# Calculus Quiz 12

**1.** (5 pts) A population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources. Let $P(t)$ is the size of the population at time $t$., we assume that $P$ satisfies following equation

$$\frac{1}{P}\frac{dP}{dt} = k\Big(1 - \frac{P}{A}\Big), \text{ where } A, k \text{ are constants}$$

**a.** By letting $Q(t) = \dfrac{A}{P(t)} - 1$, then what kind of equation that $Q$ have to be satisfied?

**b.** Prove that $P(t) = \dfrac{A}{1 + Be^{-kt}}$ for some constant $B$.

*Sol.*

**a.** Since $Q(t) = \dfrac{A}{P(t)} - 1$, so

$$\begin{aligned}\frac{dQ}{dt} &= -\frac{AP'}{P^2} = -\frac{A}{P^2}\cdot kP\Big(1 - \frac{P}{A}\Big)\\ &= -\frac{k}{P}(A - P) = -k\Big(\frac{A}{P} - 1\Big) = -kQ\end{aligned}$$

Thus $Q$ satisfies the equation $\dfrac{dQ}{dt} = -kQ$.

**b.** By **a.** and Theorem 2, we know that $Q(t) = Be^{-kt}$ for some constant $B$. Since $Q(t) = \dfrac{A}{P(t)} - 1$. So

$$\frac{A}{P(t)} - 1 = Be^{-kt} \Rightarrow P(t) = \frac{A}{1 + Be^{-kt}}$$

□

**2.** (5 pts)

**a.** For $x \geq 0$, show that

$$\sin^{-1}\Big(\frac{x-1}{x+1}\Big) - 2\tan^{-1}\sqrt{x} = C$$

for some constant $C$ and determine what value $C$ is.

**b.** Recall that $\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$ for continuous $f$ defined on $[0, \pi]$. Use this to evaluate the integral

$$\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx$$

*Sol.*

**a.** Let $f(x) = \sin^{-1}\frac{x-1}{x+1}$ and $g(x) = 2\tan^{-1}\sqrt{x}$. Then

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{1}{\sqrt{x}(x+1)}$$

$$g'(x) = \frac{2}{1+x} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}(x+1)}$$

This shows that $f$ and $g$ have the same derivative for $x \geq 0$. Hence there exists a constant $C$ such that $f(x) - g(x) = C$, that is,

$$\sin^{-1}\left(\frac{x-1}{x+1}\right) - 2\tan^{-1}\sqrt{x} = C, \ \ x \geq 0$$

By taking $x = 0$, we have that

$$C = \sin^{-1} 0 - 2\tan^{-1} 1 = -\frac{\pi}{2}$$

**b.**

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_{-1}^1 \frac{du}{1+u^2}, \text{ by letting } u = \cos x \Rightarrow du = -\sin x dx$$

$$= \frac{\pi}{2}\tan^{-1} u\Big|_{u=-1}^{u=1} = \frac{\pi}{2}\big(\tan^{-1} 1 - \tan^{-1}(-1)\big)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi^2}{4}$$

□