Calculus Quiz 7

1. (5 pts) Let $f(x) = \dfrac{(x+1)^2}{x^2+1}$

a. Find the intervals of increase or decrease.
b. Find the local maximum and minimum values.
c. Find the intervals of concavity and the inflection points.
d. Find the asymptotes.
e. Sketch the graph of $f(x)$.

Sol. Note that the denominator of $f(x)$ is always positive, so there is no vertical asymptotes of the graph of f. By evaluating the limit

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+2x+1}{x^2+1} = 1, \quad \lim_{x\to-\infty} f(x) = \lim_{x\to\infty} \frac{x^2-2x+1}{x^2+1} = 1$$

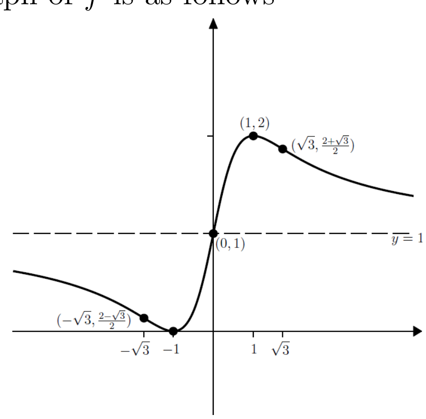
we know that there is an horizontal asymptote $y = 1$ of the graph of f. By computing the first and second derivative of f, we get

$$f'(x) = -\frac{2(x^2-1)}{(x^2+1)^2}, \qquad f''(x) = \frac{4(x^3-3x)}{(x^2+1)^3}$$

It is easy to see that $f'(x) = 0 \Leftrightarrow x = \pm 1$ and $f''(x) = 0 \Leftrightarrow x = 0$ or $x = \pm\sqrt{3}$. Observe that $f''(\pm 1) = \mp 1$, thus f has local maximum at $x = 1$ and local minimum at $x = -1$. We have the following table

x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},-1)$	1	$(-1,0)$	0	$(0,1)$	1	$(1,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
f'	−	−	−	0	+	+	+	0	+	+	+
f''	−	0	+	+	+	0	−	−	−	0	+
f	⤵	inf.pt.	↳	min.	⤴	inf.pt	↷	max.	⤵	inf.pt.	↳

And the graph of f is as follows



□

2. (5 pts)

a. Is it true that $f''(c) = 0$ implies $x = c$ is an inflection point of f? Explain your answer.

b. Show that for any $|\sin b - \sin a| \le |b - a|, \ \forall\, a, b \in \mathbb{R}$.

Sol.

a. It is not true that $f''(c) = 0$ implies $x = c$ is an inflection point of f. For example, consider $f(x) = x^4$, then

$$f'(x) = 4x^3, \qquad f''(x) = 12x^2$$

Then $f''(0) = 0$ but $x = 0$ is not inflection point of $f(x)$.

b. Since $\sin x$ is differentiable function defined on $\mathbb{R}$. For any $a, b \in \mathbb{R}$, say $a \le b$. By Mean Value Theorem, there exists $\xi \in (a, b)$ such that

$$\sin b - \sin a = \cos \xi \cdot (b - a)$$

Since $|\cos x| \le 1, \ \forall\, x$, then

$$|\sin b - \sin a| = |\cos \xi||b - a| \le |b - a|$$

Similarly, the argument also can apply to the case $a > b$, and hence the prove is complete.

□