We have established the Fourier inversion formula for Schwartz class functions. Our goal next is to show that the Fourier inversion formula holds for a larger class of functions. Motivated by the Fourier inversion formula, we would like to show, if possible, that

$$\check{\widehat{f}} = \widehat{\check{f}} = f \qquad \forall\, f \in L^1(\mathbb{R}^n) \text{ such that } \widehat{f} \in L^1(\mathbb{R}^n).$$

The above assertion cannot be true since $\hat{\hat{f}}$ and $\check{\check{f}}$ are both continuous (by Proposition 9.4) while $f \in L^1(\mathbb{R}^n)$ which is not necessary continuous. However, we will prove that the identity above holds at points of continuity of f.

Before proceeding, we establish a lemma which is very similar to Lemma 9.16.

Lemma 9.28. *Let $f \in L^1(\mathbb{R}^n)$ and $g \in \mathscr{S}(\mathbb{R}^n)$. Then $\langle \hat{f}, g\rangle = \langle f, \hat{g}\rangle$ and $\langle \check{f}, g\rangle = \langle f, \check{g}\rangle$, where $\langle f, g\rangle = \displaystyle\int_{\mathbb{R}^n} f(x)g(x)\,dx$.*

Proof. We only prove $\langle \hat{f}, g\rangle = \langle f, \hat{g}\rangle$ if $f \in L^1(\mathbb{R}^n)$ and $g \in \mathscr{S}(\mathbb{R}^n)$. By Proposition 9.4, $\hat{f}$ is bounded and continuous on $\mathbb{R}^n$; thus $\hat{f}g$ is an absolutely integrable continuous function. By Fubini's Theorem,

$$\begin{aligned}\langle \hat{f}, g\rangle &= \int_{\mathbb{R}^n}\Big(\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx\Big)g(\xi)d\xi = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x)g(\xi)e^{-ix\cdot\xi}dx\Big)d\xi \\ &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x)g(\xi)e^{-ix\cdot\xi}d\xi\Big)dx = \int_{\mathbb{R}^n} f(x)\Big(\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(\xi)e^{-ix\cdot\xi}d\xi\Big)dx\end{aligned}$$

which is exactly $\langle f, \hat{g}\rangle$. □

Recall that our goal is to show that

$$\text{if } f, \hat{f} \in L^1(\mathbb{R}^n), \text{ then } \check{\hat{f}}(x) = \hat{\check{f}}(x) = f(x) \text{ whenever } f \text{ is continuous at } x.$$

This amounts to treat $\check{\hat{f}}$ and f in the same vector space and check if $\check{\hat{f}} - f$ is the zero vector in that vector space. This underlying vector space is introduced in the following

Definition 9.29. The space $L^1_{\text{loc}}(\mathbb{R}^n)$ consists of all functions (defined on $\mathbb{R}^n$) that are absolutely integrable on all bounded open balls of $\mathbb{R}^n$. In other words,

$$L^1_{\text{loc}}(\mathbb{R}^n) = \Big\{ f : \mathbb{R}^n \to \mathbb{C} \,\Big|\, \int_{B(a,r)} f(x)\,dx \text{ is absolutely convergent for all } a \in \mathbb{R}^n \text{ and } r > 0 \Big\}.$$

Again, we emphasize that we ***abuse*** the notation $L^1_{\text{loc}}(\mathbb{R}^n)$ which in fact stands for a larger class of functions. We also note that $L^1(\mathbb{R}^n) \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$ and $\mathscr{C}_b(\mathbb{R}^n;\mathbb{C}) \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$.

How do we determine if an locally integrable function h is the zero vector in $L^1_{\text{loc}}(\mathbb{R}^n)$? Our goal is to establish an equivalent condition of that $h \equiv 0$ (or to be more precisely, $h(x) = 0$ if h is continuous at x) stated briefly as follows (the precise statement is given in Lemma 9.35):

$$\text{If } h \in L^1_{\text{loc}}(\mathbb{R}^n), \text{ then } h \equiv 0 \text{ if and only if } \langle h, g\rangle = 0 \text{ for all } g \in \mathscr{S}(\mathbb{R}^n). \tag{9.3.6}$$

We note that the difficulty here is that h and g belongs to different space so that we cannot simply let $g = \bar{h}$ to conclude that $\displaystyle\int_{\mathbb{R}^n} |h(x)|^2\, dx = 0$.

A special class of functions that will be used as the role of g in (9.3.6) is called the standard mollifiers. Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a smooth function defined by

$$\zeta(x) = \begin{cases} \exp\left(\dfrac{1}{x^2-1}\right) & \text{if } |x| < 1\,, \\ 0 & \text{if } |x| \geqslant 1\,. \end{cases}$$

For $x \in \mathbb{R}^n$, define $\eta_1(x) = C\zeta(|x|)$, where C is chosen so that $\displaystyle\int_{\mathbb{R}^n} \eta_1(x)\, dx = 1$. The change of variables formula then implies that $\eta_\varepsilon(x) \equiv \varepsilon^{-n}\eta_1(x/\varepsilon)$ has integral 1. We remark that η_ε is smooth and $\eta_\varepsilon(x) \neq 0$ if and only if $x \in B(0,\varepsilon)$; thus $\eta_\varepsilon \in \mathscr{S}(\mathbb{R}^n)$ for all $\varepsilon > 0$.

Definition 9.30. The collection of functions $\{\eta_\varepsilon\}_{\varepsilon>0}$ is called the ***standard mollifiers***.

Definition 9.31. Let f, g be functions defined on $\mathbb{R}^n$. The convolution of f and g, denoted by $f * g$, is a function defined on $\mathbb{R}^n$ given by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\, dy$$

whenever the integral makes sense for all $x \in \mathbb{R}^n$.

We note that the change of variables formula implies that $f * g = g * f$ whenever the convolution makes sense.

Remark 9.32. Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be the standard mollifiers.

1. Since $\eta_\varepsilon(y) \neq 0$ if and only if $y \in B(0,\varepsilon)$, for each $x \in \mathbb{R}^n$,

$$(\eta_\varepsilon * f)(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)f(y)\, dy = \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)f(y)\, dy$$

   and the integral exists if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

2. By the fact that η_ε is smooth, the Dominated Convergence Theorem (Theorem 6.102) shows that $\eta_\varepsilon * f$ is smooth for each $\varepsilon > 0$, and

$$D^\alpha(\eta_\varepsilon * f)(x) = \int_{\mathbb{R}^n} (D^\alpha\eta)(x-y)f(y)\, dy\,,$$

   where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index. The detail proof is left as an exercise.

Example 9.33. Let $f = \mathbf{1}_{[a,b]}$, the characteristic/indicator function of the closed interval $[a, b]$. Then for $\varepsilon \ll 1$, the function $\eta_\varepsilon * f$ is smooth and has the property that

$$(\eta_\varepsilon * f)(x) = \begin{cases} 1 & \text{if } x \in [a+\varepsilon, b-\varepsilon]\,, \\ 0 & \text{if } x \in [a-\varepsilon, b+\varepsilon]^{\complement}, \end{cases}$$

and $0 \leqslant f \leqslant 1$. Moreover, $\eta_\varepsilon * f$ converges pointwise to f on $\mathbb{R}\backslash\{a, b\}$.

The following lemma shows that $\eta_\varepsilon * f$ converges to f at points of continuity of f if $f \in L^1_{\rm loc}(\mathbb{R}^n)$.

Lemma 9.34. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and x_0 be a continuity of f. Then*

$$\lim_{\varepsilon \to 0^+} (\eta_\varepsilon * f)(x_0) = f(x_0)\,.$$

Proof. Let $\epsilon > 0$ be given. Since f is continuous at x_0, there exists $\delta > 0$ such that

$$\big|f(y) - f(x_0)\big| < \frac{\epsilon}{2} \quad \text{whenever} \quad |y - x_0| < \delta\,.$$

Therefore, by the fact that $\displaystyle\int_{\mathbb{R}^n} \eta_\varepsilon(y)\,dy = 1$, for $0 < \varepsilon < \delta$ we find that

$$\begin{aligned}\big|(\eta_\varepsilon * f)(x_0) - f(x_0)\big| &= \Big|\int_{\mathbb{R}^n} \eta_\varepsilon(y) f(x_0 - y)\,dy - \int_{\mathbb{R}^n} \eta_\varepsilon(y) f(x_0)\,dy\Big| \\ &\leqslant \int_{B(0,\varepsilon)} \eta_\varepsilon(y)\big|f(x_0 - y) - f(x_0)\big|\,dy \leqslant \frac{\epsilon}{2}\int_{B(0,\varepsilon)} \eta_\varepsilon(y)\,dy < \epsilon\,. \end{aligned}$$ □

Lemma 9.35. *Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$. If $\langle f, g\rangle = 0$ for all $g \in \mathscr{S}(\mathbb{R}^n)$, then $f(x_0) = 0$ whenever f is continuous at x_0.*

Proof. Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be the standard mollifiers, and x_0 be a point of continuity of f. Then for all $\varepsilon > 0$,

$$(\eta_\varepsilon * f)(x_0) = \int_{\mathbb{R}^n} \eta_\varepsilon(x_0 - y) f(y)\,dy = \int_{\mathbb{R}^n} f(y)\eta_\varepsilon(y - x_0)\,dy = \langle f, \tau_{x_0}\eta_\varepsilon\rangle\,,$$

where τ_{x_0} is the translation operator defined by $(\tau_{x_0}\phi)(y) = \phi(y - x_0)$. Since $\eta_\varepsilon \in \mathscr{S}(\mathbb{R}^n)$ for all $\varepsilon > 0$, the function $\tau_{x_0}\eta_\varepsilon \in \mathscr{S}(\mathbb{R}^n)$ for all $\varepsilon > 0$; thus $(\eta_\varepsilon * f)(x_0) = 0$ for all $\varepsilon > 0$. By Lemma 9.34, we conclude that

$$f(x_0) = \lim_{\varepsilon \to 0}(\eta_\varepsilon * f)(x) = 0\,.$$ □

Remark 9.36. The reverse statement of Lemma 9.35 is also true: if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ has the property that $f(x_0) = 0$ whenever f is continuous at x_0, then $\langle f, g\rangle = 0$ for all $g \in \mathscr{S}(\mathbb{R}^n)$ since if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the collection of discontinuities of f has measure zero which shows that $f(x) \neq 0$ only on a set of measure zero. Therefore, $\langle f, g\rangle = 0$ for all $g \in \mathscr{S}(\mathbb{R}^n)$. In other words, if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then

$$\langle f, g\rangle = 0 \text{ for all } g \in \mathscr{S}(\mathbb{R}^n) \text{ if and only if } f(x_0) = 0 \text{ whenever } f \text{ is continuous at } x_0.$$

Lemma 9.35 establishes the non-trival direction "⇒".

Now we are in position of showing the Fourier inversion formula for functions of more general class.

Theorem 9.37 (Fourier Inversion Formula)**.** *Let $f \in L^1(\mathbb{R}^n)$ such that $\widehat{f} \in L^1(\mathbb{R}^n)$. Then*

$$\check{\widehat{f}}(x) = \widehat{\check{f}}(x) = f(x) \qquad \textit{whenever } f \textit{ is continuous at } x.$$

Proof. Let $f : \mathbb{R}^n \to \mathbb{C}$ be such that $f, \widehat{f} \in L^1(\mathbb{R}^n)$. By the fact that $\check{f}(\xi) = \widehat{f}(-\xi)$ for all $\xi \in \mathbb{R}^n$, the change of variables formula implies that $\check{f} \in L^1(\mathbb{R}^n)$.

Let $g \in \mathscr{S}(\mathbb{R}^n)$ be given. By Lemma 9.28 and the Fourier inversion formula for Schwartz class functions (Theorem 9.19),

$$\langle \check{\widehat{f}}, g\rangle = \langle \widehat{f}, \check{g}\rangle = \langle f, \widehat{\check{g}}\rangle = \langle f, g\rangle \qquad \text{and} \qquad \langle \widehat{\check{f}}, g\rangle = \langle \check{f}, \widehat{g}\rangle = \langle f, \check{\widehat{g}}\rangle = \langle f, g\rangle\,.$$

In other words, if $f, \widehat{f} \in L^1(\mathbb{R}^n)$,

$$\langle \check{\widehat{f}} - f, g\rangle = \langle \widehat{\check{f}} - f, g\rangle = 0 \qquad \forall\, g \in \mathscr{S}(\mathbb{R}^n)\,.$$

By Proposition 9.4, $\check{\widehat{f}}, \widehat{\check{f}} \in \mathscr{C}_b(\mathbb{R}^n; \mathbb{C})$; thus the collection of points of continuities of $\check{\widehat{f}} - f$ and $\widehat{\check{f}} - f$ is identical to the collection of points of continuities of f; that is,

$$f \text{ is continuous at } x \text{ if and only if } \check{\widehat{f}} - f \text{ and } \widehat{\check{f}} - f \text{ are continuous at } x.$$

Moreover, by the fact that $\mathscr{C}_b(\mathbb{R}^n; \mathbb{C}) \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n) \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$ we find that $\check{\widehat{f}} - f, \widehat{\check{f}} - f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Therefore, the theorem is concluded by Lemma 9.35. □

Remark 9.38. Since an integrable function $f : \mathbb{R}^n \to \mathbb{R}$ must be continuous **almost everywhere** on $\mathbb{R}^n$, Theorem 9.37 implies that if $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that f, $\widehat{f} \in L^1(\mathbb{R}^n)$, then $\check{\widehat{f}} = \widehat{\check{f}} = f$ almost everywhere.

9.4 The Fourier Transform of Generalized Functions

It is often required to consider the Fourier transform of functions which do not belong to $L^1(\mathbb{R}^n)$. For example, the **normalized sinc function** sinc : $\mathbb{R} \to \mathbb{R}$ defined by

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,, \end{cases} \tag{9.4.1}$$

does not belong to $L^1(\mathbb{R})$ but it is a very important function in the study of signal processing.

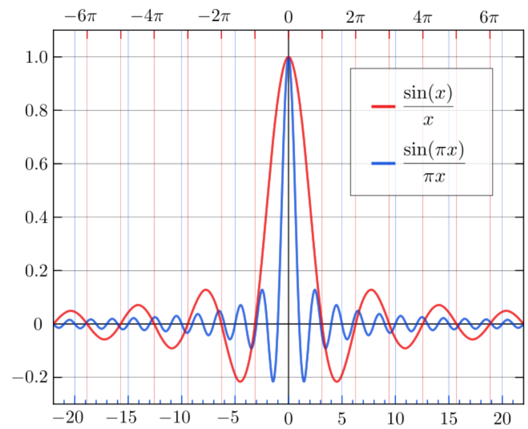


Figure 9.1: The graphs of unnormalized and normalized sinc functions (from wiki)

Moreover, there are "functions" that are not even functions in the traditional sense. For example, in physics and engineering applications the Dirac delta "function" δ is defined as the "function" which validates the relation

$$\int_{\mathbb{R}^n} \delta(x)\phi(x)\,dx = \phi(0) \qquad \forall\, \phi \in \mathscr{C}(\mathbb{R}^n)$$

In fact, there is no function (in the traditional sense) satisfying the property given above (reasoning later). Can we take the Fourier transform of those "functions" as well? To understand this topic better, it is required to study the theory of generalized functions/ distributions.

To understand the meaning of distributions, let us turn to a situation in physics: measuring the temperature. To measure the temperature T at a point x, instead of outputting the exact value of $T(x)$ the thermometer instead outputs the **overall value** of the temperature near x. In other words, the reading of the temperature is determined by a pairing of the temperature distribution with the thermometer.

In mathematical point of views, to evaluate the function value of a locally integrable function f at a point of continuity x_0, we apply Lemma 9.34 and obtain that

$$f(x_0) = \lim_{\varepsilon \to 0^+} (\eta_\varepsilon * f)(x_0) = \lim_{\varepsilon \to 0^+} \langle f, \tau_{x_0}\eta_\varepsilon \rangle, \tag{9.4.2}$$

where τ_{x_0} is the translation operator given by $(\tau_{x_0}\phi)(x) = \phi(x - x_0)$. Here η_ε can be viewed as a meter that can measure the function value of locally integrable functions, ε is a parameter that corresponds to the accuracy of this meter, and $\tau_{x_0}\eta_\varepsilon$ is a meter that locates at position x_0. Nevertheless, for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the "pairing" $\langle f, \phi \rangle$ is defined not only on functions of the form $\phi = \tau_{x_0}\eta_\varepsilon$ but also on $\phi \in \mathscr{D}(\mathbb{R}^n)$, where

$$\mathscr{D}(\mathbb{R}^n) \equiv \Big\{\phi : \mathbb{R}^n \to \mathbb{C} \,\Big|\, \phi \in \mathscr{C}^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp}(\phi) \equiv \overline{\{x \in \mathbb{R}^n \; \phi(x) \neq 0\}} \text{ is compact}\Big\}.$$

This pairing induced a (continuous) linear functional $T_f : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ defined by

$$T_f(\phi) = \langle f, \phi \rangle. \tag{9.4.3}$$

Moreover, if $T : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ is a continuous linear functional, and there exists $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $T(\phi) = \langle f, \phi \rangle$ for all $\phi \in \mathscr{D}(\mathbb{R}^n)$, then f is uniquely determined except perhaps on a set of measure zero (or to be more precise, f is uniquely determined at all points of continuity of f). Therefore, with $\mathscr{D}(\mathbb{R}^n)'$ denoting the collection of all (continuous) linear functionals defined on $\mathscr{D}(\mathbb{R}^n)$, there is a natural injection $\iota : L^1_{\text{loc}}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^n)'$ (given by $\iota(f) = T_f$).

On the other hand, a (continuous) linear functional defined on $\mathscr{D}(\mathbb{R}^n)$ might not take the form of (9.4.3). For example, there exists **no** locally integrable function f such that

$$T(\phi) = \int_{\mathbb{R}^n} f(y)\phi(y)\, dy = \phi(0) \qquad \forall\, \phi \in \mathscr{D}(\mathbb{R}^n). \tag{9.4.4}$$

To see this, suppose the contrary that there exists $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that (9.4.4) holds. Then

$$(\eta_\varepsilon * f)(x) = \int_{\mathbb{R}^n} f(y)(\tau_x\eta_\varepsilon)(y)\, dy = \eta_\varepsilon(x)$$

which vanishes if $x \notin B(0, \varepsilon)$. This implies that $f = 0$ almost everywhere so that $\langle f, \phi \rangle = 0$ for all $\phi \in \mathscr{D}(\mathbb{R}^n)$, a contradiction. Therefore, $\iota : L^1_{\text{loc}}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^n)'$ (given by $\iota(f) = T_f$) is not surjective; thus (continuous) linear functionals on $\mathscr{D}(\mathbb{R}^n)$ defines more "functions" than $L^1_{\text{loc}}(\mathbb{R}^n)$. Such kind of linear functionals are called generalized functions or distributions.

The Fourier transform can be defined on a smaller class of generalized functions, the space of tempered distributions. A tempered distribution is a continuous linear functional on $\mathscr{S}(\mathbb{R}^n)$. In other words, T is a tempered distribution if

$$T : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C},\ T(c\phi + \psi) = cT(\phi) + T(\psi) \text{ for all } c \in \mathbb{C} \text{ and } \phi, \psi \in \mathscr{S}(\mathbb{R}^n),$$
$$\text{and } \lim_{j\to\infty} T(\phi_j) = T(\phi) \text{ if } \{\phi_j\}_{j=1}^{\infty} \subseteq \mathscr{S}(\mathbb{R}^n) \text{ and } \phi_j \to \phi \text{ in } \mathscr{S}(\mathbb{R}^n).$$

The convergence in $\mathscr{S}(\mathbb{R}^n)$ is described by semi-norms, and is given in the following

Definition 9.39 (Convergence in $\mathscr{S}(\mathbb{R}^n)$)**.** For each $k \in \mathbb{N} \cup \{0\}$, define the semi-norm

$$p_k(\phi) = \sup_{x\in\mathbb{R}^n, |\alpha|\leqslant k} \langle x\rangle^k |D^\alpha \phi(x)|\,,$$

where $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$. A sequence $\{\phi_j\}_{j=1}^{\infty} \subseteq \mathscr{S}(\mathbb{R}^n)$ is said to converge to ϕ in $\mathscr{S}(\mathbb{R}^n)$ if $p_k(\phi_j - \phi) \to 0$ as $j \to \infty$ for all $k \in \mathbb{N} \cup \{0\}$.

We note that $p_k(\phi) \leqslant p_{k+1}(\phi)$, so $\{\phi_j\}_{j=1}^{\infty} \subseteq \mathscr{S}(\mathbb{R}^n)$ converges to ϕ in $\mathscr{S}(\mathbb{R}^n)$ whenever $p_k(\phi_j - \phi) \to 0$ as $j \to \infty$ for $k \gg 1$. We also note that if $\{\phi_j\}_{j=1}^{\infty}$ converge to ϕ in $\mathscr{S}(\mathbb{R}^n)$, then $\{\phi_j\}_{j=1}^{\infty}$ converges uniformly to ϕ on $\mathbb{R}^n$.

Definition 9.40 (Tempered Distributions)**.** A linear map $T : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous if there exists $N \in \mathbb{N}$ such that for each $k \geqslant N$, there exists a constant C_k such that

$$\big|T(\phi)\big| \leqslant C_k p_k(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

The collection of continuous linear functionals on $\mathscr{S}(\mathbb{R}^n)$ is denoted by $\mathscr{S}(\mathbb{R}^n)'$. Elements of $\mathscr{S}(\mathbb{R}^n)'$ are called ***tempered distributions***.

Example 9.41. For $1 \leqslant p < \infty$, let $L^p(\mathbb{R}^n)$ denote the collection of Riemann measurable functions whose p-th power is integrable; that is,

$$L^p(\mathbb{R}^n) = \Big\{ f : \mathbb{R}^n \to \mathbb{C} \,\Big|\, f \text{ is Riemann measurable and } \int_{\mathbb{R}^n} \big|f(x)\big|^p\,dx < \infty \Big\}\,,$$

and let $L^\infty(\mathbb{R}^n)$ denote the collection of bounded Riemann measurable functions. Every L^p-function $f : \mathbb{R}^n \to \mathbb{C}$ can be viewed as a tempered distribution for all $p \in [1, \infty]$. In fact, the tempered distribution T_f associated with f is defined by

$$T_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)\,dx \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,. \tag{9.4.5}$$

Now we show that T_f given by (9.4.5) is indeed a tempered distribution. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be given. Then $\|\phi\|_{L^\infty(\mathbb{R}^n)} \leqslant p_k(\phi)$ for all $k \in \mathbb{N}$, while for $1 \leqslant q < \infty$ and $k > \dfrac{n}{q}$,

$$\begin{aligned}\|\phi\|_{L^q(\mathbb{R}^n)} \equiv \Big(\int_{\mathbb{R}^n} \big|\phi(x)\big|^q\,dx\Big)^{\frac{1}{q}} &= \Big(\int_{\mathbb{R}^n} \langle x\rangle^{-kq}\big[\langle x\rangle^k|\phi(x)|\big]^q\,dx\Big)^{\frac{1}{q}} \leqslant \Big(\int_{\mathbb{R}^n} \langle x\rangle^{-kq}\,dx\Big)^{\frac{1}{q}} p_k(\phi)\\ &\leqslant \Big(\omega_{n-1}\int_0^\infty (1+r^2)^{-\frac{kq}{2}} r^{n-1} dr\Big)^{\frac{1}{q}} p_k(\phi)\,.\end{aligned}$$

Note that $\displaystyle\int_0^\infty (1+r^2)^{-\frac{kq}{2}} r^{n-1} dr < \infty$ if $k > \dfrac{n}{q}$; thus for all $q \in [1,\infty]$, there exists $C_{k,q,n} > 0$ such that

$$\|\phi\|_{L^q(\mathbb{R}^n)} \leqslant C_{k,q,n} p_k(\phi) \qquad \forall\, k \gg 1\,. \tag{9.4.6}$$

Therefore, if $f \in L^p(\mathbb{R}^n)$, by the Hölder inequality we have

$$\big|\langle f,\phi\rangle\big| \leqslant \|f\|_{L^p(\mathbb{R}^n)}\|\phi\|_{L^{p'}(\mathbb{R}^n)} \leqslant C_{k,p',n}\|f\|_{L^p(\mathbb{R}^n)} p_k(\phi) \qquad \forall\, k \gg 1\,,$$

where $p' \in [1,\infty]$ is the Hölder conjugate of p satisfying $\dfrac{1}{p} + \dfrac{1}{p'} = 1$; thus $T_f \in \mathscr{S}(\mathbb{R}^n)'$ if $f \in L^p(\mathbb{R}^n)$. Note that the sinc function belongs to $L^2(\mathbb{R})$ so that $T_{\rm sinc} \in \mathscr{S}(\mathbb{R})'$.

Example 9.42. Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic, Riemann measurable function such that $\displaystyle\int_{-\pi}^{\pi} |f(x)|\,dx < \infty$, and $\phi \in \mathscr{S}(\mathbb{R})$. By the definition of the p_k semi-norm,

$$|x|^2\big|\phi(x)\big| \leqslant p_2(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n\,.$$

Therefore,

$$\begin{aligned}\big|\langle f,\phi\rangle\big| &= \Big|\sum_{k=-\infty}^{\infty}\int_{-\pi+2k\pi}^{\pi+2k\pi} f(x)\phi(x)\,dx\Big| \leqslant \sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi}\big|f(x)\big|\big|\phi(x+2k\pi)\big|\,dx\\ &= \int_{-\pi}^{\pi}\big|f(x)\big|\big|\phi(x)\big|\,dx + \sum_{|k|\geqslant 1}\int_{-\pi}^{\pi}\big|f(x)\big|\big|\phi(x+2k\pi)\big|\,dx\\ &\leqslant p_0(\phi)\int_{-\pi}^{\pi}\big|f(x)\big|\,dx + \sum_{|k|\geqslant 1}\int_{-\pi}^{\pi}\big|f(x)\big|\frac{1}{|x+2k\pi|^2}p_2(\phi)\,dx\\ &\leqslant \Big(\int_{-\pi}^{\pi}\big|f(x)\big|\,dx\Big)\Big(1 + 2\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\Big)p_2(\phi)\end{aligned}$$

which implies that T_f is a tempered distribution. In particular, $T_c \in \mathscr{S}(\mathbb{R})'$ for all constant $c \in \mathbb{R}$.

From now on, we identify f with the tempered distribution T_f if f is an ordinary function (or a function defined pointwise). In other words, if $T \in \mathscr{S}(\mathbb{R}^n)'$ and $f : \mathbb{R}^n \to \mathbb{C}$ is a function, we say that $T = f$ in $\mathscr{S}(\mathbb{R}^n)'$ if $T = T_f$, where T_f is the tempered distribution associated with the function f. Moreover, if $T \in \mathscr{S}(\mathbb{R}^n)'$ and $\phi \in \mathscr{S}(\mathbb{R}^n)$, $T(\phi)$ is also expressed as $\langle T, \phi \rangle$.

Example 9.43 (Dirac delta function)**.** Consider the map $\delta : \mathscr{C}(\mathbb{R}^n) \to \mathbb{R}$ defined by $\delta(\phi) = \phi(0)$. Then $\big|\delta(\phi)\big| \leqslant p_0(\phi) \leqslant p_k(\phi)$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $k \in \mathbb{N} \cup \{0\}$; thus $\delta \in \mathscr{S}(\mathbb{R}^n)'$. Therefore, we also write $\delta(\phi)$ as $\langle \delta, \phi \rangle$. This explains why the Dirac delta function has the property that

$$\int_{\mathbb{R}^n} \delta(x)\phi(x)\,dx = \phi(0)$$

since the integral above is an informal expression of $\langle \delta, \phi \rangle$.

Similarly, the Dirac delta function at a point ω defined by $\langle \delta_\omega, \phi \rangle = \phi(\omega)$ is also a tempered distribution.

Remark 9.44. Not all ordinary functions are tempered distributions. For example, the function $f(x) = e^{x^4}$ is locally integrable (since it is continuous), but $\displaystyle\int_{\mathbb{R}} f(x)e^{-x^2}\,dx = \infty$. Therefore, being in $L^1_{\text{loc}}(\mathbb{R}^n)$ is not good enough to generate elements in $\mathscr{S}(\mathbb{R}^n)'$, and it requires that $|f(x)| \leqslant C(1 + |x|^N)$ for any $N \in \mathbb{N}$. In such a case, $T_f \in \mathscr{S}(\mathbb{R}^n)'$ is well-defined.

As shown in the example above, a tempered distribution might not be defined in the pointwise sense. Therefore, how to define usual operations such as translation, dilation, and reflection on generalized functions should be answered prior to define the Fourier transform of tempered distributions. For completeness, let us start from providing the definitions of translation, dilation and reflection operators.

Definition 9.45 (Translation, dilation, and reflection)**.** Let $f : \mathbb{R}^n \to \mathbb{C}$ be a function.

1. For $h \in \mathbb{R}^n$, the translation operator τ_h maps f to $\tau_h f$ given by $(\tau_h f)(x) = f(x - h)$.

2. For $\lambda > 0$, the dilation operator $d_\lambda : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ maps f to $d_\lambda f$ given by $(d_\lambda f)(x) = f(\lambda^{-1}x)$.

3. The reflection operator $\widetilde{\ }$ maps f to $\widetilde{f}$ given by $\widetilde{f}(x) = f(-x)$.

Now suppose that $T \in \mathscr{S}(\mathbb{R}^n)'$. We expect that $\tau_h T$, $d_\lambda T$ and $\widetilde{T}$ are also tempered distributions, so we need to provide the values of $\langle \tau_h T, \phi\rangle$, $\langle d_\lambda T, \phi\rangle$ and $\langle \widetilde{T}, \phi\rangle$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$. If $T = T_f$ is the tempered distribution associated with $f \in L^1(\mathbb{R}^n)$, then for $g \in \mathscr{S}(\mathbb{R}^n)$, the change of variable formula implies that

$$\begin{aligned}
\langle \tau_h f, g\rangle &= \int_{\mathbb{R}^n} f(x-h)g(x)\,dx = \int_{\mathbb{R}^n} f(x)g(x+h)\,dx = \langle f, \tau_{-h} g\rangle\,,\\
\langle d_\lambda f, g\rangle &= \int_{\mathbb{R}^n} f(\lambda^{-1}x)g(x)\,dx = \int_{\mathbb{R}^n} f(x)g(\lambda x)\lambda^n\,dx = \langle f, \lambda^n d_{\lambda^{-1}} g\rangle\,,\\
\langle \widetilde{f}, g\rangle &= \int_{\mathbb{R}^n} f(-x)g(x)\,dx = \int_{\mathbb{R}^n} f(x)g(-x)\,dx = \langle f, \widetilde{g}\rangle\,.
\end{aligned}$$

The computations above motivate the following

Definition 9.46. Let $h \in \mathbb{R}^n$, $\lambda > 0$, and τ_h and d_λ be the translation and dilation operator given in Definition 9.45. For $T \in \mathscr{S}(\mathbb{R}^n)'$, $\tau_h T$, $d_\lambda T$ and $\widetilde{T}$ are the tempered distributions defined by

$$\langle \tau_h T, \phi\rangle = \langle T, \tau_{-h}\phi\rangle\,,\quad \langle d_\lambda T, \phi\rangle = \langle T, \lambda^n d_{\lambda^{-1}}\phi\rangle \quad\text{and}\quad \langle \widetilde{T}, \phi\rangle = \langle T, \widetilde{\phi}\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

We note that $\tau_h T$, $d_\lambda T$ and $\widetilde{T}$ are tempered distributions since

$$\begin{aligned}
p_k(\tau_{-h}\phi) &\leqslant \sup_{x\in\mathbb{R}^n, |\alpha|\leqslant k} \langle x\rangle^k \big|D^\alpha\phi(x-h)\big| \leqslant \sup_{x\in\mathbb{R}^n, |\alpha|\leqslant k} \langle x+h\rangle^k \big|D^\alpha\phi(x)\big|\\
&\leqslant \Big(\sup_{x\in\mathbb{R}^n} \frac{\langle x+h\rangle^2}{\langle x\rangle^2}\Big)^{\frac{k}{2}} p_k(\phi) \leqslant (1+|h|)^k p_k(\phi)\,,\\
p_k(\lambda^n d_{\lambda^{-1}}\phi) &\leqslant \lambda^n \sup_{x\in\mathbb{R}^n, |\alpha|\leqslant k} \langle x\rangle^k \lambda^{|\alpha|} \big|(D^\alpha\phi)(\lambda x)\big| \leqslant \lambda^n \max\{\lambda^k, \lambda^{-k}\} p_k(\phi)\,,\\
p_k(\widetilde{\phi}) &= p_k(\phi)
\end{aligned}$$

so that by the fact that $\big|\langle T, \phi\rangle\big| \leqslant C_k p_k(\phi)$ for $k \gg 1$, for all $\phi \in \mathscr{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\big|\langle \tau_h T, \phi\rangle\big| &= \big|\langle T, \tau_{-h}\phi\rangle\big| \leqslant C_k (1+|h|)^k p_k(\phi) = \widetilde{C}_k p_k(\phi)\,,\\
\big|\langle d_\lambda T, \phi\rangle\big| &= \big|\langle T, \lambda^n d_{\lambda^{-1}}\phi\rangle\big| \leqslant C_k \lambda^n \max\{\lambda^k, \lambda^{-k}\} p_k(\phi) = \widetilde{C}_k p_k(\phi)\,,\\
\big|\langle \widetilde{T}, \phi\rangle\big| &= \big|\langle T, \widetilde{\phi}\rangle\big| \leqslant C_k p_k(\phi)\,.
\end{aligned}$$

Example 9.47. Let $\omega, h \in \mathbb{R}^n$ and $\lambda > 0$.

1. $\tau_h \delta_\omega = \delta_{\omega+h}$ since if $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\langle \tau_h \delta_\omega, \phi\rangle = \langle \delta_\omega, \tau_{-h}\phi\rangle = \phi(\omega+h) = \langle \delta_{\omega+h}, \phi\rangle$.

2. $d_\lambda \delta_\omega = \lambda^n \delta_{\lambda\omega}$ since if $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\langle d_\lambda \delta_\omega, \phi\rangle = \langle \delta_\omega, \lambda^n d_{1/\lambda}\phi\rangle = \lambda^n \phi(\lambda\omega) = \langle \lambda^n \delta_{\lambda\omega}, \phi\rangle$.

3. $\widetilde{\delta_\omega} = \delta_{-\omega}$ since if $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\langle \widetilde{\delta_\omega}, \phi\rangle = \langle \delta_\omega, \widetilde{\phi}\rangle = \phi(-\omega) = \langle \delta_{-\omega}, \phi\rangle$.

From the experience of defining the translation, dilation and reflection of tempered distributions, now we can talk about how to defined Fourier transform of tempered distributions. Recall that in Lemma 9.28 we have established that

$$\langle \widehat{f}, g\rangle = \langle f, \widehat{g}\rangle \quad \text{and} \quad \langle \check{f}, g\rangle = \langle f, \check{g}\rangle \qquad \forall\, f, g \in L^1(\mathbb{R}^n)\,.$$

Since the identities above hold for all L^1-functions f (and L^1-functions corresponds to tempered distributions T_f through (9.4.5)), we expect that the Fourier transform of tempered distributions has to satisfy the identities above as well. Let $T \in \mathscr{S}(\mathbb{R}^n)'$ be given, and define $\widehat{T} : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ by

$$\widehat{T}(\phi) = \langle \widehat{T}, \phi\rangle \equiv \langle T, \widehat{\phi}\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,. \tag{9.4.7}$$

Let $k \geqslant 2$ and k is a multiple of 4. Then

$$\begin{aligned}
p_k(\widehat{\phi}) &= \sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} \langle \xi\rangle^k \big|D^\alpha \widehat{\phi}(\xi)\big| = \sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} \langle \xi\rangle^k \Big|\mathscr{F}_x\big[x^\alpha \phi(x)\big](\xi)\Big| \\
&\leqslant \sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} (n+1)^{\frac{k}{2}-1}\big(1 + |\xi_1|^k + \cdots + |\xi_n|^k\big)\Big|\mathscr{F}_x\big[x^\alpha \phi(x)\big](\xi)\Big| \\
&\leqslant (n+1)^{\frac{k}{2}-1} \sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} \Big|\mathscr{F}_x\big[(1 + \partial^k_{x_1} + \cdots + \partial^k_{x_n})\big(x^\alpha\phi(x)\big)\big](\xi)\Big|.
\end{aligned}$$

Since

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} \Big|\mathscr{F}_x\big[x^\alpha \phi(x)\big](\xi)\Big| &\leqslant \sup_{|\alpha|\leqslant k} \int_{\mathbb{R}^n} \big|x^\alpha \phi(x)\big|\, dx \leqslant \int_{\mathbb{R}^n} \langle x\rangle^k \big|\phi(x)\big|\, dx \\
&\leqslant \big\|\langle\cdot\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n} \langle x\rangle^{n+k+1}\big|\phi(x)\big| \leqslant \big\|\langle\cdot\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)}\, p_{n+k+1}(\phi)
\end{aligned}$$

and for $1 \leqslant j \leqslant n$,

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} \Big|\mathscr{F}_x\big[\partial^k_{x_j}(x^\alpha \phi(x)\big](\xi)\Big| \leqslant \sum_{\ell=0}^{k} C^k_\ell \sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} \Big|\mathscr{F}_x\big[(\partial^{k-\ell}_{x_j} x^\alpha)\partial^\ell_{x_j}\phi(x)\big](\xi)\Big| \\
&\quad\leqslant \sum_{\ell=0}^{k} C^k_\ell \sup_{|\alpha|\leqslant k} \int_{\mathbb{R}^n} \big|(\partial^{k-\ell}_{x_j} x^\alpha)\partial^\ell_{x_j}\phi(x)\big|\, dx \leqslant \sum_{\ell=0}^{k} C^k_\ell \sup_{|\alpha|\leqslant k} |\alpha|! \big\|\langle\cdot\rangle^{|\alpha|-k+\ell}\partial^\ell_{x_j}\phi(\cdot)\big\|_{L^1(\mathbb{R}^n)} \\
&\quad\leqslant \sum_{\ell=0}^{k} C^k_\ell k! \sup_{|\beta|=\ell} \big\|\langle\cdot\rangle^\ell D^\beta\phi(\cdot)\big\|_{L^1(\mathbb{R}^n)} \leqslant k! \sum_{\ell=0}^{k} C^k_\ell \big\|\langle\cdot\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)} p_{n+\ell+1}(\phi) \\
&\quad\leqslant k! \big\|\langle\cdot\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)}\, p_{n+k+1}(\phi) \sum_{\ell=0}^{k} C^k_\ell = k! 2^k \big\|\langle\cdot\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)}\, p_{n+k+1}(\phi)\,,
\end{aligned}$$

we conclude that

$$p_k(\widehat{\phi}) \leqslant (n+1)^{\frac{k}{2}-1}(1+nk!2^k)\big\|\langle\cdot\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)}p_{n+k+1}(\phi) = \bar{C}(n,k)p_{n+k+1}(\phi)\,. \tag{9.4.8}$$

Therefore,

$$\big|\langle\widehat{T},\phi\rangle\big| = \big|\langle T,\widehat{\phi}\rangle\big| \leqslant C_k p_k(\widehat{\phi}) \leqslant C_k\bar{C}(n,k)p_{k+n+1}(\phi) \qquad \forall\, k \gg 1 \tag{9.4.9}$$

which shows that $\widehat{T}$ defined by (9.4.7) is a tempered distribution. Similarly, $\check{T}:\mathscr{S}(\mathbb{R}^n)\to\mathbb{C}$ defined by $\langle\check{T},\phi\rangle = \langle T,\check{\phi}\rangle$ for all $\phi\in\mathscr{S}(\mathbb{R}^n)$ is also a tempered distribution. The discussion above leads to the following

Definition 9.48. Let $T\in\mathscr{S}(\mathbb{R}^n)'$. The Fourier transform of T and the Fourier $*$ transform of T, denoted by $\widehat{T}$ and $\check{T}$ respectively, are tempered distributions given by

$$\langle\widehat{T},\phi\rangle = \langle T,\widehat{\phi}\rangle \qquad \text{and} \qquad \langle\check{T},\phi\rangle = \langle T,\check{\phi}\rangle \qquad \forall\,\phi\in\mathscr{S}(\mathbb{R}^n)\,.$$

In other words, if $T\in\mathscr{S}(\mathbb{R}^n)'$, then $\widehat{T},\check{T}\in\mathscr{S}(\mathbb{R}^n)'$ as well and the actions of $\widehat{T}$, $\check{T}$ on $\phi\in\mathscr{S}(\mathbb{R}^n)$ are given in the relations above.

Example 9.49 (The Fourier transform of the Dirac delta function)**.** Consider the Dirac delta function $\delta:\mathscr{S}(\mathbb{R}^n)\to\mathbb{C}$ defined in Example 9.43. Then for $\phi\in\mathscr{S}(\mathbb{R}^n)$,

$$\langle\delta,\widehat{\phi}\rangle = \widehat{\phi}(0) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\phi(x)e^{-ix\cdot 0}\,dx = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\phi(x)\,dx = \langle\frac{1}{\sqrt{2\pi}^n},\phi\rangle;$$

thus the Fourier transform of the Dirac delta function is a constant function and $\widehat{\delta}(\xi) = \dfrac{1}{\sqrt{2\pi}^n}$. Similarly, $\check{\delta}(\xi) = \dfrac{1}{\sqrt{2\pi}^n}$, so $\widehat{\delta} = \check{\delta}$.

Next we consider the Fourier transform of δ_ω, the Dirac delta function at point $\omega\in\mathbb{R}^n$. Note that for $\phi\in\mathscr{S}(\mathbb{R}^n)$,

$$\langle\delta_\omega,\widehat{\phi}\rangle = \widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\phi(x)e^{-ix\cdot\omega}\,dx = \Big\langle\frac{e^{-ix\cdot\omega}}{\sqrt{2\pi}^n},\phi\Big\rangle \equiv \langle\widehat{\delta_\omega},\phi\rangle;$$

thus the Fourier transform of the Dirac delta function at point ω is the function $\widehat{\delta_\omega}(\xi) = \dfrac{e^{-i\xi\cdot\omega}}{\sqrt{2\pi}^n}$. The inverse Fourier transform of δ_ω can be computed in the same fashion and we have $\check{\delta_\omega}(\xi) = \dfrac{e^{i\xi\cdot\omega}}{\sqrt{2\pi}^n}$. We note that $\check{\delta_\omega} = \widehat{\widehat{\widehat{\delta_\omega}}} = \widehat{\widehat{\widehat{\delta_\omega}}}$.

Symbolically, "assuming" that $\langle\delta_\omega,\phi\rangle = \phi(\omega)$ for all continuous function ϕ,

$$\widehat{\delta_\omega}(\xi) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\delta_\omega(x)e^{-ix\cdot\xi}\,dx = \frac{1}{\sqrt{2\pi}^n}e^{-ix\cdot\xi}\Big|_{x=\omega} = \frac{e^{-i\xi\cdot\omega}}{\sqrt{2\pi}^n}$$

and

$$\check{\delta_\omega}(\xi) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\delta_\omega(x)e^{ix\cdot\xi}\,dx = \frac{1}{\sqrt{2\pi}^n}e^{ix\cdot\xi}\Big|_{x=\omega} = \frac{e^{i\xi\cdot\omega}}{\sqrt{2\pi}^n}\,.$$

Example 9.50 (The Fourier transform of $e^{ix\cdot\omega}$)**.** By "definition" and the Fourier inversion formula, for $\phi \in \mathscr{S}(\mathbb{R}^n)$ we have

$$\langle e^{ix\cdot\omega}, \widehat{\phi}\rangle = \int_{\mathbb{R}^n} e^{ix\cdot\omega}\widehat{\phi}(x)\,dx = \sqrt{2\pi}^n\cdot\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\widehat{\phi}(x)e^{ix\cdot\omega}\,dx = \sqrt{2\pi}^n\check{\widehat{\phi}}(\omega) = \sqrt{2\pi}^n\phi(\omega)\,;$$

thus

$$\langle e^{ix\cdot\omega}, \widehat{\phi}\rangle = \sqrt{2\pi}^n\phi(\omega) = \langle\sqrt{2\pi}^n\delta_\omega, \phi\rangle\,.$$

Therefore, the Fourier transform of the function $s(x) = e^{ix\cdot\omega}$ is $\sqrt{2\pi}^n\delta_\omega$, where δ_ω is the Dirac delta function at point ω introduced in Example 9.49. We note that this result also implies that

$$\widehat{\check{\delta_\omega}} = \delta_\omega \qquad \forall\,\omega\in\mathbb{R}^n\,.$$

Similarly, $\check{\widehat{\delta_\omega}} = \delta_\omega$ for all $\omega \in \mathbb{R}^n$; thus the Fourier inversion formula is also valid for the Dirac δ function.

Example 9.51 (The Fourier Transform of the Sine function)**.** Let $s(x) = \sin\omega x$, where ω denotes the frequency of this sine wave. Since $\sin\omega x = \dfrac{e^{i\omega x} - e^{-i\omega x}}{2i}$, we conclude that the Fourier transform of $s(x) = \sin\omega x$ is

$$\frac{\sqrt{2\pi}}{2i}\big(\delta_\omega - \delta_{-\omega}\big)$$

since if T_1, T_2 are tempered distributions, then $T = T_1 + T_2$ satisfies

$$\langle\widehat{T}, \phi\rangle = \langle T_1 + T_2, \widehat{\phi}\rangle = \langle T_1, \widehat{\phi}\rangle + \langle T_2, \widehat{\phi}\rangle = \langle\widehat{T_1}, \phi\rangle + \langle\widehat{T_2}, \phi\rangle = \langle\widehat{T_1} + \widehat{T_2}, \phi\rangle \qquad \forall\,\phi\in\mathscr{S}(\mathbb{R}^n)$$

which shows that $\widehat{T} = \widehat{T_1} + \widehat{T_2}$.

Theorem 9.52. *Let $T \in \mathscr{S}(\mathbb{R}^n)'$. Then $\check{\widehat{T}} = \widehat{\check{T}} = T$.*

Proof. To see that $\check{\widehat{T}}$ and T are the same tempered distribution, we need to show that $\langle\check{\widehat{T}}, \phi\rangle = \langle T, \phi\rangle$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$. Nevertheless, by the definition of the Fourier transform and the inverse Fourier transform of tempered distributions,

$$\langle\check{\widehat{T}}, \phi\rangle = \langle\widehat{T}, \check{\phi}\rangle = \langle T, \widehat{\check{\phi}}\rangle = \langle T, \phi\rangle \qquad \forall\,\phi\in\mathscr{S}(\mathbb{R}^n)\,.$$

The identity $\widehat{\check{T}} = T$ can be proved in the same fashion. □

Example 9.53 (The Fourier Transform of the sinc function)**.** The rect/rectangle function, also called the gate function or windows function, is a function $\Pi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Pi(x) = \begin{cases} 1 & \text{if } |x| < 1\,, \\ 0 & \text{if } |x| \geqslant 1\,. \end{cases}$$

Since $\Pi \in L^1(\mathbb{R})$, we can compute its (inverse) Fourier transform in the usual way, and we have

$$\widehat{\Pi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Pi(x) e^{-ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-ix\xi}}{-i\xi}\Big|_{x=-1}^{x=1} = \sqrt{\frac{2}{\pi}} \frac{\sin\xi}{\xi} \qquad \forall\, \xi \neq 0$$

and $\widehat{\Pi}(0) = \sqrt{\dfrac{2}{\pi}}$. Define the ***unnormalized sinc function*** $\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,. \end{cases}$
Then $\widehat{\Pi}(\xi) = \sqrt{\dfrac{2}{\pi}}\operatorname{sinc}(\xi)$. Similar computation shows that $\check{\Pi}(\xi) = \widehat{\Pi}(\xi) = \sqrt{\dfrac{2}{\pi}}\operatorname{sinc}(\xi)$.

Even though the sinc function is not integrable, we can apply Theorem 9.52 and see that

$$\widehat{\operatorname{sinc}}(\xi) = \widecheck{\operatorname{sinc}}(\xi) = \sqrt{\frac{\pi}{2}}\,\Pi(\xi) \qquad \forall\, \xi \in \mathbb{R}\,.$$

Theorem 9.54. *Let* $T \in \mathscr{S}(\mathbb{R}^n)'$. *Then*

$$\langle \widehat{\tau_h T}, \phi \rangle = \langle \widehat{T}(\xi), \phi(\xi) e^{-i\xi\cdot h} \rangle, \quad \langle \widehat{d_\lambda T}, \phi \rangle = \langle \widehat{T}, d_\lambda \phi \rangle \ \ \textit{and} \ \ \langle \widehat{\widehat{T}}, \phi \rangle = \langle \check{T}, \phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

A short-hand notation for identities above are $\widehat{\tau_h T}(\xi) = \widehat{T}(\xi) e^{-i\xi\cdot h}$, $\widehat{d_\lambda T}(\xi) = \lambda^n \widehat{T}(\lambda\xi)$, *and* $\widehat{\widehat{T}}(\xi) = \check{T}(\xi)$.

Proof. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. For $h \in \mathbb{R}^n$, define $\phi_h(x) = \phi(x) e^{-ix\cdot h}$. Then

$$(\tau_{-h}\widehat{\phi})(\xi) = \widehat{\phi}(\xi + h) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \phi(x) e^{-ix\cdot(\xi+h)}\, dx = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \phi(x) e^{-ix\cdot h} e^{-ix\cdot\xi}\, dx = \widehat{\phi_h}(\xi)\,.$$

By the definition of the Fourier transform of tempered distribution and the translation operator,

$$\langle \widehat{\tau_h T}, \phi \rangle = \langle T, \tau_{-h}\widehat{\phi} \rangle = \langle T, \widehat{\phi_h} \rangle = \langle \widehat{T}(x), \phi(x) e^{-ix\cdot h} \rangle = \langle \widehat{T}(\xi), \phi(\xi) e^{-i\xi\cdot h} \rangle\,.$$

On the other hand, for $\lambda > 0$,

$$(d_{\lambda^{-1}}\widehat{\phi})(\xi) = \widehat{\phi}(\lambda\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \phi(x) e^{-ix\cdot(\lambda\xi)}\, dx = \lambda^{-n} \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \phi\big(\frac{x}{\lambda}\big) e^{-ix\cdot\xi}\, dx = \lambda^{-n} \widehat{d_\lambda \phi}(\xi)\,.$$

Therefore,

$$\langle \widehat{d_\lambda T}, \phi \rangle = \langle T, \lambda^n d_{\lambda^{-1}} \widehat{\phi} \rangle = \langle T, \widehat{d_\lambda \phi} \rangle = \langle \widehat{T}, d_\lambda \phi \rangle = \langle \lambda^n d_{\lambda^{-1}} \widehat{T}, \phi \rangle .$$

The identity $\langle \widehat{\widehat{T}}, \phi \rangle = \langle \check{T}, \phi \rangle$ follows from that $\widehat{\widehat{\phi}} = \check{\phi}$, and the detail proof is left to the readers. □

Remark 9.55. One can check easily that $\widehat{\tau_h f}(\xi) = \widehat{f}(\xi) e^{-i\xi \cdot h}$ and $\widehat{d_\lambda f}(\xi) = \lambda^n \widehat{f}(\lambda \xi)$ if $f \in L^1(\mathbb{R}^n)$.

Next we define the convolution of a tempered distribution and a Schwartz function. Before proceeding, we note that if $f, g, \phi \in \mathscr{S}(\mathbb{R}^n)$, then the Fubini Theorem implies that

$$\begin{aligned} \langle f * g, \phi \rangle &= \int_{\mathbb{R}^n} (f * g)(x) \phi(x) \, dx = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} f(y) g(x - y) \, dy \Big) \phi(x) \, dx \\ &= \int_{\mathbb{R}^n} f(y) \Big(\int_{\mathbb{R}^n} g(x - y) \phi(x) \, dx \Big) \, dy \\ &= \int_{\mathbb{R}^n} f(y) \Big(\int_{\mathbb{R}^n} \widetilde{g}(y - x) \phi(x) \, dx \Big) \, dy = \langle f, \widetilde{g} * \phi \rangle . \end{aligned}$$

The change of variable formula further shows that

$$\begin{aligned} (\widetilde{g} * \phi)(y) &= \Big(\int_{\mathbb{R}^n} \widetilde{g}(x) \phi(y - x) \, dx = \int_{\mathbb{R}^n} \widetilde{g}(-x) \phi(y + x) \, dx \\ &= \int_{\mathbb{R}^n} g(x) \widetilde{\phi}(-y - x) \, dx = (g * \widetilde{\phi})(-y) = \widetilde{g * \widetilde{\phi}}(y) ; \end{aligned}$$

thus

$$\langle f * g, \phi \rangle = \langle f, \widetilde{g} * \phi \rangle = \langle f, \widetilde{g * \widetilde{\phi}} \rangle = \langle \widetilde{f}, g * \widetilde{\phi} \rangle \qquad \forall \, f, g, \phi \in \mathscr{S}(\mathbb{R}^n) .$$

The identity above serves as the origin of the convolution of a tempered distribution and a Schwartz function.

Definition 9.56 (Convolution)**.** Let $T \in \mathscr{S}(\mathbb{R}^n)'$ and $g \in \mathscr{S}(\mathbb{R}^n)$. The convolution of T and g, denoted by $T * g$, is the tempered distribution given by

$$\langle T * g, \phi \rangle = \langle T, \widetilde{g} * \phi \rangle = \langle \widetilde{T}, g * \widetilde{\phi} \rangle \qquad \forall \, \phi \in \mathscr{S}(\mathbb{R}^n) ,$$

where $\widetilde{T}$ is the tempered distribution given in Definition 9.46.

Remark 9.57. We will explain why $T \ast g \in \mathscr{S}(\mathbb{R}^n)'$ (if $T \in \mathscr{S}(\mathbb{R}^n)'$ and $g \in \mathscr{S}(\mathbb{R}^n)$) later in Remark 9.60. For the time being we can temporarily treat the convolution given above as a "computational" definition (without knowing that if $T \ast g$ is continuous); that is, $T \ast g : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ is defined by

$$(T \ast g)(\phi) \equiv \langle T, \widetilde{g} \ast \phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)$$

since $\langle T, \widetilde{g} \ast \phi \rangle$ is well-defined, and $\langle T \ast g, \phi \rangle$ is another expression of $(T \ast g)(\phi)$.

Example 9.58. Let δ_ω be the Dirac delta function at point $\omega \in \mathbb{R}^n$, and $g \in \mathscr{S}(\mathbb{R}^n)$. Then $\delta_\omega \ast g = \tau_\omega g$ since if $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle \delta_\omega, \widetilde{g} \ast \phi \rangle = (\widetilde{g} \ast \phi)(\omega) = \int_{\mathbb{R}^n} \widetilde{g}(y)\phi(\omega - y)\, dy = \int_{\mathbb{R}^n} g(z-\omega)\phi(z)\, dz = \big\langle \tau_\omega g, \phi \big\rangle$$

In symbol,

$$(\delta_\omega \ast g)(x) = \int_{\mathbb{R}^n} \delta_\omega(y) g(x-y)\, dy = g(x-\omega) = (\tau_\omega g)(x)\,. \tag{9.4.10}$$

Similar to Theorem 9.26 and Corollary 9.27, the product and the convolutions of functions are related under Fourier transform.

Theorem 9.59. *Let $T \in \mathscr{S}(\mathbb{R}^n)'$ and $g \in \mathscr{S}(\mathbb{R}^n)$. Then*

$$\langle T \star g, \widehat{\phi} \rangle = \langle \widehat{T}, \widehat{g}\phi \rangle \quad \textit{and} \quad \langle T \star g, \check{\phi} \rangle = \langle \check{T}, \check{g}\phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,,$$

and

$$\langle \widehat{T} \star \widehat{g}, \phi \rangle = \langle T, g\widehat{\phi} \rangle \quad \textit{and} \quad \langle \check{T} \star \check{g}, \phi \rangle = \langle T, g\check{\phi} \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,,$$

where $S \star h = \dfrac{1}{\sqrt{2\pi}^{\,n}}(S \ast h)$ *if* $S \in \mathscr{S}(\mathbb{R}^n)'$ *and* $h \in \mathscr{S}(\mathbb{R}^n)$.

Proof. Note that the "convolution" $\star$ also satisfies that

$$\langle T \star g, \phi \rangle = \langle T, \widetilde{g} \star \phi \rangle = \langle \widetilde{T}, g \star \widetilde{\phi} \rangle\,.$$

By Theorem 9.26 and Corollary 9.27,

$$\langle T \star g, \widehat{\phi} \rangle = \langle T, \widetilde{g} \star \widehat{\phi} \rangle = \langle \check{\widehat{T}}, \widetilde{g} \star \widehat{\phi} \rangle = \langle \widehat{T}, \mathscr{F}^*[\widetilde{g} \star \widehat{\phi}] \rangle = \big\langle \widehat{T}, \check{\widetilde{g}}\check{\widehat{\phi}} \big\rangle = \langle \widehat{T}, \widehat{g}\phi \rangle$$

and by the definition of the convolution of tempered distributions and Schwartz functions,

$$\langle T, g\widehat{\phi} \rangle = \langle \check{\widehat{T}}, g\widehat{\phi} \rangle = \langle \widehat{T}, \mathscr{F}^*(g\widehat{\phi}) \rangle = \langle \widehat{T}, \check{g} \star \phi \rangle = \langle \widehat{T}, \widetilde{\widehat{g}} \star \phi \rangle = \langle \widehat{T} \star \widehat{g}, \phi \rangle\,.$$

The counterpart for the inverse Fourier transform can be proved similarly. $\square$

Remark 9.60. Let $g, \phi \in \mathscr{S}(\mathbb{R}^n)$, and $T \in \mathscr{S}(\mathbb{R}^n)'$ satisfy $\big|\langle T, u\rangle\big| \leqslant C_k p_k(u)$ for all $u \in \mathscr{S}(\mathbb{R}^n)$ and $k \gg 1$. By Theorem 9.59, we find that

$$\langle T * g, \phi\rangle = \sqrt{2\pi}^{\,n}\langle T \star g, \widehat{\check{\phi}}\rangle = \sqrt{2\pi}^{\,n}\langle \widehat{T}, \widehat{g}\check{\phi}\rangle .$$

By the fact that

$$\begin{aligned} p_k(gh) &= \sup_{x\in\mathbb{R}^n, |\alpha|\leqslant k} \langle x\rangle^k \big|D^\alpha(gh)(x)\big| \leqslant \sup_{x\in\mathbb{R}^n, |\alpha|\leqslant k} \sum_{0\leqslant\beta\leqslant\alpha} C^\alpha_\beta \langle x\rangle^k \big|D^{\alpha-\beta}g(x)D^\beta h(x)\big| \\ &\leqslant \Big(\sup_{|\alpha|\leqslant k} \sum_{0\leqslant\beta\leqslant\alpha} C^\alpha_\beta\Big) p_k(g)p_k(h) \equiv M_k p_k(g)p_k(h) \qquad \forall\, g, h \in \mathscr{S}(\mathbb{R}^n)\,, \end{aligned}$$

we conclude from (9.4.8) and (9.4.9) that for $k \gg 1$,

$$\begin{aligned} \big|\langle T * g, \phi\rangle\big| &\leqslant \sqrt{2\pi}^{\,n} C_k \bar{C}(n,k) p_{k+n+1}\big(\widehat{g}\check{\phi}\big) \leqslant \sqrt{2\pi}^{\,n} C_k \bar{C}(n,k) M_k p_{k+n+1}\big(\widehat{g}\big) p_{k+n+1}\big(\widehat{\check{\phi}}\big) \\ &\leqslant \sqrt{2\pi}^{\,n} C_k M_k \bar{C}(n,k)\bar{C}(n,k+n+1)^2 p_{k+2n+2}(g) p_{k+2n+2}\big(\widetilde{\phi}\big) \\ &= \widetilde{C}(n,k) p_{k+2n+2}(g) p_{k+2n+2}(\phi)\,. \end{aligned}$$

Therefore, $T * g$ is a tempered distribution.

Remark 9.61. For $T \in \mathscr{S}(\mathbb{R}^n)'$ and $g \in \mathscr{S}(\mathbb{R}^n)$, define $gT : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ by

$$\langle gT, \phi\rangle = \langle T, g\phi\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

Then the fact that $T * g \in \mathscr{S}(\mathbb{R}^n)'$ (from Remark 9.60) and Theorem 9.59 show that

$$\langle \widehat{T \star g}, \phi\rangle = \langle \widehat{g}\widehat{T}, \phi\rangle \quad \text{and} \quad \langle \widecheck{T \star g}, \phi\rangle = \langle \check{g}\check{T}, \phi\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,,$$

and

$$\langle \widehat{T} \star \widehat{g}, \phi\rangle = \langle \widehat{gT}, \phi\rangle \quad \text{and} \quad \langle \check{T} \star \check{g}, \phi\rangle = \langle \widecheck{gT}, \phi\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,,$$

In other words, we have $\widehat{T \star g} = \widehat{g}\widehat{T}$, $\widecheck{T \star g} = \check{g}\check{T}$, $\widehat{gT} = \widehat{T} \star \widehat{g}$ and $\widecheck{gT} = \check{T} \star \check{g}$ in $\mathscr{S}(\mathbb{R}^n)'$. Therefore, Theorem 9.59 can be viewed as the generalization of Theorem 9.26 and Corollary 9.27.

Remark 9.62. If $S \in \mathscr{S}(\mathbb{R}^n)'$ satisfies that $S * \phi \in \mathscr{S}(\mathbb{R}^n)$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$, we can also define the convolution of T and S by

$$\langle T * S, \phi\rangle = \langle \widetilde{T}, S * \widetilde{\phi}\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

In other words, it is possible to define the convolution of two tempered distributions.

For example, from Example 9.58 we find that $\delta_\omega * \phi = \tau_\omega \phi$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$; thus $\delta_\omega * \phi \in \mathscr{S}(\mathbb{R}^n)$ for all $\mathscr{S}(\mathbb{R}^n)$ (and $\omega \in \mathbb{R}^n$). Therefore, if T is a tempered distribution, $T * \delta_\omega$ is also a tempered distribution and is given by

$$\langle T * \delta_\omega, \phi \rangle = \big\langle \widetilde{T}, \tau_\omega \widetilde{\phi} \big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

Further computation shows that

$$\langle T * \delta_\omega, \phi \rangle = \big\langle \widetilde{T}, \widetilde{\tau_{-\omega}\phi} \big\rangle = \big\langle T, \tau_{-\omega}\phi \big\rangle = \big\langle \tau_\omega T, \phi \big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

The identity above shows that $T * \delta_\omega = \tau_\omega T$ for all $T \in \mathscr{S}(\mathbb{R}^n)'$. This formula agrees with (9.4.10).

Index

Accumulation Points, 81, 92
Affine Hyperplanes, 36
Analytic Function, 149
Archimedean Property, 8
Arzelà Bounded Convergence Theorem, 186, 187
Arzelà-Ascoli Theorem, 236

Banach Spaces, 52
Bernstein Polynomial, 238
Bijections, iii
Bilinear Map, Bilinear Form, 134
Bolzano-Weierstrass Property, 21, 76
Bolzano-Weierstrass Theorem, 49
Boundary of Sets, 60
Bounded Linear Map, 112
Bounded Sets, 12, 46

Cantor Set, 85
Cauchy Criterion, 53
Cauchy Sequences, 23, 51
Cauchy-Schwarz Inequality, 42
Cesàro Mean, 266
Chain Rule, 131
Clairaut's Theorem, 142
Closed Sets, 60
Closure of Sets, 58
Cluster Points, 21, 49
Compact Sets, 74
Comparison Test
  for Improper Integrals, 201
  for Series, 54
Completeness, 27
Connected Component, 90
Connected Sets, 76, 77
Continuity, Countinuous Maps, 92
Continuously Differentiable Functions, 128
Convolution, 259, 294, 300, 313
Countable Sets, iv
Cover, Subcover, Finite Subcover, 74

Darboux Integrable Functions, 153
De Morgan's Law, ii
Dense Subsets, 59
Denumerable Sets, iv
Derivative, 117, 132
Derived Sets, 81, 92
Diagonal Process, 234
Diameter of Sets, 89
Differentiability, 117, 132
Dini Theorem, 245
Directional Derivative, 118
Dirichlet Kernel, 259
Dirichlet Test
  for Series, 54

Disconnect Sets, 76
Discrete Fourier Transform, 281
Discrete Metric, 43
Dominated Convergence Theorem, 203

Equi-Continuity, 230
Euclidean Spaces, 34
Exterior of Sets, 58
Exterior Points, 58
Extreme Value Theorem, 101

Fast Fourier Transform, 282
Fejér Kernel, 267
Fixed-Points, 106
Fourier Coefficients, 255
Fourier Inversion Formula, 295, 302
Fourier Series, 255
Fourier Transform, 288
Fubini Theorem, 176, 208

Gamma Function, 197
Gibbs Phenomenon, 272
Gradient of Functions, 125
Greatest Lower Bound, 16

Hölder Continuity, 261
Heine-Borel Theorem, 69
Helly Selection Theorem, 251
Hessian, Hessian Matrix, 141
Hilbert Spaces, 52
Hyperplanes, 36

Improper Integrals, 190
Infimum, 16
Injections, Injective functions, iii
Inner Products, Inner Product Spaces, 40
Interior of Sets, 58
Interior Points, 58
Intermediate Value Theorem, 105
Interval of Convergence, Convergence Interval, 224
Isolated Points, 81, 92

Jacobian Matrix, 123

Least Upper Bound, 16
Least Upper Bound Property, 19
Lebesgue Number, 76
Lebesgue Theorem, 164
Lebesgue's Differentiation Theorem, 168
Limit of Sequences, 9, 45
Limit Points, 58
Limit Superior, Limit Inferior, 29
Line Segment Principle, 84
Linear Map, 112
Linear Order, 3
Lower Bounds, 16
Lower Integral, Upper Integral, 153
Lower Sum, Upper Sum, 153

Measure Zero, 160
Mesh Size of Partitions, 151, 152
Metric, Metric Spaces, 43
Minkowski's Inequality, 37
Mollifier, 300
Monotone Convergence Theorem, 185, 186, 204
Monotone Sequence Property, 14
Monotone Sequences, 14
Multi-index, 146

Nested Sets Property, 87
Norm, Normed Vector Spaces, 36

Open Cover, 74
Open Sets, 64
Ordered Fields, 3
Oscillation, 163

Partial Order, 3
Partition, 150, 151
Perfect Set, 85
Permutation, 10
Plancherel formula, 297
Pointwise Compact, Pointwise Bounded, Pointwise Pre-Compact, 235
Pointwise Convergence, 210
Power Series, 223
Pre-Compact Sets, 76

Radius of Convergence, 224
Ratio Test, 57
    for Series, 54
Refinement of Partitions, 154
Riemann Condition for Integrability, 155
Riemann Integrable Functions, 153
Riemann Measurable Functions, 189
Riemann Measurable Sets, 189
Riemann Sums, 152
Root Test
    for Series, 54

Sample Points of Partitions, 152
Sampling Frequency, 282
Sandwich Lemma, 11
Schwartz space $\mathscr{S}(\mathbb{R}^n)$, 289
Separable Sets, 86
Sequences
    in Metric Spaces, 45
    in Ordered Fields, 9
Sequentially Compact Sets, 68
Series, 52
Simple Function, 239
sinc Function
    Normalized, 303
    Unnormalized, 312
Stone-Weierstrass Theorem, 237, 241
Subsequences, 19
Subspace Topology, 78
Sup-norm of Continuous Mappings, 227
Support, 289
Supremum, 16
Surjections, Surjective functions, iii

Taylor Theorem, 144, 145
Tempered Distributions, 305
Tonelli Theorem, 207
Total Order, 3
Totally Bounded Sets, 70

Uniform Continuity, 106
Uniform Convergence, 210
Upper Bounds, 16
Upper Integral, Lower Integral, 153
Upper Sum, Lower Sum, 153

Vector Spaces, 34
Vector Subspaces, 35
Volume of Sets, 159

Weierstrass M-Test, 221