Calculus Quiz 9

1. (5 pts) *This example shows that the Newton's method can not apply when the root has vertical tangent. The approximation does not converge and will getting worse and worse.*
Apply Newton's method to $f(x) = x^{\frac{1}{3}}$ with $x_0 = 1$ and calculate x_1, x_2, x_3, and x_4. Find a formula for $|x_n|$. What happen to $|x_n|$ as $n \to \infty$?
Sol. Since $f(x) = x^{\frac{1}{3}}$, then $f'(x) = \dfrac{1}{3}x^{-\frac{2}{3}}$. By applying Newton's method, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} = x_n - 3x_n = -2x_n$$

Now $x_0 = 1$, it is clear that $x_1 = -2,\ x_2 = 4,\ x_3 = -8,\ x_4 = 16$. In fact according to above equality, we have that

$$|x_n| = 2^n, \text{ when } x_0 = 1$$

Thus we may conclude that $|x_n| \to \infty$ as $n \to \infty$. □

2. (5 pts) Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and its downward acceleration is

$$a = \begin{cases} 9 - 0.9t & \text{if } 0 \le t \le 10 \\ 0 & \text{if } t > 10 \end{cases}$$

If the raindrop is initially 500 m above the ground, how long does it take to fall?
Sol. Taking the upward direction to be positive. Let $a_1(t)$ denote the acceleration for first 10 seconds. By definition of a, we have that

$$a_1(t) = -9 + 0.9t =: v_1'(t)$$

Then $v_1(t) = -9t + 0.45t^2 + v_0$ for some constant v_0. Since the initial downward velocity is 10 m/s, so $v_1(0) = v_0 = -10$. Thus the velocity function v_1 for first 10 second is

$$v_1(t) = -10 - 9t + 0.45t^2 =: s_1'(t)$$

Then $s_1(t) = -10t - 0.45t^2 + 0.15t^3 + s_0$ for some constant s_0. It is obvious that $s_1(0) = s_0 = 500$. Thus the displacement function s_1 for first 10 second is

$$s_1(t) = 500 - 10t - 0.45t^2 + 0.15t^3$$

Note that $s_1(10) = 100$, so it takes more than 10 seconds for the raindrop to fall. Now for $t > 10$, $a(t) = 0 = v'(t)$ which implies

$$v(t) = \text{constant} = v(10) = -55$$

Hence the last 100 m will take $\dfrac{100}{55} \approx 1.82$ s to fall. Therefore, the total time is $10 + \frac{100}{55} \approx 11.82$ s. $\square$