Calculus Quiz 13

1. (5 pts)

a. Evaluate the limit $\lim_{x\to 0} \frac{1}{ax^3}\int_0^{bx} \sin(ct^2)dt,\ abc \neq 0.$

b. Find r, s such that $\lim_{x\to 0}\left(x^{-3}\sin 3x + rx^{-2} + s\right) = 0.$

Sol.

a. It is clear that $\lim_{x\to 0} ax^3 = 0 = \lim_{x\to 0}\int_0^{bx}\sin(ct^2)dt$. By L'Hospital's rule and FTC, we have that

$$\lim_{x\to 0}\frac{1}{ax^3}\int_0^{bx}\sin(ct^2)dt = \lim_{x\to 0}\frac{b\sin(b^2cx^2)}{3ax^2} = \frac{b}{3a}\lim_{x\to 0}\frac{\sin(b^2cx^2)}{x^2}$$
$$= \frac{b^3c}{3a}\lim_{x\to 0}\frac{\sin\left(b^2cx^2\right)}{b^2cx^2} = \frac{b^3c}{3a}$$

b. Let $L = \lim_{x\to 0}\left(x^{-3}\sin 3x + rx^{-2} + s\right) = \lim_{x\to 0}\frac{\sin 3x + rx + sx^3}{x^3}$. By L'Hospital's rule

$$L = \lim_{x\to 0}\frac{3\cos 3x + r + 3sx^2}{3x^2}$$

Since $\lim_{x\to 0} 3x^2 = 0$, the existence of L implies that

$$\lim_{x\to 0}\left(3\cos 3x + r + 3sx^2\right) = 3 + r = 0 \Rightarrow r = -3$$

Since $L = 0$, so

$$s = -\lim_{x\to 0}\frac{\sin 3x - 3x}{x^3} = \lim_{x\to 0}\frac{1-\cos 3x}{x^2}$$
$$= \frac{3}{2}\lim_{x\to 0}\frac{\sin 3x}{x} = \frac{9}{2}\lim_{x\to 0}\cos 3x = \frac{9}{2}$$

□

1. (5 pts)

a. Evaluate the indefinite integral $\int \sin(\ln x)dx$

b. If $p(x)$ is a polynomial with $\deg p = n$. Show that

$$\int p(x)e^{ax}dx = e^{ax}\sum_{k=0}^{n}(-1)^k\frac{p^{(k)}(x)}{a^{k+1}} + C$$

Sol.

a.

$$\int \sin(\ln x)dx = \int e^y \sin y dy, \text{ by letting } {\scriptstyle y=\ln x \Rightarrow x=e^y,\ dy=\frac{dx}{x}=\frac{dx}{e^y} \Rightarrow e^y dy=dx}$$

$$= -e^y \cos y + \int e^y \cos y dy, \text{ by letting } {\scriptstyle \begin{matrix} u=e^y, & dv=\sin y dy \\ du=e^y dy, & v=-\cos y \end{matrix}}$$

$$= -e^y \cos y + e^y \sin y - \int e^y \sin y dy, \text{ by letting } {\scriptstyle \begin{matrix} u=e^y, & dv=\cos y dy \\ du=e^y dy, & v=\sin y \end{matrix}}$$

This shows that $\int e^y \sin y dy = \frac{e^y}{2} \sin y - \frac{e^y}{2} \cos y + C$. Thus

$$\int \sin(\ln x)dx = \frac{x}{2} \sin(\ln x) - \frac{x}{2} \cos(\ln x) + C$$

b. We prove the state by induction on n. For $n = 0$, then $p(x) = p_0$ for some constant p_0 and

$$\int p(x)e^{ax}dx = p_0 \int e^{ax}dx = \frac{p_0}{a}e^{ax} + C$$

Suppose that the equality hold for polynomials p with $\deg p < k$. Then if $p(x)$ is a polynomial with $\deg p = k$,

$$\int p(x)e^{ax}dx = \frac{p(x)}{a}e^{ax} - \frac{1}{a}\int p'(x)e^{ax}dx, \text{ by letting } {\scriptstyle \begin{matrix} u=p(x), & dv=e^{ax}dx \\ du=p'(x)dx, & v=\frac{1}{a}e^{ax} \end{matrix}}$$

Write $q(x) = p'(x)$, note that $\deg q = \deg p' = n - 1$, so

$$\int p'(x)e^{ax}dx = \int q(x)e^{ax}dx = e^{ax}\sum_{k=0}^{n-1}(-1)^k \frac{q^{(k)}(x)}{a^{k+1}} + C$$

Hence

$$\int p(x)e^{ax}dx = \frac{p(x)}{a}e^{ax} - \frac{e^{ax}}{a}\sum_{k=0}^{n-1}(-1)^k \frac{q^{(k)}(x)}{a^{k+1}} + C$$

$$= e^{ax}\Big(\frac{p(x)}{a} + \sum_{k=0}^{n-1}(-1)^{k+1}\frac{p^{(k+1)}(x)}{a^{k+2}}\Big) + C$$

$$= e^{ax}\Big(\frac{p(x)}{a} + \sum_{k=1}^{n}(-1)^{k}\frac{p^{(k)}(x)}{a^{k+2}}\Big) + C$$

$$= e^{ax}\sum_{k=0}^{n}(-1)^k\frac{p^{(k)}(x)}{a^{k+1}} + C$$

By mathematical induction hypothesis, the equality holds for all $n \in \mathbb{N}$.

$\square$