微積分 MA1001-A 上課筆記（精簡版）

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Theorem 5.7: Inverse Function Differentiation

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))} \qquad \text{for all } x \text{ with } f'(g(x)) \neq 0.$$

Definition 5.50

The arcsin, arccos, and arctan functions are the inverse functions of the function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$, $g : [0, \pi] \to \mathbb{R}$, and $h : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, respectively, where $f(x) = \sin x$, $g(x) = \cos x$ and $h(x) = \tan x$. In other words,

1. $y = \arcsin x$ if and only if $\sin y = x$, where $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, $-1 \leqslant x \leqslant 1$.
2. $y = \arccos x$ if and only if $\cos y = x$, where $0 \leqslant y \leqslant \pi$, $-1 \leqslant x \leqslant 1$.
3. $y = \arctan x$ if and only if $\tan y = x$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$, $-\infty < x < \infty$.

Remark 5.51. Since arcsin, arccos and arctan look like the inverse function of sin, cos and tan, respectively, often times we also write arcsin as $\sin^{-1}$, arccos as $\cos^{-1}$, and arctan as $\tan^{-1}$.

Example 5.53. If $y = \arcsin x$, then $\cos y = \sqrt{1 - x^2}$. In other words, $\cos(\arcsin x) = \sqrt{1 - x^2}$. If $y = \arccos x$, then $\sin y = \sqrt{1 - x^2}$; that is, $\sin(\arccos x) = \sqrt{1 - x^2}$.

Example 5.54. Suppose that $y = \arctan x$ for some $x \in \mathbb{R}$. Then $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ which implies that $\cos y > 0$. Therefore,

$$\cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + \tan^2 y}} = \frac{1}{\sqrt{1 + x^2}}.$$

As for $\sin y$, we note that $y > 0$ if and only if $x > 0$; thus $\sin y = \frac{x}{\sqrt{1 + x^2}}$ $\left(\text{instead of } \frac{-x}{\sqrt{1 + x^2}}\right)$. Therefore,

$$\sin(\arctan x) = \frac{x}{\sqrt{1 + x^2}} \quad \text{and} \quad \cos(\arctan x) = \frac{1}{\sqrt{1 + x^2}}.$$

Theorem 5.55: Differentiation of Inverse Trigonometric Functions

1. $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$ for all $-1 < x < 1$.

2. $\dfrac{d}{dx}\arccos x = -\dfrac{1}{\sqrt{1-x^2}}$ for all $-1 < x < 1$.

3. $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$ for all $x \in \mathbb{R}$.

Proof. By Inverse Function Differentiation,

$$\frac{d}{dx}\arcsin x = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}} \qquad \forall\, x \in (-1,1)\,,$$
$$\frac{d}{dx}\arccos x = \frac{1}{-\sin(\arccos x)} = -\frac{1}{\sqrt{1-x^2}} \qquad \forall\, x \in (-1,1)\,,$$

and

$$\frac{d}{dx}\arctan x = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2} \qquad \forall\, x \in \mathbb{R}\,. \qquad \square$$

Remark 5.56. By Theorem 5.55,

$$\frac{d}{dx}\big(\arcsin x + \arccos x\big) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0 \qquad \forall\, -1 < x < 1\,.$$

Therefore, the function $y = \arcsin x + \arccos x$ is constant on the interval $(-1,1)$. The constant can be obtained by testing with $x = 0$ and we find that

$$\arcsin x + \arccos x = \frac{\pi}{2} \qquad \forall\, x \in [-1,1]\,, \tag{5.7.1}$$

where the value of the left-hand side at $x = \pm 1$ are computed separately.

Example 5.57. Find the derivative of $y = \arcsin x + x\sqrt{1-x^2}$.

By Theorem 5.55 and the chain rule, for $-1 < x < 1$ we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} - x \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(2x) = 2\sqrt{1-x^2}\,.$$

Example 5.58. Find the derivative of $y = \arctan\sqrt{x}$.

By the chain rule,

$$\frac{dy}{dx} = \frac{1}{1+\sqrt{x}^2}\frac{d}{dx}\sqrt{x} = \frac{1}{1+x}\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}\,.$$

5.8 Inverse Trigonometric Functions: Integration

Theorem 5.59

Let a be a positive real number. Then

1. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C.$ 2. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C.$

Proof. 1. Let $x = a \sin u$. Then $dx = a \cos u du$; thus

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos u}{\sqrt{a^2(1 - \sin^2 u)}} du = \int du = u + C = \arcsin \frac{x}{a} + C .$$

2. Let $x = a \tan u$. Then $dx = a \sec^2 u du$; thus

$$\int \frac{dx}{a^2 + x^2} = \int \frac{a \sec^2 u}{a^2(1 + \tan^2 u)} du = \frac{1}{a} \int du = \frac{u}{a} + C = \frac{1}{a} \arctan \frac{x}{a} + C .$$ □

Rule of Thumb: During the process of computing anti-derivatives, it is a good idea to try

1. the substitution $x = a \sin u$ when seeing $\sqrt{a^2 - x^2}$ in the integrand;
2. the substitution $x = a \tan u$ when seeing $a^2 + x^2$ in the denominator;
3. the substitution $x = a \sec u$ when seeing $x^2 - a^2$ in the denominator.

Example 5.60. Find the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$ is a constant.

Let $x = a \sec u$. Then $dx = a \sec u \tan u du$; thus

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 - a^2}} &= \int \frac{a \sec u \tan u}{\sqrt{a^2(\sec^2 u - 1)}} du = \int \sec u \, du = \ln |\sec u + \tan u| + C \\ &= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C = \ln \left| x + \sqrt{x^2 - a^2} \right| + C .\end{aligned}$$

Example 5.61. Find the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}}$, where $a > 0$ is a constant.

Let $x = a \tan u$. Then $dx = a \sec^2 u du$; thus

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2 + x^2}} &= \int \frac{a \sec^2 u}{\sqrt{a^2(\tan^2 u + 1)}} du = \int \sec u \, du = \ln |\sec u + \tan u| + C \\ &= \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| + C = \ln \left| x + \sqrt{x^2 + a^2} \right| + C .\end{aligned}$$

Example 5.62. Find the indefinite integral $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}}$.

Let $x = a\sec u$. Then $dx = a\sec u\tan u$; thus

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec u\tan u}{a\sec u\sqrt{a^2(\sec^2 u-1)}} = \frac{1}{a}\int du = \frac{u}{a} + C\,.$$

If $x = a\sec u$, then $\tan u = \dfrac{\sqrt{x^2-a^2}}{a}$; thus $u = \arctan\dfrac{\sqrt{x^2-a^2}}{a}$ which implies that

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arctan\frac{\sqrt{x^2-a^2}}{a} + C\,.$$

Example 5.63. Find the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Let $u = e^x$. Then $du = e^x dx$; thus $dx = \dfrac{du}{u}$ which implies that

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{du}{u\sqrt{u^2-1}} = \operatorname{arctan}\sqrt{u^2-1} + C = \operatorname{arctan}\sqrt{e^{2x}-1} + C\,.$$

Example 5.64. Find the indefinite integral $\displaystyle\int \frac{x+2}{\sqrt{4-x^2}}\,dx$.

Let $x = 2\sin u$. Then $dx = 2\cos u du$; thus

$$\begin{aligned}\int \frac{x+2}{\sqrt{4-x^2}}\,dx &= \int \frac{2\sin u+2}{\sqrt{4-4\sin^2 u}}\cdot 2\cos u\,du = \int (2\sin u+2)\,du = 2u - 2\cos u + C\\ &= 2\arcsin\frac{x}{2} - 2\sqrt{1-\left(\frac{x}{2}\right)^2} + C = 2\arcsin\frac{x}{2} - \sqrt{4-x^2} + C\,.\end{aligned}$$

Example 5.65. Find the indefinite integral $\displaystyle\int \frac{dx}{x^2-4x+7}$.

First we complete the square and obtain that $x^2-4x+7 = (x-2)^2+3$. Let $x-2 = \sqrt{3}\tan u$. Then $dx = \sqrt{3}\sec^2 u du$; thus

$$\int \frac{dx}{x^2-4x+7} = \int \frac{\sqrt{3}\sec^2 u}{3\tan^2 u+3}\,du = \frac{1}{\sqrt{3}}\int du = \frac{1}{\sqrt{3}}u + C = \frac{1}{\sqrt{3}}\arctan\frac{x-2}{\sqrt{3}} + C\,.$$

Example 5.66. Find the indefinite integral $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,dx$.

Let $x = \sin u$. Then $dx = \cos u du$; thus

$$\begin{aligned}\int \sqrt{\frac{1-x}{1+x}}\,dx &= \int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{(1-\sin u)\cos u}{\sqrt{1-\sin^2 u}}\,du = \int (1-\sin u)\,du\\ &= u + \cos u + C = \arcsin x + \sqrt{1-x^2} + C\,.\end{aligned}$$

Example 5.67. In this example, we compute $\displaystyle\int \arcsin x\, dx$. Note the by the substitution $x = \sin u$,

$$\int \arcsin x\, dx = \int u \cos u\, du\,;$$

thus it suffices to compute the anti-derivative of the function $y = x\cos x$. We first compute the definite integral $\displaystyle\int_0^a x\cos x\, dx$.

By Example 4.12, for $0 < x < \pi$ we have

$$\sum_{i=1}^{n} \sin(ix) = \frac{1}{2\sin\frac{x}{2}}\Big[\cos\frac{x}{2} - \cos\big((n+\frac{1}{2})x\big)\Big]\,.$$

Therefore, if $0 < x < \pi$,

$$\begin{aligned}\sum_{i=1}^{n} i\cos(ix) &= \frac{d}{dx}\sum_{i=1}^{n}\sin(ix) = \frac{d}{dx}\frac{1}{2\sin\frac{x}{2}}\Big[\cos\frac{x}{2} - \cos\big((n+\frac{1}{2})x\big)\Big]\\ &= \frac{-\cos\frac{x}{2}}{4\sin^2\frac{x}{2}}\Big[\cos\frac{x}{2} - \cos\big((n+\frac{1}{2})x\big)\Big]\\ &\quad + \frac{1}{2\sin\frac{x}{2}}\Big[-\frac{1}{2}\sin\frac{x}{2} + (n+\frac{1}{2})\sin\big((n+\frac{1}{2})x\big)\Big]\,.\end{aligned}$$

By partitioning $[0,a]$ into n sub-intervals with equal length, the Riemann sum of $y = x\cos x$ for this partition given by the right end-point rule is

$$I_n = \sum_{i=1}^{n} \frac{ia}{n}\cos\frac{ia}{n}\frac{a}{n} = \frac{a^2}{n^2}\sum_{i=1}^{n} i\cos\frac{ia}{n}\,.$$

Letting $r = \dfrac{a}{2n}$, we find that

$$\begin{aligned}I_n &= 4r^2\sum_{i=1}^{n} i\cos(2ir)\\ &= \frac{-r^2\cos r}{\sin^2 r}\Big[\cos r - \cos(a+r)\Big] + \frac{r}{\sin r}\Big[-r\sin r + (a+r)\sin(a+r)\Big]\end{aligned}$$

which, by the fact that $\dfrac{\sin x}{x} \to 1$ as $x \to 0$ and $r \to 0$ as $n \to \infty$, implies that

$$\int_0^a x\cos x\, dx = \lim_{n\to\infty} I_n = -(1-\cos a) + a\sin a = a\sin a + \cos a - 1\,.$$

The identity above further implies that

$$\int x\cos x\, dx = x\sin x + \cos x + C\,;$$

thus with the substitution $x = \sin u$,

$$\int \arcsin x \, dx = \int u \cos u \, du = u \sin u + \cos u + C = x \arcsin x + \sqrt{1 - x^2} + C\,.$$

Using (5.7.1), we also find that

$$\begin{aligned}\int \arccos x \, dx &= \int \bigl(\frac{\pi}{2} - \arcsin x\bigr) \, dx = \frac{\pi}{2} x - x \arcsin x - \sqrt{1 - x^2} + C \\ &= x\bigl(\frac{\pi}{2} - \arcsin x\bigr) - \sqrt{1 - x^2} + C = x \arccos x - \sqrt{1 - x^2} + C\,.\end{aligned}$$