Exercise Problem Sets 11

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Problem 1. Determine the discrete Fourier transform of the sequence $(1,0,0,-1)$. Check the inversion formula.

Solution. For $N=4$, $\omega_N \equiv \exp\big(-\frac{2\pi i}{N}\big) = \exp\big(-\frac{\pi i}{2}\big) = -i$. Then

$$F_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_N & \omega_N^2 & \omega_N^3 \\ 1 & \omega_N^2 & \omega_N^4 & \omega_N^6 \\ 1 & \omega_N^3 & \omega_N^6 & \omega_N^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}.$$

Therefore, the DFT $\boldsymbol{X}$ of the given sequence $\boldsymbol{x} = [1,0,0,-1]^{\mathrm{T}}$ is

$$\boldsymbol{X} = F_N \boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1-i \\ 2 \\ 1+i \end{bmatrix}.$$ □

Problem 2. Determine the discrete Fourier transform of the N-periodic sequence $x_n = \sin\frac{n\pi}{N}$, $n = 0, \cdots, N-1$.

Problem 3. Give a version of the Parseval identity for discrete Fourier transform.

Solution. For a fixed $N \in \mathbb{N}$, define

$$\widetilde{F}_N = \frac{1}{\sqrt{N}} F_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-\frac{2\pi i}{N}} & e^{-\frac{4\pi i}{N}} & \cdots & e^{-\frac{2\pi(N-1)i}{N}} \\ 1 & e^{-\frac{4\pi i}{N}} & e^{-\frac{8\pi i}{N}} & \cdots & e^{-\frac{4\pi(N-1)i}{N}} \\ \vdots & & & \ddots & \vdots \\ 1 & e^{-\frac{2\pi(N-1)i}{N}} & e^{-\frac{4\pi(N-1)i}{N}} & \cdots & e^{-\frac{2\pi(N-1)^2 i}{N}} \end{bmatrix};$$

that is, the (k,ℓ)-entry of $\widetilde{F}_N$ is $\omega_N^{(k-1)(\ell-1)}$. Then by the fact that $F_N^{-1} = \frac{1}{N}F_N^*$, we find that

$$\widetilde{F}_N^{-1} = \Big(\frac{1}{\sqrt{N}}F_N\Big)^{-1} = \sqrt{N}F_N^{-1} = \sqrt{N}\frac{1}{N}F_N^* = \frac{1}{\sqrt{N}}F_N^* = \Big(\frac{1}{\sqrt{N}}F_N\Big)^* = \widetilde{F}_N^*.$$

Therefore, $\widetilde{F}_N$ is unitary so for any vectors $\boldsymbol{x} \in \mathbb{C}^N$, we have $\|\widetilde{F}_N\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$. Therefore, if $\boldsymbol{x} \in \mathbb{C}^N$ and $\boldsymbol{X}$ is the DFT of $\boldsymbol{x}$, then

$$\|\boldsymbol{X}\|_2^2 = \|F_N\boldsymbol{x}\|_2^2 = \|\sqrt{N}\widetilde{F}_N\boldsymbol{x}\|_2^2 = N\|\widetilde{F}_N\boldsymbol{x}\|_2^2 = N\|\boldsymbol{x}\|_2^2$$

which gives the Parseval identity

$$\sum_{k=0}^{N-1} |X_k|^2 = N\sum_{k=0}^{N-1} |x_k|^2$$

for the discrete Fourier transform. □

Problem 4. Let $x(n)$ be N-periodic, and

$$x(n) = \begin{cases} 1 & \text{if } 0 \leqslant n \leqslant k-1\,, \\ 0 & \text{if } k \leqslant n \leqslant N-1\,. \end{cases}$$

Compute the discrete Fourier transform. Using the Parseval identity, compute the sum

$$\sum_{n=1}^{N-1} \frac{1-\cos\frac{2\pi nk}{N}}{1-\cos\frac{2\pi n}{N}}\,.$$

Problem 5. Find the Fourier transform of the following functions.

1. $f:\mathbb{R}\to\mathbb{R}$ given by $f(x) = xe^{-tx^2}$ for $t>0$.
2. $f:\mathbb{R}\to\mathbb{R}$ given by $f(x) = \chi_{(-a,a)}(x)$, the characteristic (indicator) function of the set $(-a,a)$.
3. $f:\mathbb{R}\to\mathbb{R}$ given by $f(x) = \begin{cases} e^{-tx} & \text{if } x>0\,, \\ 0 & \text{if } x \leqslant 0\,, \end{cases}$ where $t>0$.

Problem 6. Compute the Fourier transform of the following functions.

(a) $y = x\chi_{[-1,1]}(x)$;

(b) $y = \sin x\chi_{[-\pi,\pi]}(x)$;

(c) $y = e^{-x}H(x)$;

(d) $y = e^{-|x|}\cos x$;

(e) $y = \dfrac{1}{x^2+6x+13}$;

(f) $y = \dfrac{x}{(x^2+1)^2}$;

(g) $y = \dfrac{1}{(x^2+1)^2}$.

Here, H is the Heaviside function (that is, $H(x) = \mathbf{1}_{[1,\infty)}(x)$) and $\chi_{[a,b]}(x)$ is the characteristic function of $[a,b]$; that is, $\chi_{[a,b]}(x) = 1$ for $x\in[a,b]$ and 0 else.

Problem 7. Suppose that the Fourier transform of a Schwartz function $f \in \mathscr{S}(\mathbb{R})$ is $\widehat{f}(\xi)$. Find the Fourier transform of the function $y = f(2x+1)\cos x$.

Solution. By the Euler identity,

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}\,.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}} f(2x+1)\cos x e^{-ix\xi}\,dx &= \int_{\mathbb{R}} f(2x+1)\frac{e^{ix}+e^{-ix}}{2}e^{-ix\xi}\,dx \\
&= \frac{1}{2}\Big(\int_{\mathbb{R}} f(2x+1)e^{-ix(\xi-1)}\,dx + \int_{\mathbb{R}} f(2x+1)e^{-ix(\xi+1)}\,dx\Big) \\
&= \frac{1}{4}\Big(\int_{\mathbb{R}} f(t)e^{-i\frac{t-1}{2}(\xi-1)}\,dt + \int_{\mathbb{R}} f(t)e^{-i\frac{t-1}{2}(\xi+1)}\,dt\Big) \\
&= \frac{1}{4}\Big(e^{i\frac{\xi-1}{2}}\int_{\mathbb{R}} f(t)e^{-it\frac{\xi-1}{2}}\,dt + e^{i\frac{\xi+1}{2}}\int_{\mathbb{R}} f(t)e^{-it\frac{\xi+1}{2}}\,dt\Big)
\end{aligned}$$

which shows that the Fourier of $y = f(2x+1)\cos x$ is $\frac{1}{4}\Big[e^{i\frac{\xi-1}{2}}\widehat{f}\big(\frac{\xi-1}{2}\big) + e^{i\frac{\xi+1}{2}}\widehat{f}\big(\frac{\xi+1}{2}\big)\Big]$. □

Problem 8. Let $\alpha > 0$ be given. Show that the Fourier transform of the function

$$f(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}e^{-\frac{|x|^2}{t}}\,dt\,, \qquad x \in \mathbb{R}^n\,,$$

is positive.

Proof. For $\xi \in \mathbb{R}^n$, define $g(x,t) = t^{\alpha-1}e^{-t}e^{-\frac{|x|^2}{t}}e^{ix\cdot\xi}$. By the Tonelli Theorem,

$$\begin{aligned}\int_{\mathbb{R}^n\times(0,\infty)} \big|g(x,t)\big|\,d(x,t) &= \int_{\mathbb{R}}\int_0^\infty t^{\alpha-1}e^{-t}e^{-\frac{|x|^2}{t}}\,dtdx = \int_0^\infty t^{\alpha-1}e^{-t}\Big(\int_{\mathbb{R}} e^{-\frac{|x|^2}{t}}\,dx\Big)dt \\ &= \int_0^\infty t^{\alpha-1}e^{-t}\sqrt{\pi t}^{\,n}\,dt = \sqrt{\pi}^{\,n}\int_0^\infty t^{\frac{n}{2}+\alpha-1}e^{-t}\,dt = \sqrt{\pi}^{\,n}\Gamma(\frac{n}{2}+\alpha-1) < \infty\,.\end{aligned}$$

The computation above also shows that $f \in L^1(\mathbb{R}^n)$. Therefore, the Fubini Theorem implies that

$$\begin{aligned}\Gamma(\alpha)\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}^{\,n}}\int_{\mathbb{R}^n}\Big(\int_0^\infty t^{\alpha-1}e^{-t}e^{-\frac{|x|^2}{t}}\,dt\Big)e^{-ix\cdot\xi}dx \\ &= \frac{1}{\sqrt{2\pi}^{\,n}}\int_{\mathbb{R}^n}\Big(\int_0^\infty t^{\alpha-1}e^{-t}e^{-\frac{|x|^2}{t}}e^{-ix\cdot\xi}dt\Big)dx = \int_0^\infty t^{\alpha-1}e^{-t}\Big(\frac{1}{\sqrt{2\pi}^{\,n}}\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{t}}e^{-ix\cdot\xi}dx\Big)dt \\ &= \int_0^\infty t^{\alpha-1}e^{-t}\mathscr{F}_x[e^{-\frac{|x|^2}{t}}](\xi)\,dt = \int_0^\infty t^{\alpha-1}e^{-t}\sqrt{\frac{t}{2}}^{\,n}e^{-4t|\xi|^2}\,dt > 0\,.\end{aligned}$$

The positivity of $\widehat{f}$ then follows from the fact that $\Gamma(\alpha) > 0$. □