Calculus Quiz 3

1. (5 pts)

a. Find the derivative of the function $g(x) = \dfrac{1}{\sqrt{x}}$ by using the definition of derivative.

b. Let f be a smooth function defined on $\mathbb{R}$ and $c \in \mathbb{R}$. If $f'(c) = a,\ f''(c) = b$. Evaluate the following limit

$$\lim_{h\to 0}\left[\frac{2f(c+h) - 4f(c) + 2f(c-h)}{3h^2} + \frac{f(c+h) - f(c-h)}{h}\right]$$

Sol.

a.

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0}\frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x(x+h)}} \\
&= \lim_{h\to 0}\frac{x - (x+h)}{h\sqrt{x(x+h)}\left(\sqrt{x} + \sqrt{x+h}\right)} = \lim_{h\to 0}\frac{-1}{\sqrt{x(x+h)}\left(\sqrt{x} + \sqrt{x+h}\right)} \\
&= \frac{-1}{2\sqrt{x^2}\cdot\sqrt{x}} = \frac{-1}{2x^{\frac{3}{2}}}
\end{aligned}$$

b. Let $L = \lim\limits_{h\to 0}\left[\dfrac{2f(c+h) - 4f(c) + 2f(c-h)}{3h^2} + \dfrac{f(c+h) - f(c-h)}{h}\right]$.
Note that

$$\begin{aligned}
\frac{f'(c) - f'(c-h)}{h} &= \frac{\lim\limits_{k\to 0}\frac{f(c+k)-f(c)}{k} - \lim\limits_{k\to 0}\frac{f(c-h+k)-f(c-h)}{k}}{h} \\
&= \lim_{k\to 0}\frac{f(c+k) - f(c) - f(c-h+k) + f(c-h)}{hk}
\end{aligned}$$

Then

$$\begin{aligned}
f''(c) &= \lim_{h\to 0}\frac{f'(c) - f'(c-h)}{h} \\
&= \lim_{h\to 0}\lim_{k\to 0}\frac{f(c+k) - f(c) - f(c-h+k) + f(c-h)}{hk}
\end{aligned}$$

Since $k \to 0$ arbitrarily, by taking $k = h$, we have that

$$f''(c) = \lim_{h\to 0}\frac{f(c+h) - 2f(c) + f(c-h)}{h^2}$$

Hence

$$
\begin{aligned}
L &= \frac{2}{3}\lim_{h\to 0}\frac{f(c+h)-2f(c)+f(c-h)}{h^2} + \lim_{h\to 0}\frac{f(c+h)-f(c-h)}{h} \\
&= \frac{2}{3}f''(c) + \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h} + \frac{f(c)-f(c-h)}{h}\right] \\
&= \frac{2}{3}f''(c) + \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} + \lim_{h\to 0}\frac{f(c)-f(c-h)}{h} \\
&= \frac{2}{3}f''(c) + 2f'(c) = \frac{2}{3}b + 2a
\end{aligned}
$$

$\square$

2. (5 pts)

a. Let $f(x)$ be a function satisfying $|f(x)| \leq x^2$ for $-1 \leq x \leq 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

b. Show that

$$
f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}
$$

is differentiable at $x = 0$ and find $f'(0)$.

Proof.

a. Since $|f(x)| \leq x^2$ for $-1 \leq x \leq 1$, then for $x = 0$, we have that $|f(0)| \leq 0$ which implies $f(0) = 0$. Also, we have that $-x \leq \left|\dfrac{f(x)}{x}\right| \leq x,\ \forall -1 \leq x \leq 1$. Since $\lim_{x\to 0} x = \lim_{x\to 0}(-x) = 0$. By Squeeze Theorem,

$$
|f'(0)| = \left|\lim_{h\to 0}\frac{f(h)-f(0)}{h}\right| = \lim_{h\to 0}\left|\frac{f(h)}{h}\right| = 0
$$

This implies that $f'(0) = 0$.

b. Since $|\sin y| \leq 1$, $\forall y$. So $\left|x^2 \sin\dfrac{1}{x}\right| \leq x^2$, $\forall x$. In particular for all $-1 \leq x \leq 1$. Also, since $f(0) = 0$. By argument in **a.**, we can conclude that f is differentiable at $x = 0$ and $f'(0) = 0$.

$\square$