Calculus MA1001-A Midterm 2

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Problem 1. (15%) State and prove the Fundamental Theorem of Calculus.

Problem 2. (10%) Show that $\frac{d}{dx}\int_{\sqrt{\log_\pi(e+x^2)}}^{\sqrt{\log_\pi 2}} \pi^{u^2}\,du = -\frac{x}{\sqrt{\ln(e+x^2)\cdot\ln\pi}}$.

Proof. Let $f(x) = \int_{\sqrt{\log_\pi 2}}^{x} \pi^{t^2}\,dt$, and $g(x) = \sqrt{\log_\pi(e+x^2)}$. Then

$$\int_{\sqrt{\log_\pi(e+x^2)}}^{\sqrt{\log_\pi 2}} \pi^{u^2}\,du = -(f\circ g)(x).$$

By the Fundamental Theorem of Calculus, $f'(x) = \pi^{x^2}$. Moreover,

$$\begin{aligned} g'(x) &= \frac{1}{2}\big[\log_\pi(e+x^2)\big]^{-\frac{1}{2}}\frac{d}{dx}\log_\pi(e+x^2) = \frac{1}{2\sqrt{\log_\pi(e+x^2)}}\frac{2x}{e+x^2}\frac{1}{\ln\pi} \\ &= \frac{\sqrt{\ln\pi}}{\sqrt{\ln(e+x^2)}}\frac{x}{e+x^2}\frac{1}{\ln\pi} = \frac{x}{\sqrt{\ln(e+x^2)\cdot\ln\pi}}\frac{1}{e+x^2}\,, \end{aligned}$$

where we have used the change of base formula for the logarithm to conclude the equality. Therefore, the chain rule implies that

$$\begin{aligned} \frac{d}{dx}\int_{\sqrt{\log_\pi(e+x^2)}}^{\sqrt{\log_\pi 2}} \pi^{u^2}\,du &= -f'(g(x))g'(x) = -\pi^{\log_\pi(e+x^2)}\cdot\frac{x}{\sqrt{\ln(e+x^2)\cdot\ln\pi}}\frac{1}{e+x^2} \\ &= -(e+x^2)\frac{x}{\sqrt{\ln(e+x^2)\cdot\ln\pi}}\frac{1}{e+x^2} = -\frac{x}{\sqrt{\ln(e+x^2)\cdot\ln\pi}}\,. \end{aligned}$$ □

Problem 3. (10%) Find all values of α such that the limit $\lim\limits_{x\to 0^+}\frac{x^\alpha}{x^x-1}$ exists.

Solution. Since $\lim\limits_{x\to 1} x^x = 1$, we must have $\alpha > 0$ otherwise the limit will not exist. Therefore, we assume that $\alpha > 0$.

Let $f(x) = x^\alpha$ and $g(x) = x^x - 1$. Then $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} g(x) = 0$. Moreover, $f'(x) = \alpha x^{\alpha-1}$ and $g'(x) = x^x(\ln x + 1)$ so that $\frac{f'(x)}{g'(x)} = \frac{\alpha}{x^x}\frac{\ln x}{\ln x+1}\frac{x^{\alpha-1}}{\ln x}$ for $x \neq 0$. Note that

$$\lim_{x\to 0^+}\frac{\alpha}{x^x} = \alpha \qquad \text{and} \qquad \lim_{x\to 0^+}\frac{\ln x}{\ln x+1} = 1\,.$$

1. If $\alpha \geqslant 1$, then $\lim\limits_{x\to 0^+}\frac{x^{\alpha-1}}{\ln x} = 0$. Therefore, L'Hôspital's rule implies that $\lim\limits_{x\to 0^+}\frac{f(x)}{g(x)} = \lim\limits_{x\to 0^+}\frac{f(x)}{g(x)} = 0$.

2. If $0 < \alpha < 1$, then

   $$\frac{x^{\alpha-1}}{\ln x} = \frac{1}{x^{1-\alpha}\ln x} \to -\infty \text{ as } x \to 0^+$$

   which implies that

   $$\lim_{x\to 0^+}\frac{g'(x)}{f'(x)} = 0\,.$$

Therefore, L'Hôspital's rule implies that $\lim\limits_{x\to 0^+}\dfrac{g(x)}{f(x)} = 0$. This implies that $\lim\limits_{x\to 0^+}\dfrac{f(x)}{g(x)}$ does not exist for otherwise it will implies that

$$1 = \lim_{x\to 0^+}\frac{f(x)}{g(x)}\frac{g(x)}{f(x)} = \lim_{x\to 0^+}\frac{f(x)}{g(x)}\lim_{x\to 0^+}\frac{g(x)}{f(x)} = 0\,,$$

a contradiction.

Therefore, the desired limit to exist if and only if $\alpha \geqslant 1$. □

Problem 4. (10%) Find the indefinite integral $\displaystyle\int \frac{4}{(e^x+e^{-x})^3}\,dx$.

Solution. Let $u = e^x$. Then $du = e^x\,dx$ or $\dfrac{du}{u} = dx$; thus

$$\begin{aligned}\int \frac{4}{(e^x+e^{-x})^3}\,dx &= \int \frac{4}{(u+u^{-1})^3}\frac{du}{u} = \int \frac{4u^2}{(1+u^2)^3}\,du \overset{(u=\tan t)}{=} \int \frac{4\tan^2 t}{\sec^6 t}\sec^2 t\,dt = \int 4\sin^2 t\cos^2 t\,dt \\ &= \int \sin^2 2t\,dt = \int \frac{1-\cos 4t}{4}\,dt = \frac{1}{4}\Big(t - \frac{\sin 4t}{4}\Big) + C \\ &= \frac{1}{16}\Big[4\arctan u - \sin(4\arctan u)\Big] + C \\ &= \frac{1}{16}\Big[4\arctan e^x - \sin(4\arctan e^x)\Big] + C\,.\end{aligned}$$

□

Problem 5. (10%) Find the definite integral $\displaystyle\int_0^{\frac{\pi}{4}} \sec^3 x\,dx$ using the substitution of variable $t = \sec x + \tan x$.

Hint: Show that $\dfrac{1}{t} = \sec x - \tan x$.

Solution. Let $t = \sec x + \tan x = \dfrac{1+\sin x}{\cos x} = \dfrac{\cos x}{1-\sin x}$. Then

$$\frac{1}{t} = \frac{\cos x}{1+\sin x} = \frac{1-\sin x}{\cos x} = \sec x - \tan x\,;$$

thus

$$\sec x = \frac{1}{2}\Big(t+\frac{1}{t}\Big) = \frac{1+t^2}{2t} \qquad \text{and} \qquad \tan x = \frac{1}{2}\Big(t-\frac{1}{t}\Big) = \frac{t^2-1}{2t}\,.$$

Moreover,

$$dt = \sec x(\tan x + \sec x)\,dx = \frac{1+t^2}{2}\,dx \qquad \text{or equivalently,} \qquad \frac{2}{1+t^2}\,dt = dx\,.$$

Since $\sec 0 + \tan 0 = 1$ and $\sec\dfrac{\pi}{4} + \tan\dfrac{\pi}{4} = 1+\sqrt{2}$, the substitution of variable formula implies that

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sec^3 x\,dx &= \int_1^{1+\sqrt{2}} \frac{(1+t^2)^3}{8t^3}\frac{2}{1+t^2}\,dt = \frac{1}{4}\int_1^{1+\sqrt{2}} \frac{1+2t^2+t^4}{t^3}\,dt \\ &= \frac{1}{4}\Big(\frac{-1}{2}t^{-2} + 2\ln t + \frac{1}{2}t^2\Big)\Big|_{t=1}^{t=1+\sqrt{2}} = \frac{1}{4}\Big(\frac{1}{2}\big[(1+\sqrt{2})^2 - (1+\sqrt{2})^{-2}\big] + 2\ln(1+\sqrt{2})\Big) \\ &= \frac{1}{8}\Big[(1+\sqrt{2})^2 - (1-\sqrt{2})^2\Big] + \frac{1}{2}\ln(1+\sqrt{2}) = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2})\,.\end{aligned}$$

□

Problem 6. (10%) Show that $\arctan x < \sum_{k=0}^{2n} \frac{(-1)^k x^{2k+1}}{2k+1}$ for all $x > 0$ and $n \in \mathbb{N}$.

Proof. Let $f(x) = \sum_{k=0}^{2n} \frac{(-1)^k x^{2k+1}}{2k+1} - \arctan x$. Then

$$\begin{aligned} f'(x) &= \sum_{k=0}^{2n} (-1)^k x^{2k} - \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + x^{4n-4} - x^{4n-2} + x^{4n} - \frac{1}{1+x^2} \\ &= \frac{1 - x^2 + \cdots + x^{4n} + x^2(1 - x^2 + \cdots - x^{4n-2} + x^{4n}) - 1}{1+x^2} = \frac{x^{4n+2}}{1+x^2} \end{aligned}$$

which is positive when $x > 0$. Therefore, f is strictly increasing on $[0, \infty)$; thus

$$f(x) > f(0) = 0 \qquad \text{if} \quad x > 0$$

which concludes the proof. □

Problem 7. Let a be a real number and $a \neq 1$, and $f(x) = x^a \ln x$. Compute $\int f(x)\,dx$ by completing the following.

1. (2%) Let $b > 1$, and $\mathcal{P} = \{1 = x_0 < x_1 < \cdots < x_n = b\}$, where $x_k = r^k$, be the "geometric" partition of $[1, b]$. Show that the Riemann sum of f for $\mathcal{P}$ using the right end-point rule is
$$I_n = (r-1)\ln r \sum_{k=1}^{n} k r^{(a+1)k-1}.$$
2. (3%) Show that $\sum_{k=1}^{n} k r^{(a+1)k-1} = \frac{1}{(r^{a+1}-1)^2}\Big[n r^{(a+1)(n+1)+a} - (n+1) r^{(a+1)(n+1)-1} + r^a\Big]$.
3. (10%) Find $\int_1^b x^a \ln x\,dx$ for $b > 1$ by computing the limit $\lim_{n\to\infty} I_n$.
4. (5%) Find $\int x^a \ln x\,dx$.

Proof. First we consider the definite integral $\int_1^b x^a \ln x\,dx$ for $b > 1$. Let $r = b^{\frac{1}{n}}$, and $x_k = r^k$. Then the Riemann sum of $y = x^a \ln x$ for partition $\mathcal{P} = \{1 = x_0 < x_1 < \cdots < x_n = a\}$ given by the right end-point rule is given by

$$\sum_{k=1}^{n} x_k^a \ln x_k (x_k - x_{k-1}) = \sum_{k=1}^{n} r^{ka} \ln(r^k)(r^k - r^{k-1}) = (r-1)\ln r \sum_{i=1}^{n} k r^{(a+1)k-1}.$$

Since $\sum_{k=1}^{n} r^{(a+1)k} = \frac{r^{(a+1)(n+1)} - r^{a+1}}{r^{a+1}-1}$ if $a \neq -1$,

$$\begin{aligned} \frac{d}{dr}\sum_{k=1}^{n} r^{(a+1)k} &= \frac{\big[(a+1)(n+1)r^{(a+1)(n+1)-1} - (a+1)r^a\big](r^{a+1}-1) - (a+1)r^a\big[r^{(a+1)(n+1)} - r^{a+1}\big]}{(r^{a+1}-1)^2} \\ &= \frac{a+1}{(r^{a+1}-1)^2}\Big[n r^{(a+1)(n+1)+a} - (n+1) r^{(a+1)(n+1)-1} + r^a\Big]; \end{aligned}$$

thus by the fact that $\frac{d}{dr}\sum\limits_{k=1}^{n} r^{(a+1)k} = (a+1)\sum\limits_{k=1}^{n} kr^{(a+1)k-1}$ and $r^n = b$, we find that

$$\begin{aligned}
\sum_{k=1}^{n} r^{ka}\ln(r^k)(r^k - r^{k-1}) &= \frac{(r-1)\ln r}{(r^{a+1}-1)^2}\Big[nr^{(a+1)(n+1)+a} - (n+1)r^{(a+1)(n+1)-1} + r^a\Big] \\
&= \frac{r-1}{(r^{a+1}-1)^2}\Big[nb^{a+1}r^{2a+1}\ln r - b^{a+1}r^a(n+1)\ln r + r^a\ln r\Big] \\
&= \frac{r-1}{(r^{a+1}-1)^2}\Big[b^{a+1}(r^{2a+1} - r^a)\ln b + (1-b^{a+1})r^a\ln r\Big] \\
&= \frac{r-1}{r^{a+1}-1}b^{a+1}r^a\ln b + \frac{(r-1)\ln r}{(r^{a+1}-1)^2}(1-b^{a+1})r^a\,.
\end{aligned}$$

Since $r \to 1$ if and only if $n \to \infty$, L'Hôspital's Rule implies that

$$\lim_{n\to\infty}\frac{r-1}{r^{a+1}-1} = \lim_{r\to 1}\frac{r-1}{r^{a+1}-1} = \lim_{r\to 1}\frac{1}{(a+1)r^a} = \frac{1}{a+1}$$

and

$$\lim_{n\to\infty}\frac{(r-1)\ln r}{(r^{a+1}-1)^2} = \lim_{r\to 1}\frac{(r-1)\ln r}{(r^{a+1}-1)^2} = \lim_{r\to 1}\frac{\ln r + \frac{r-1}{r}}{2(a+1)(r^{a+1}-1)} = \lim_{r\to 1}\frac{\frac{1}{r} + \frac{1}{r^2}}{2(a+1)^2r^a} = \frac{1}{(a+1)^2}\,.$$

Therefore,

$$\int_1^b x^a\ln x\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} r^{ka}\ln(r^k)(r^k - r^{k-1}) = \frac{b^{a+1}\ln b}{a+1} + \frac{1-b^{a+1}}{(a+1)^2}$$

which suggests that

$$\int x^a\ln x\,dx = \frac{x^{a+1}\ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2} + C\,. \qquad (\star)$$

To verify this, we differentiate the right-hand side and find that

$$\frac{d}{dx}\Big[\frac{x^{a+1}\ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2}\Big] = x^a\ln x + \frac{x^a}{a+1} - \frac{x^a}{a+1} = x^a\ln x$$

which indeed shows $(\star)$. □

Problem 8. (15%) Let y be a function satisfying

$$\sqrt{1+x^2}\,y' + y = x \qquad \textit{and} \qquad y(0) = 0\,.$$

Find $y(1)$.

Solution. First we rewrite the differential equation $\sqrt{1+x^2}\,y' + y = x$ as

$$y' + \frac{1}{\sqrt{1+x^2}}y = \frac{x}{\sqrt{1+x^2}} \qquad (\star)$$

Since

$$\begin{aligned}
\int\frac{1}{\sqrt{1+x^2}}\,dx &\overset{(x=\tan u)}{=} \int\frac{\sec^2 u}{\sqrt{1+\tan^2 u}}\,du = \int\sec u\,du = \ln\big|\sec u + \tan u\big| + C \\
&= \ln\big(x + \sqrt{x^2+1}\big) + C\,,
\end{aligned}$$

an integrating factor of $(\star)$ is

$$e^{\ln(x+\sqrt{x^2+1})} = x + \sqrt{x^2+1}\,;$$

thus

$$\frac{d}{dx}\big[(x+\sqrt{x^2+1})y\big] = (x+\sqrt{x^2+1})\cdot\frac{x}{\sqrt{1+x^2}} = \sqrt{1+x^2} - \frac{1}{\sqrt{1+x^2}} + x$$

which implies that

$$\begin{aligned}(x+\sqrt{x^2+1})y(x) - y(0) &= \int_0^x \Big[\sqrt{1+t^2} - \frac{1}{\sqrt{1+t^2}} + t\Big]\,dt \\ &= \int_0^x \sqrt{1+t^2}\,dt - \ln\big(x+\sqrt{x^2+1}\big) + \frac{x^2}{2}\,.\end{aligned}$$

Therefore, by the fact that $y(0)=0$, the identity above implies that

$$(1+\sqrt{2})y(1) = \int_0^1 \sqrt{1+t^2}\,dt - \ln(1+\sqrt{2}) + \frac{1}{2} \overset{(t=\tan u)}{=} \int_0^{\frac{\pi}{4}} \sec^3 u\,du - \ln(1+\sqrt{2}) + \frac{1}{2}$$

and Problem 5 further shows that

$$(1+\sqrt{2})y(1) = \frac{1+\sqrt{2}}{2} - \frac{1}{2}\ln(1+\sqrt{2})\,.$$

Therefore,

$$y(1) = \frac{1}{2} - \frac{1}{2(1+\sqrt{2})}\ln(1+\sqrt{2}) = \frac{1}{2} - \frac{\sqrt{2}-1}{2}\ln(1+\sqrt{2})\,.$$

□