Calculus Quiz 4

1. (5 pts)

a. Suppose $f\left(\frac{\pi}{3}\right) = 4$ and $f'\left(\frac{\pi}{3}\right) = -2$ and let

$$g(x) = f(x)\sin x, \qquad h(x) = \frac{\cos x}{f(x)}$$

Find $g'\left(\frac{\pi}{3}\right)$ and $h'\left(\frac{\pi}{3}\right)$.

b. Let $p(x) = ax^2 + bx,\ q(x) = cx^2 + dx$ and let

$$f(x) = p(x)\cos x + q(x)\sin x$$

Determine a, b, c, d such that $f(x)$ satisfies the equation

$$f''(x) + f(x) = x\sin x$$

Sol.

a. Since $g(x) = f(x)\sin x$ and $h(x) = \dfrac{\cos x}{f(x)}$, then

$$g'(x) = f'(x)\sin x + f(x)\cos x$$
$$h'(x) = \frac{-f(x)\sin x - f'(x)\cos x}{f(x)^2}$$

Hence

$$g'\left(\tfrac{\pi}{3}\right) = f'\left(\tfrac{\pi}{3}\right)\sin\tfrac{\pi}{3} + f\left(\tfrac{\pi}{3}\right)\cos\tfrac{\pi}{3} = 2 - \sqrt{3}$$
$$h'\left(\tfrac{\pi}{3}\right) = \frac{-f\left(\frac{\pi}{3}\right)\sin\frac{\pi}{3} - f'\left(\frac{\pi}{3}\right)\cos\frac{\pi}{3}}{f\left(\frac{\pi}{3}\right)^2} = \frac{1 - 2\sqrt{3}}{16}$$

b. Observe that

$$\begin{aligned}
f'(x) &= p'(x)\cos x - p(x)\sin x + q'(x)\sin x + q(x)\cos x \\
&= \big(p'(x) + q(x)\big)\cos x + \big(q'(x) - p(x)\big)\sin x \\
f''(x) &= \big(p''(x) + q'(x)\big)\cos x - \big(p'(x) + q(x)\big)\sin x \\
&\quad + \big(q''(x) - p'(x)\big)\sin x + \big(q'(x) - p(x)\big)\cos x \\
&= \big(p''(x) + 2q'(x) - p(x)\big)\cos x + \big(q''(x) - 2p'(x) - q(x)\big)\sin x
\end{aligned}$$

Thus

$$f''(x) + f(x) = \big(p''(x) + 2q'(x)\big)\cos x + \big(q''(x) - 2p'(x)\big)\sin x$$

Since $p(x) = ax^2 + bx,\ q(x) = cx^2 + dx$, then

$$p'(x) = 2ax + b, \qquad p''(x) = 2a$$
$$q'(x) = 2cx + d, \qquad q''(x) = 2c$$

Since $f''(x) + f(x) = x \sin x$ then

$$(4cx + 2a + 2d)\cos x + (-4ax - 2b + 2c)\sin x = x \sin x$$

This implies that

$$\begin{cases} 4cx + 2a + 2d = 0 \\ -4ax - 2b + 2c = x \end{cases} \Rightarrow \begin{cases} 4c = 0 \\ 2a + 2d = 0 \\ -4a = 1 \\ -2b + 2c = 0 \end{cases}$$

and therefore $a = -\frac{1}{4},\ b = 0,\ c = 0,\ d = \frac{1}{4}$.

□

2. (5 pts)

a. For what values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?

b. Find equation of the lines both tangent to parabolas $y = x^2$ and $y = -x^2 + 6x - 17$.

Sol.

a. Since $f'(c)$ represent the slope of the tangent line of $f(x)$ at the point $x = c$, so $f(x)$ has a horizontal tangent line if and only if $f'(x) = 0$. Note that $f'(x) = 1 + 2\cos x$, so

$$f'(x) = 0 \Leftrightarrow \cos x = -\frac{1}{2} \Leftrightarrow x = (2n+1)\pi \pm \frac{\pi}{3},\ n \in \mathbb{Z}$$

Therefore $f(x)$ has horizontal tangent at $x = (2n+1)\pi \pm \frac{\pi}{3},\ n \in \mathbb{Z}$.

b. Denote the tangent line by L. Let $f(x) = x^2,\ g(x) = -x^2 + 6x - 17$. Suppose L tangent to $f(x)$ and the point $(x_1, f(x_1))$ and tangent to $g(x)$ at the point $(x_2, g(x_2))$. Then the equation of L can be expressed as

$$\begin{aligned} y &= f'(x_1)(x - x_1) + f(x_1) \\ &= 2x_1(x - x_1) + x_1^2 = 2x_1 x - x_1^2 \\ y &= g'(x_2)(x - x_2) + g(x_2) \\ &= (-2x_2 + 6)(x - x_2) - x_2^2 + 6x_2 - 17 = (-2x_2 + 6)x + x_2^2 - 17 \end{aligned}$$

This implies that

$$\begin{cases} 2x_1 = -2x_2 + 6 \\ -x_1^2 = x_2^2 - 17 \end{cases} \Rightarrow \begin{cases} x_1 = -x_2 + 3 \\ x_1^2 + x_2^2 = 17 \end{cases}$$

$$\Rightarrow x_2^2 - 3x^2 - 4 = (x_2 + 1)(x_2 - 4) = 0$$

Hence $x_2 = -1$ or $x_2 = 4$.
Therefore, the equation of the line tangent to both $y = x^2$ and $y = -x^2 + 6x - 17$ are

$$y = 8x - 16 \text{ and } y = -2x - 1$$

$\square$