Calculus Quiz 11

1. (5 pts) Let $f(x) = \int_0^x \frac{dt}{\sqrt{1+t^3}}$, $x > 0$.

a. Show that $f(x)$ has inverse function.

b. Let $g(x)$ be the inverse function of $f(x)$. Show that g has the following property

$$g''(x) = cg(x)^2$$

and determine the value c.

Sol.

a. Note that $f'(x) = \frac{1}{\sqrt{1+x^3}} > 0$ for $x > 0$. This implies that f is strictly increasing, and hence is one-to-one. Therefore it has inverse function.

b. Since g is the inverse function of f, we have that

$$g'(x) = \frac{1}{f'(g(x))} = \sqrt{1+g(x)^3}$$

Thus by chain rule,

$$g''(x) = \frac{3g(x)^2 g'(x)}{2\sqrt{1+g(x)^3}} = \frac{3}{2}g(x)^2 \frac{\sqrt{1+g(x)^3}}{\sqrt{1+g(x)^3}} = \frac{3}{2}g(x)^2$$

This shows that $g''(x) = cg(x)^3$ with $c = \frac{3}{2}$.

$\square$

1. (5 pts)

a. The *error function* is defined as $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. Show that the function $y = e^{x^2}\operatorname{erf}(x)$ satisfies the differential equation

$$y' = 2xy + \frac{2}{\sqrt{\pi}}$$

b. Evaluate the indefinite integral $\int \frac{dx}{\csc x - 2\cot x}$.

Sol.

a. Since $y = e^{x^2}\text{erf}(x)$. Then by Product Rule and FTC,

$$y' = 2xe^{x^2}\text{erf}(x) + e^{x^2}\text{erf}'(x) = 2xy + e^{x^2} \cdot \frac{2}{\sqrt{\pi}}e^{-x^2}$$
$$= 2xy + \frac{2}{\sqrt{\pi}}$$

b.

$$\int \frac{dx}{\csc x - 2\cot x} = \int \frac{\sin x}{1 - 2\cos x}dx$$
$$= \frac{1}{2}\int \frac{du}{u}, \text{ by letting } u{=}1{-}2\cos x{\Rightarrow}du{=}2\sin x dx$$
$$= \frac{1}{2}\ln|u| + C = \ln\sqrt{1 - 2\cos x} + C$$

$\square$