Calculus Quiz 14

1. (5 pts)

a. Evaluate the integral $\displaystyle\int \cos^{2n+1} x \sin^m x dx$ for $m, n \geq 0$.

b. A finite *Fourier series* is given by the sum $f(x) = \sum_{n=1}^{N} a_n \sin nx$.

Show that the mth coefficient a_m is given by the formula

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx dx, \;\; m = 1, \cdots, N.$$

Sol.

a.

$$\begin{aligned}
\int \cos^{2n+1} x \sin^m x dx &= \int \left(\cos^2 x\right)^n \sin^m x \cos x dx = \int (1-\sin^2 x)^n \sin^m x \cos x dx \\
&= \int (1-u^2)^n u^m du, \;\; \text{by letting } {\scriptstyle u=\sin x \Rightarrow du=\cos x dx} \\
&= \int \sum_{k=1}^{n} \binom{n}{k} (-1)^k u^{2k+m} du = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \int u^{2k+m} du \\
&= \sum_{k=1}^{n} \frac{(-1)^k}{2k+m+1} \binom{n}{k} u^{2k+m+1} + C \\
&= \sum_{k=1}^{n} \frac{(-1)^k}{2k+m+1} \binom{n}{k} \sin^{2k+m+1} x + C
\end{aligned}$$

b. Observe that if $n \neq m$, then

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin nx \sin mx dx &= \frac{1}{2}\int_{-\pi}^{\pi} \big(\cos[(n-m)x] - \cos[(n+m)x]\big) dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \cos[(n-m)x] dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos[(n+m)x] dx \\
&= \frac{1}{2(n-m)} \int_{-(n-m)\pi}^{(n-m)\pi} \cos u du - \frac{1}{2(n+m)} \int_{-(n+m)\pi}^{(n+m)\pi} \cos v dv, \\
&\quad \text{by letting } {\scriptstyle u=(n-m)x \Rightarrow du=(n-m)dx, \;\; v=(n+m)x \Rightarrow dv=(n+m)dx} \\
&= \frac{\sin u}{2(n-m)}\bigg|_{u=-(n-m)\pi}^{x=(n-m)\pi} - \frac{\sin u}{2(n+m)}\bigg|_{u=-(n+m)\pi}^{x=(n+m)\pi} \\
&= \frac{\sin[(n-m)\pi]}{n-m} - \frac{\sin[(n+m)\pi]}{n+m} = 0
\end{aligned}$$

If $n = m$, then

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \int_{-\pi}^{\pi} \sin^2 nx dx = \int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2} dx$$
$$= \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos 2nx dx$$
$$= \frac{x}{2}\Big|_{x=-\pi}^{x=\pi} - \frac{1}{4n}\int_{-2n\pi}^{2n\pi} \cos u du, \text{ by letting } u=2nx \Rightarrow du=2ndx$$
$$= \pi - \frac{\sin u}{4n}\Big|_{u=-2n\pi}^{u=2n\pi} = \pi - \frac{\sin 2n\pi}{2n} = \pi$$

Thus, we can conclude that $\int \sin nx \sin mx dx = \begin{cases} 0, & \text{if } n \neq m \\ \pi, & \text{if } n = m \end{cases}$.

Hence, for $m = 1, \cdots, N$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \Big(\sum_{n=1}^{N} a_n \sin nx\Big) \sin mx dx$$
$$= \frac{1}{\pi}\sum_{n=1}^{N} a_n \Big(\int_{-\pi}^{\pi} \sin nx \sin mx dx\Big) = \frac{1}{\pi} \cdot a_m \pi = a_m$$

□

2. (5 pts) Evaluate the following integrals

a. $\int \sqrt{1+e^{2x}} dx$.

b. $\int \frac{x}{x^2+x+1} dx$.

Sol.

a.

$$\int \sqrt{1+e^{2x}} dx = \int \frac{\sqrt{1+y^2}}{y} dy, \text{ by letting } y=e^x \Rightarrow dy=e^x dx \Rightarrow dx=\tfrac{dy}{e^x}=\tfrac{dy}{y}$$
$$= \int \frac{\sec u}{\tan u} \sec^2 u du, \text{ by letting } y=\tan u \Rightarrow dy=\sec^2 u du$$
$$= \int \frac{\sec u}{\tan u}(1+\tan^2 u) du = \int \Big(\frac{\sec u}{\tan u} + \sec u \tan u\Big) du$$
$$= \int \frac{du}{\sin u} + \int \sec u \tan u du = \sec u + \int \csc u du$$
$$= \sec u + \int \frac{\csc u \cot u + \csc^2 u}{\csc u + \cot u} du$$
$$= \sec u - \int \frac{dw}{w}, \text{ by letting } w=\csc u+\cot u \Rightarrow dw=-(\csc u \cot u+\csc^2 u)du$$

$$
\begin{aligned}
&= \sec u - \ln|w| + C = \sec u - \ln|\csc u + \cot u| + C \\
&= \sqrt{y^2+1} - \ln\left|\frac{\sqrt{y^2+1}+1}{y}\right| + C \\
&= \sqrt{e^{2x}+1} - \ln\left(\sqrt{e^{2x}+1}+1\right) + x + C
\end{aligned}
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b.

$$
\begin{aligned}
\int \frac{x}{x^2+x+1}dx &= \int \frac{x+\frac{1}{2}-\frac{1}{2}}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{dx}{x^2+x+1} \\
&= \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}, \text{ by letting } u=x^2+x+1 \Rightarrow du=(2x+1)dx \\
&= \frac{1}{2}\ln|u| - \frac{2}{3}\int \frac{dx}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1} \\
&= \frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\int \frac{dv}{v^2+1}, \text{ by letting } v=\frac{2x+1}{\sqrt{3}} \Rightarrow dv=\frac{2}{\sqrt{3}}dx \\
&= \ln\sqrt{x^2+x+1} - \frac{1}{\sqrt{3}}\tan^{-1} v + C \\
&= \ln\sqrt{x^2+x+1} - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C
\end{aligned}
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□