Exercise Problem Sets 8

Nov. 8. 2019

Problem 1. In class we have talked about how to prove the following theorem

> Let $f : [a, b] \to \mathbb{R}$ be a function. If f is Riemann integrable on $[a, b]$, then f is bounded on $[a, b]$; that is, there exists $M > 0$ such that
>
> $$\big|f(x)\big| \leqslant M \qquad \text{whenever} \quad x \in [a, b]\,.$$

Read the following proof again and try to prove this theorem directly (that is, without proving by contradiction or by contrapositive/contraposition).

Proof. Let f be Riemann integrable on $[a, b]$. Then there exists $A \in \mathbb{R}$ and $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ belongs to $(A - 1, A + 1)$. Choose $n \in \mathbb{N}$ so that $\dfrac{b-a}{n} < \delta$. Then the regular partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$, where $x_i = a + \dfrac{b-a}{n} i$, satisfies $\|\mathcal{P}\| < \delta$.

Suppose the contrary that f is not bounded. Then there exists $x^* \in [a, b]$ such that

$$\big|f(x^*)\big| > \frac{n(|A| + 1)}{b - a} + \sum_{i=1}^{n} \big|f(x_i)\big|\,.$$

Suppose that $x^* \in [x_{k-1}, x_k]$. By the fact that $\sum\limits_{\substack{i=1\\ i\neq k}}^{n} f(x_i)(x_i - x_{i-1}) + f(x^*)(x_k - x_{k-1})$ is a Riemann sum of f for $\mathcal{P}$, we have

$$A - 1 < \sum_{\substack{i=1\\ i\neq k}}^{n} f(x_i)(x_i - x_{i-1}) + f(x^*)(x_k - x_{k-1}) < A + 1\,.$$

Since $x_i - x_{i-1} = \dfrac{b-1}{n}$ for all $1 \leqslant i \leqslant n$, the inequality above shows that

$$\frac{n(A-1)}{b-a} - \sum_{\substack{i=1\\ i\neq k}}^{n} f(x_i) < f(x^*) < \frac{n(A+1)}{b-a} - \sum_{\substack{i=1\\ i\neq k}}^{n} f(x_i)$$

and the triangle inequality further implies that

$$-\Big[\frac{n(|A|+1)}{b-a} + \sum_{\substack{i=1\\ i\neq k}}^{n} \big|f(x_i)\big|\Big] < f(x^*) < \frac{n(|A|+1)}{b-a} + \sum_{\substack{i=1\\ i\neq k}}^{n} \big|f(x_i)\big|\,.$$

Therefore, we conclude that

$$\big|f(x^*)\big| < \frac{n(|A|+1)}{b-a} + \sum_{\substack{i=1\\ i\neq k}}^{n} \big|f(x_i)\big| \leqslant \frac{n(|A|+1)}{b-a} + \sum_{i=1}^{n} \big|f(x_i)\big|\,,$$

a contradiction. □

Problem 2. Recall that in Exercise 4 Problem 5 we have "shown" that there exists a number $e > 1$ such that

$$\frac{d}{dx}\log_e x = \frac{1}{x} \qquad \forall\, x > 0\,.$$

In this example you need to compute $\displaystyle\int_1^b \log_e x\, dx$ by the following steps.

(a) Partition $[1, b]$ into n sub-intervals by $x_i = r^i$, where $1 \leqslant i \leqslant n$ and $r = b^{\frac{1}{n}}$. Show that the Riemann sum given by the right end-point rule is

$$(r-1)\log_e r \sum_{i=1}^{n} i r^{i-1}\,. \qquad (\diamond)$$

(b) Use $(\diamond)$ and the formula in Problem 4 of Exercise 4 to simplify the Riemann sum given above and show that the Riemann sum is

$$\frac{nbr - nb - b + 1}{n(r-1)}\log_e b = \left[b - \frac{b-1}{n(r-1)}\right]\log_e b\,.$$

(c) Pass the Riemann sum above to the limit as $n \to \infty$ to show that

$$\int_1^b \log_e x\, dx = b\log_e b - b + 1\,.$$

(d) Verify that $f(x) = x\log_e x - x$ is an anti-derivative of $y = \log_e x$.

Problem 3. Use Problem 2 in Exercise 5 to find the integral $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2+1}\, dx$.

Problem 4. Find an anti-derivative of the function $y = x\sin x$ (using Riemann sums).
Hint: See Problem 4 in Exercise 7 for reference.