Exercise Problem Sets 6

Oct. 18. 2019

Problem 1. 1. Let $f, g : (a, b) \to \mathbb{R}$ be functions and $f'(x) = g'(x)$. Show that there exists a constant C such that $f(x) = g(x) + C$.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying that $f'(x) = 3x^2 + 4\cos x$ and $f(0) = 0$. Find $f(x)$.

Problem 2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that f has only one critical point $c \in (a, b)$.

1. Show that if $f(c)$ is a local extremum of f, then $f(c)$ is an absolute extremum of f.

2. Show that if $f(c)$ is the absolute minimum of f, then $f(x) > f(c)$ for all $x \in [a, b]$ and $x \neq c$. Similarly, show that if $f(c)$ is the absolute maximum of f, then $f(x) < f(c)$ for all $x \in [a, b]$ and $x \neq c$.

Problem 3. Let I, J be intervals, $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be increasing functions. Show that if J contains the range of g, then $f \circ g$ is increasing on I.

Problem 4. 1. If the function $f(x) = x^3 + ax^2 + bx$ has the local minimum value $-\dfrac{2\sqrt{3}}{9}$ at $x = \dfrac{1}{\sqrt{3}}$, what are the values of a and b?

2. Which of the tangent lines to the curve in part (1) has the smallest slope?

Problem 5. A number a is called a fixed point of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

Problem 6. Suppose f is an odd function (that is, $f(-x) = -f(x)$ for all $x \in \mathbb{R}$) and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = \dfrac{f(b)}{b}$.

Problem 7. Show that $2\sqrt{x} > 3 - \dfrac{1}{x}$ for all $x > 1$.

Problem 8. Show that $\sqrt{b} - \sqrt{a} < \dfrac{b-a}{2\sqrt{a}}$ for all $0 < a < b$.

Problem 9. Show that for all (rational numbers) $p, q \in (1, \infty)$ satisfying $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we have

$$ac + bd \leqslant (a^p + b^p)^{\frac{1}{p}}(c^q + d^q)^{\frac{1}{q}} \qquad \forall\, a, b, c, d > 0\,.$$

Hint: Let $x = \dfrac{a}{b}$ and $y = \dfrac{d}{c}$.

Problem 10. Show that for all $k \in \mathbb{N} \cup \{0\}$,

$$x - \frac{x^3}{3!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+3}}{(4k+3)!} \leqslant \sin x \leqslant x - \frac{x^3}{3!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} \qquad \forall\, x \geqslant 0\,,$$
$$1 - \frac{x^2}{2!} + \cdots + \frac{x^{4k}}{(4k)!} - \frac{x^{4k+2}}{(4k+2)!} \leqslant \cos x \leqslant 1 - \frac{x^2}{2} + \cdots + \frac{x^{4k}}{(4k)!} \qquad \forall\, x \geqslant 0\,.$$

Problem 11. （不要用交叉相乘）Show that for all $k \in \mathbb{N} \cup \{0\}$,

$$1 - x + x^2 - x^3 + \cdots + x^{2k} - x^{2k+1} \leqslant \frac{1}{1+x} \leqslant 1 - x + x^2 - x^3 + \cdots + x^{2k} \qquad \forall\, x \geqslant 0\,.$$

Problem 12. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying that $f'(x) = f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$.

1. （不要試著找出 f 而是直接用 f 的性質）Show that f is increasing on $\mathbb{R}$.
2. Show that if $k \in \mathbb{N} \cup \{0\}$, then $f(x) \geqslant 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^k}{k!}$ for all $x \geqslant 0$.
3. Show that if $k \in \mathbb{N} \cup \{0\}$, then

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2k}}{(2k)!} + \frac{x^{2k+1}}{(2k+1)!} \leqslant f(x) \leqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2k}}{(2k)!} \qquad \forall\, x \leqslant 0\,.$$

Hint: 1. Show that f^2 is increasing on $\mathbb{R}$ and argue that f is also increasing on $\mathbb{R}$.

Problem 13. Find the minimum value of

$$\left|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x\right|$$

for real numbers x.
Hint: Let $t = \sin x + \cos x$.

Problem 14. Let $f, g : (a, b) \to \mathbb{R}$ be twice differentiable functions such that $f''(x) \neq 0$ and $g''(x) \neq 0$ for all $x \in (a, b)$. Prove that if f and g are positive, increasing, and concave upward on the interval (a, b), then fg is also concave upward on (a, b).

Problem 15. For what values of a and b is $(2, 2.5)$ an inflection point of the curve $x^2 + ax + by = 0$? What additional inflection points does the curve have?