微積分 MA1002-A 上課筆記（精簡版）

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Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. If f is differentiable at $(x_0, y_0) \in R$, the tangent plane of the graph of f at $(x_0, y_0, f(x_0, y_0))$ is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)\,,$$

and the vector $(f_x(x_0, y_0), f_y(x_0, y_0), -1)$ is a normal vector to the graph of f at $(x_0, y_0, f(x_0, y_0))$.

Let $w = F(x, y, z)$ be a function of three variables such that F_x, F_y and F_z are continuous. If $(\nabla F)(x_0, y_0, z_0) \neq \mathbf{0}$, then the tangent plane of the level surface $F(x, y, z) = F(x_0, y_0, z_0)$ at (x_0, y_0, z_0) is given by

$$(\nabla F)(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0\,,$$

and the vector $(\nabla F)(x_0, y_0, z_0)$ is a normal vector to the level surface $F(x, y, z) = F(x_0, y_0, z_0)$.

Theorem 13.51

Let F be a function of three variables. If F has continuous first partial derivatives F_x, F_y, F_z in a neighborhood of (x_0, y_0, z_0) and $(\nabla F)(x_0, y_0, z_0) \neq \mathbf{0}$, then $(\nabla F)(x_0, y_0, z_0)$ is perpendicular/normal to the level surface $F(x, y, z) = F(x_0, y_0, z_0)$ at (x_0, y_0, z_0). Moreover, the value of F at (x_0, y_0, z_0) increase most rapidly in the direction $\dfrac{(\nabla F)(x_0, y_0, z_0)}{\|(\nabla F)(x_0, y_0, z_0)\|}$ and decreases most rapidly in the direction $-\dfrac{(\nabla F)(x_0, y_0, z_0)}{\|(\nabla F)(x_0, y_0, z_0)\|}$, where $\|\cdot\|$ denotes the length of the vector.

Theorem 13.54

Let f be a function of two variables. If f has continuous first partial derivatives f_x and f_y in a neighborhood of (x_0, y_0) and $(\nabla f)(x_0, y_0) \neq \mathbf{0}$, then $(\nabla f)(x_0, y_0)$ is perpendicular/normal to the level curve $f(x, y) = f(x_0, y_0)$ at (x_0, y_0). Moreover, the value of f at (x_0, y_0) increase most rapidly in the direction $\dfrac{(\nabla f)(x_0, y_0)}{\|(\nabla f)(x_0, y_0)\|}$ and decreases most rapidly in the direction $-\dfrac{(\nabla f)(x_0, y_0)}{\|(\nabla f)(x_0, y_0)\|}$, where $\|\cdot\|$ denotes the length of the vector.

Example 13.56. A heat-seeking particle is located at the point $(2,-3)$ on a metal plate whose temperature at (x,y) is $T(x,y)=20-4x^2-y^2$. Find the path of the particle as it continuously moves in the direction of maximum temperature increase.

Suppose the path of the particle is given by $(x(t),y(t))$. Then

$$(x'(t),y'(t))/\!/\,(\nabla T)(x(t),y(t))=\big(-8x(t),-2y(t)\big)\,.$$

Therefore, there exists a function $k(t)$ such that $-8x=k\dfrac{dx}{dt}$ and $-2y=k\dfrac{dy}{dt}$; thus

$$\frac{d}{dt}\big(\ln|x|-4\ln|y|\big)=0\,.$$

Then $|x||y|^{-4}=C$. Since $(x(t),y(t))$ passes through $(2,-3)$, we find that $C=\dfrac{2}{81}$; thus (x,y) satisfies $x=\dfrac{2}{81}y^4$.

Example 13.57. Consider the normal line of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at $(a\cos\theta,b\sin\theta)$.

Let $f(x,y)=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$. Then the given ellipse is the level curve $f(x,y)=1=f(a\cos\theta,b\sin\theta)$; thus Theorem 13.54 implies that the normal "direction" of this ellipse at point $(a\cos\theta,b\sin\theta)$ is given by

$$\big(f_x(a\cos\theta,b\sin\theta),f_y(a\cos\theta,b\sin\theta)\big)=\Big(\frac{2\cos\theta}{a},\frac{2\sin\theta}{b}\Big)\,.$$

Therefore, the normal line is given by

$$\Big(-\frac{2\sin\theta}{b},\frac{2\cos\theta}{a}\Big)\cdot(x-a\cos\theta,y-b\sin\theta)=0\,.$$

13.8 Extrema of Functions of Several Variables

13.8.1 Relative extrema

Similar to the case of functions of one variable, we define the relative extrema as follows.

Definition 13.58: Relative Extrema

Let f be a function defined on a region R containing (x_0,y_0).

1. The function f has a relative minimum at (x_0,y_0) if $f(x,y)\geqslant f(x_0,y_0)$ for all (x,y) in an open disk containing (x_0,y_0).

2. The function f has a relative maximum at (x_0,y_0) if $f(x,y)\leqslant f(x_0,y_0)$ for all (x,y) in an open disk containing (x_0,y_0).

Similar to the critical points for functions of one variable defined in Definition 3.4,we have the following

Definition 13.59: Critical Points

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a critical point of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$;
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Similar to Theorem 3.5, we have the following necessary condition for points where f attains its relative extrema.

Theorem 13.60

Let R be an open region in the plane, and $f : R \to \mathbb{R}$ be continuous. If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

Example 13.61. Determine the relative extrema of the function

$$f(x, y) = -x^3 + 4xy - 2y^2 + 1\,.$$

First we find the critical points of f. Since f is differentiable, the critical points are those points at which the gradient of f is the zero vector. Since $f_x(x, y) = -3x^2 + 4y$ and $f_y(x, y) = 4x - 4y$, if (a, b) is a critical point of f, then $-3a^2 + 4b = 4a - 4b = 0$. Therefore, $(0, 0)$ and $\big(\frac{4}{3}, \frac{4}{3}\big)$ are the only critical points of f.

Note that $(0, 0)$ is not a relative extremum of f since $f(x, 0)$ does not attain its extremum at $x = 0$. Near $\big(\frac{4}{3}, \frac{4}{3}\big)$, we find that if $|h|, |k| \ll 1$,

$$\begin{aligned} f\Big(\frac{4}{3} + h, \frac{4}{3} + k\Big) &= -\Big(h + \frac{4}{3}\Big)^3 + 4\Big(\frac{4}{3} + h\Big)\Big(\frac{4}{3} + k\Big) - 2\Big(k + \frac{4}{3}\Big)^2 + 1 \\ &= -h^3 - 4h^2 - \frac{16h}{3} - \frac{64}{27} + 4\Big(\frac{16}{9} + \frac{4}{3}h + \frac{4}{3}k + hk\Big) - 2\Big(k^2 + \frac{8}{3}k + \frac{16}{9}\Big) + 1 \\ &= -h^3 - 4h^2 + 4hk - 2k^2 + f\Big(\frac{4}{3}, \frac{4}{3}\Big) \\ &= f\Big(\frac{4}{3}, \frac{4}{3}\Big) - 2(k - h)^2 - h^2(2 + h) \leqslant f\Big(\frac{4}{3}, \frac{4}{3}\Big)\,. \end{aligned}$$

Therefore, f has a relative maximum at $\big(\frac{4}{3}, \frac{4}{3}\big)$.

13.8.2 The second partials test

A critical point of a function of two variables do not always yield relative maxima or minima.

Definition 13.62

Let f be a function of two variables. A point (x_0, y_0) is a saddle point of f if (x_0, y_0) is a critical point of f but f does not attain its extrema at (x_0, y_0).

Theorem 13.63

Suppose that a function f of two variables has continuous second partial derivatives on an open region containing a point (x_0, y_0) for which $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Let

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}.$$

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).
3. If $D < 0$, then (x_0, y_0) is a saddle point of f.
4. The test is inconclusive if $D = 0$.

Example 13.64. Consider the relative extrema of the function given in Example 13.61. We have computed that $(0, 0)$ and $\left(\frac{4}{3}, \frac{4}{3}\right)$ are the only critical points of f.

1. The point $(0, 0)$: we compute the second partial derivatives and obtain that

$$f_{xx}(0, 0) = 0\,, \quad f_{xy}(0, 0) = 4 \quad \text{and} \quad f_{yy}(0, 0) = -4\,.$$

   Therefore, $D = -16 < 0$ which implies that $(0, 0)$ is a saddle point.

2. The point $\left(\frac{4}{3}, \frac{4}{3}\right)$: we compute the second partial derivatives and obtain that

$$f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -8\,, \quad f_{xy}\left(\frac{4}{3}, \frac{4}{3}\right) = 4 \quad \text{and} \quad f_{yy}\left(\frac{4}{3}, \frac{4}{3}\right) = -4\,.$$

   Therefore, $D = 16 > 0$. Since $f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) < 0$, f has a relative maximum at $\left(\frac{4}{3}, \frac{4}{3}\right)$.