微積分 MA1001-A 上課筆記（精簡版）

2019.01.07.

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5.9 Hyperbolic Functions

Definition 5.68: Hyperbolic Functions

The hyperbolic functions sinh, cosh, tanh, coth, sech and csch are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}\,, \qquad \cosh x = \frac{e^x + e^{-x}}{2}\,, \qquad \tanh x = \frac{\sinh x}{\cosh x}\,,$$
$$\coth x = \frac{1}{\tanh x}\,, \qquad \operatorname{sech} x = \frac{1}{\cosh x}\,, \qquad \operatorname{csch} x = \frac{1}{\sinh x}\,.$$

Motivation: The Euler identity provides the following relation

$$e^{ix} = \cos x + i \sin x \qquad \forall\, x \in \mathbb{R}\,. \tag{5.9.1}$$

This implies that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \text{and} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \forall\, x \in \mathbb{R}\,.$$

For a complex number $z = x + iy$, where $x, y \in \mathbb{R}$, define $\sin z$ and $\cos z$ by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \equiv \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i}\,, \tag{5.9.2a}$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \equiv \frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2}\,. \tag{5.9.2b}$$

Then on the imaginary axis, we have

$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i} = i \sinh y \quad \text{and} \quad \cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh y \qquad \forall\, y \in \mathbb{R}\,. \tag{5.9.3}$$

The hyperbolic functions, roughly speaking, can be viewed as trigonometric functions on the imaginary axis (by ignoring i in the output).

We also note that by definition, for $z = x + iy$ with $x, y \in \mathbb{R}$,

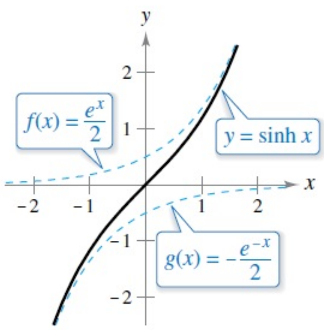
$$\sin^2 z + \cos^2 z = \frac{e^{-2y}e^{2ix} - 2 + e^{2y}e^{-2ix}}{-4} + \frac{e^{-2y}e^{2ix} + 2 + e^{2y}e^{-2ix}}{4} = 1\,.$$

Moreover, if z_1, z_2 are complex numbers,

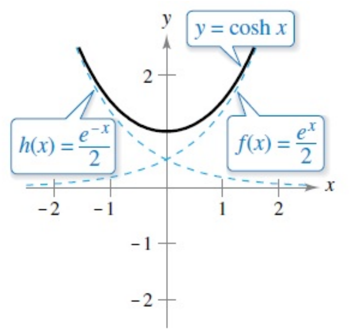
$$\begin{aligned}
&\cos z_1 \cos z_2 - \sin z_1 \sin z_2 \\
&\quad = \frac{e^{iz_1} + e^{-iz_1}}{2} \cdot \frac{e^{iz_2} + e^{-iz_2}}{2} - \frac{e^{iz_1} - e^{-iz_1}}{2i} \cdot \frac{e^{iz_2} - e^{-iz_2}}{2i} \\
&\quad = \frac{e^{i(z_1+z_2)} + e^{i(z_1-z_2)} + e^{i(z_2-z_1)} + e^{-i(z_1+z_2)}}{4} + \frac{e^{i(z_1+z_2)} - e^{i(z_1-z_2)} - e^{i(z_2-z_1)} + e^{-i(z_1+z_2)}}{4} \\
&\quad = \frac{e^{i(z_1+z_2)} + e^{i(z_1+z_2)}}{2} = \cos(z_1 + z_2)\,.
\end{aligned}$$

The above computations show that trigonometric identities are still valid even for complex arguments.

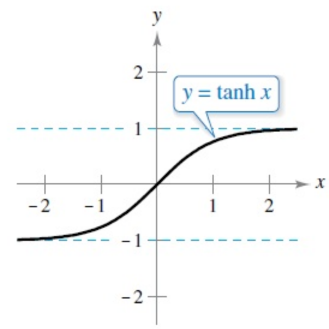
- **The graph of hyperbolic functions**



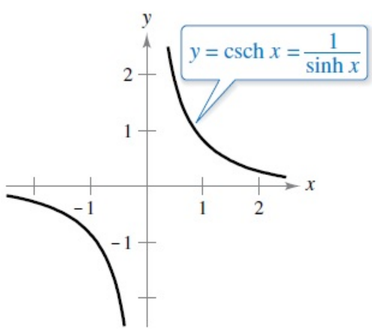
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



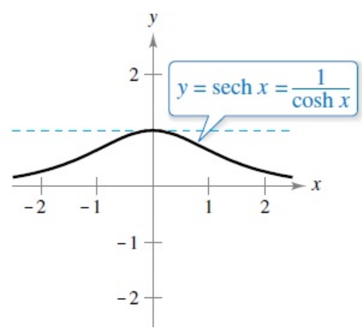
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



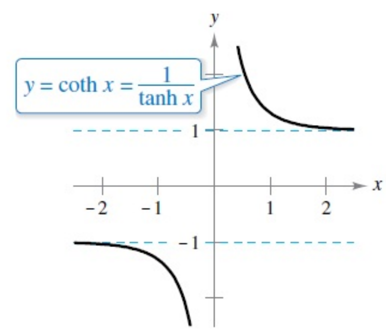
Domain: $(-\infty, \infty)$
Range: $(-1, 1)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(-\infty, \infty)$
Range: $(0, 1]$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -1) \cup (1, \infty)$

Theorem 5.69: Hyperbolic identities

1. $\cosh^2 x - \sinh^2 x = 1$; 2. $\tanh^2 x + \operatorname{sech}^2 x = 1$;
3. $\coth^2 x - \operatorname{csch}^2 x = 1$;
4. $\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x$;
5. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$;
6. $\sinh^2 = \frac{-1 + \cosh(2x)}{2}$; $\cosh^2 x = \frac{1 + \cosh(2x)}{2}$;
7. $\sinh(2x) = 2 \sinh x \cosh x$; $\cosh(2x) = \cosh^2 x + \sinh^2 x$.

Remark 5.70. By the definition (5.9.2), one can easily check that $\sin^2 z + \cos^2 z = 1$ for all complex z and this further implies that

$$1 = \sin^2(iy) + \cos^2(iy) = (i \sinh y)^2 + \cosh^2 y = \cosh^2 y - \sinh^2 y \qquad \forall\, y \in \mathbb{R}\,.$$

All the other hyperbolic identities can be memorized/derived in the same way.

Theorem 5.71: Differentiation and integration of hyperbolic functions

1. $\dfrac{d}{dx} \sinh x = \cosh x$; $\qquad \displaystyle\int \cosh x\, dx = \sinh x + C$;
2. $\dfrac{d}{dx} \cosh x = \sinh x$; $\qquad \displaystyle\int \sinh x\, dx = \cosh x + C$;
3. $\dfrac{d}{dx} \tanh x = \operatorname{sech}^2 x$; $\qquad \displaystyle\int \operatorname{sech}^2 x\, dx = \tanh x + C$;
4. $\dfrac{d}{dx} \coth x = -\operatorname{csch}^2 x$; $\qquad \displaystyle\int \operatorname{csch}^2 x\, dx = -\coth x + C$;
5. $\dfrac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$; $\qquad \displaystyle\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$;
6. $\dfrac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$; $\qquad \displaystyle\int \operatorname{csch} x \coth x\, dx = \operatorname{csch} x + C$;
7. $\displaystyle\int \tanh x\, dx = \ln \cosh x + C$;
8. $\displaystyle\int \operatorname{sech} x\, dx = 2 \arctan e^x + C$.

Proof. We only prove 7 and 8. By Theorem **??**, it is easy to see that

$$\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x}\, dx = \int \frac{\frac{d}{dx}\cosh x}{\cosh x}\, dx = \ln \cosh x + C\,,$$

so we focus on 8.

Let $u = e^x$. Then $du = e^x\, dx$ or equivalently, $\dfrac{du}{u} = dx$; thus

$$\int \operatorname{sech} x\, dx = \int \frac{2}{u + u^{-1}} \cdot \frac{du}{u} = \int \frac{2}{u^2 + 1}\, du = 2 \arctan u + C = 2 \arctan e^x + C\,. \qquad \square$$

Remark 5.72. Assuming that one knows that $\frac{d}{dx}f(ix) = if'(ix)$ (that is, the rule of differentiation $\frac{d}{dx}f(ax) = af'(ax)$ can also be applied for complex a), we have

$$\begin{aligned}\frac{d}{dx}\tanh x &= \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{1}{i}\frac{d}{dx}\frac{\sin(ix)}{\cos(ix)} = \frac{1}{i}\tan(ix) = \sec^2(ix)\\ &= \frac{1}{\cos^2(ix)} = \frac{1}{\cosh^2 x} = \cosh^2 x\,.\end{aligned}$$

All the other derivatives of hyperbolic functions can be memorized/derived in the same way.

- **Inverse hyperbolic functions**

Similar to inverse trigonometric functions, we can also talk about the inverse function of hyperbolic functions. Note that

$$\begin{aligned}&\sinh : (-\infty,\infty) \xrightarrow[onto]{1-1} (-\infty,\infty)\,,\\ &\tanh : (-\infty,\infty) \xrightarrow[onto]{1-1} (-1,1)\,,\end{aligned}$$

while

$$\begin{aligned}&\cosh : (-\infty,\infty) \to [1,\infty) \text{ is onto but not one-to-one}\,,\\ &\text{sech} : (-\infty,\infty) \to (0,1] \text{ is onto but not one-to-one}\,.\end{aligned}$$

We first find the inverse function of sinh and tanh.

1. Let $y = \sinh x = \frac{e^x - e^{-x}}{2}$. Then $e^{2x} - 2ye^x - 1 = 0$; thus by the fact that $e^x > 0$,

   $$e^x = \frac{2y + \sqrt{4y^2+4}}{2} = y + \sqrt{y^2+1}$$

   which further implies that $x = \ln(y + \sqrt{y^2+1})$. Therefore,

   $$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad \forall\, x \in \mathbb{R}\,.$$

2. Let $y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Then $e^{2x}(1-y) = 1+y$; thus $x = \frac{1}{2}\ln\frac{1+y}{1-y}$. Therefore,

   $$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x} \qquad \forall\, x \in (-1,1)\,.$$

To find the inverse of cosh, we note that $\cosh : [0,\infty) \xrightarrow[onto]{1-1} [1,\infty)$. Let $x \geqslant 0$ and $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$. Then $e^{2x} - 2ye^x + 1 = 0$ which implies that

$$e^x = y + \sqrt{y^2 - 1}\,.$$

As a consequence,

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \qquad \forall\, x \in [1,\infty)\,.$$

Since $\operatorname{sech} x = \dfrac{1}{\cosh x}$, we find that

$$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x} = \ln\Big(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\Big) = \ln\frac{1 + \sqrt{1 - x^2}}{x}\,.$$

We summarize these inverse hyperbolic functions in the following

Theorem 5.73

1. $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \qquad \forall\, x \in \mathbb{R}$.
2. $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \qquad \forall\, x \in [1,\infty)$.
3. $\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x} \qquad \forall\, x \in (-1,1)$.
4. $\operatorname{sech}^{-1} x = \ln\dfrac{1 + \sqrt{1 - x^2}}{x} \qquad \forall\, x \in (0,1]$.

• **Differentiation and integration of inverse hyperbolic functions**

Theorem 5.74

1. $\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{x^2+1}}; \qquad \displaystyle\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C;$
2. $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}; \qquad \displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C;$
3. $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}; \qquad \displaystyle\int \frac{dx}{1-x^2} = \tanh^{-1} x + C.$

Proof. By the chain rule,

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}(x+\sqrt{x^2+1}) = \frac{1}{\sqrt{x^2+1}}\,,$$
$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{x+\sqrt{x^2-1}}\frac{d}{dx}(x+\sqrt{x^2-1}) = \frac{1}{\sqrt{x^2-1}}\,,$$

as well as

$$\frac{d}{dx}\tanh^{-1}x = \frac{1}{2}\Big[\frac{1}{1+x}+\frac{1}{1-x}\Big] = \frac{1}{1-x^2}\,.$$

□

Example 5.75. Find the indefinite integral $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}}$, where $a>0$.

First we use trigonometric substitution $x = a\cos u$ to compute the integral. Since $dx = -a\sin u\,du$, we have

$$\begin{aligned}\int \frac{dx}{x\sqrt{a^2-x^2}} &= \int \frac{-a\sin u}{a\cos u\cdot a\sin u}\,du = -\frac{1}{a}\int \sec u\,du = -\frac{1}{a}\ln|\sec u+\tan u| + C\\ &= -\frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{|x|} + C\,.\end{aligned}$$

Now we use hyperbolic functions substitution to compute the integral. Let $x = a\,\mathrm{sech}\,u$ (we note that when using this substitution, we have already restrict ourself to the case $x>0$). Then $dx = -a\,\mathrm{sech}\,u\tanh u\,du$; thus

$$\begin{aligned}\int \frac{dx}{x\sqrt{a^2-x^2}} &= \int \frac{-a\,\mathrm{sech}\,u\tanh u}{a\,\mathrm{sech}\,u\sqrt{a^2-a^2\,\mathrm{sech}^2u}}\,du = -\frac{1}{a}\int \frac{\mathrm{sech}\,u\tanh}{\mathrm{sech}\,u\sqrt{1-\mathrm{sech}^2u}}\,du\\ &= -\frac{1}{a}\int \frac{\mathrm{sech}\,u\tanh u}{\mathrm{sech}\,u\sqrt{a^2\tanh^2u}}\,du = -\frac{1}{a}\int du = -\frac{1}{a}u + C\\ &= \frac{1}{a}\,\mathrm{sech}^{-1}\frac{x}{a} + C = \frac{1}{a}\ln\frac{1+\sqrt{1-\frac{x^2}{a^2}}}{\frac{x}{a}} + C\\ &= -\frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x} + C\,.\end{aligned}$$