Exercise Problem Sets 9

Nov. 18. 2019

Problem 1. Compute $\int_{-\sqrt{\frac{3}{2}}}^{1} \frac{dx}{\sqrt{2-x^2}}$ using the following substitution of variables:

1. $x = \sqrt{2}\sin t$.
2. $x = -\sqrt{2}\sin t$.
3. $x = \sqrt{2}\cos t$.
4. $x = -\sqrt{2}\cos t$.

Problem 2. Find the definite integral $\int_0^{\frac{\pi}{2}} \frac{\sin x dx}{1+\cos^2 x}$.

Problem 3. Find the following indefinite integrals.

1. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$.
2. $\int \frac{1}{x^3}\cos^2\frac{1}{x^2}\, dx$.
3. $\int \frac{1}{x^2}\cos^3\frac{1}{x}\, dx$.
4. $\int \frac{\sin\sqrt{x}}{\sqrt{x\cos^3\sqrt{x}}}\, dx$.

Problem 4. Find the following indefinite integrals.

1. $\int \tan^3 x \sec^2 x\, dx$, $\int \frac{\sec^2 x}{\tan^2 x}\, dx$, $\int \tan^2 x \sec^2 x\, dx$. Use your experience on these three integrals to find $\int \tan^m x \sec^2 x\, dx$ for $m \neq -1$.
2. $\int \sec^3 x \tan x\, dx$, $\int \sec^5 x \tan x\, dx$, $\int \frac{\tan x}{\sec^3 x}\, dx$. Use your experience on these three integrals to find $\int \sec^m x \tan x\, dx$ for $m \neq 0$.

Problem 5. Compute the indefinite integral $\int \sin^6 x\, dx$ by the following steps:

1. Write $\sin^6 x = \sin^3 x \cdot \sin^3 x$, and use the triple and double angle formula, as well as the product to sum formula, to show that
$$\sin^6 x = -\frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x - \frac{15}{32}\cos 2x + \frac{5}{16}\,.$$
2. Find the indefinite integral $\int \sin^6 x\, dx$ using the identity above.

Problem 6. Compute the indefinite integral $\int \cos^5 x\, dx$ by the following steps:

1. Write $\cos^5 x = \cos^3 x \cdot \cos^2 x$, and use the triple and double angle formula, as well as the product to sum formula, to show that
$$\cos^5 x = \frac{1}{16}\cos 5x + \frac{5}{16}\cos 3x + \frac{5}{8}\cos x\,.$$
2. Find the indefinite integral $\int \cos^5 x\, dx$ using the identity above.

Problem 7. Let y be a twice differentiable function satisfying
$$\frac{d^2y}{dx^2} = 4\sec^{-2} 2x \tan 2x\,, \quad y(0) = -1, y'(0) = 4\,.$$
Find y.

Problem 8. 1. Let $f : [0, a] \to \mathbb{R}$ be a continuous function. Find $\displaystyle\int_0^a \frac{f(x)}{f(x) + f(a-x)}\, dx$.

2. Find $\displaystyle\int_0^1 \frac{\sin x}{\sin x + \sin(1-x)}\, dx$. 3. Find $\displaystyle\int_0^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{3-x}}\, dx$.

Problem 9. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function.

1. Show that

$$\int_0^{\frac{\pi}{2}} f(\sin x)\, dx = \int_0^{\frac{\pi}{2}} f(\cos x)\, dx\,.$$

2. Use the identity above to find the integrals $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$ and $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$.

Problem 10. Let a and b be positive (rational) numbers. Show that

$$\int_0^1 x^a(1-x)^b\, dx = \int_0^1 x^b(1-x)^a\, dx\,.$$

Problem 11. Let $a_0,\ a_1,\ \cdots,\ a_n$ be real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0\,.$$

Show that the polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ has at least one real zero.