# Exercise Problem Sets 8

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**Problem 1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that

$$f(x,y) + f(y,z) + f(z,x) = 0 \qquad \forall\, x, y, z \in \mathbb{R}\,.$$

Show that there exists $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x,y) = g(x) - g(y) \qquad \forall\, x, y \in \mathbb{R}\,.$$

**Problem 2.** In the following sub-problems, find the limit if it exists or explain why it does not exist.

(1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y}$ (2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2}$ (3) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ (4) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

(5) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ (6) $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$ (7) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$

(8) $\displaystyle\lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}$ (9) $\displaystyle\lim_{(x,y)\to(0,0)} x\cos\frac{1}{y}$ (10) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

(11) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+zx}{x^2+y^2+z^2}$ (12) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$ (13) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \arctan\frac{1}{x^2+y^2+z^2}$

**Problem 3.** Discuss the continuity of the functions given below.

1. $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{xy} & \text{if } xy \neq 0\,, \\ 1 & \text{if } xy = 0\,. \end{cases}$

2. $f(x,y) = \begin{cases} \dfrac{e^{-x^2-y^2}-1}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 1 & \text{if } (x,y) = (0,0)\,. \end{cases}$

3. $f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$

**Problem 4.** Let $f(x,y) = \begin{cases} 0 & \text{if } y \leqslant 0 \text{ or } y \geqslant x^4\,, \\ 1 & \text{if } 0 < y < x^4\,. \end{cases}$

1. Show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along any path through $(0,0)$ of the form $y = mx^\alpha$ with $0 < \alpha < 4$.

2. Show that $f$ is discontinuous on two entire curves.

**Problem 5.** Find $\displaystyle\frac{\partial}{\partial x}\bigg|_{(x,y,z)=(\ln 4,\ln 9,2)} \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!z^n}$. Do not write the answer in terms of an infinite sum.

**Problem 6.** Let $f(x,y) = (x^2+y^2)^{\frac{2}{3}}$. Find the partial derivative $\dfrac{\partial f}{\partial x}$.

**Problem 7.** Let $f(x,y,z) = xy^2z^3 + \arcsin(x\sqrt{z})$. Find $f_{xzy}$ in the region $\big\{(x,y,z) \,\big|\, |x^2z| < 1\big\}$.

**Problem 8.** Let $\vec{\boldsymbol{a}} = (a_1, a_2, \cdots, a_n)$ be a unit vector, $\vec{\boldsymbol{x}} = (x_1, x_2, \cdots, x_n)$, and $f(x_1, x_2, \cdots, x_n) = \exp(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{x}})$. Show that

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = f\,.$$

**Problem 9.** Let $f(x,y) = x(x^2+y^2)^{-\frac{3}{2}} e^{\sin(x^2y)}$. Find $f_x(1,0)$.

**Problem 10.** Let $f(x,y) = \displaystyle\int_1^y \frac{dt}{\sqrt{1-x^3t^3}}$. Show that

$$f_x(x,y) = \int_1^y \Big(\frac{\partial}{\partial x}\frac{1}{\sqrt{1-x^3t^3}}\Big)\,dt$$

in the region $\big\{(x,y) \,\big|\, x<1, y>1 \text{ and } xy<1\big\}$.

**Problem 11.** The gas law for a fixed mass $m$ of an ideal gas at absolute temperature $T$, pressure $P$, and volume $V$ is $PV = mRT$, where $R$ is the gas constant. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1\,.$$

**Problem 12.** The total resistance $R$ produced by three conductors with resistances $R_1$, $R_2$, $R_3$ connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\,.$$

Find $\dfrac{\partial R}{\partial R_1}$ by directly taking the partial derivative of the equation above.

**Problem 13.** Find the value of $\dfrac{\partial z}{\partial x}$ at the point $(1,1,1)$ if the equation

$$xy + z^3x - 2yz = 0$$

defines $z$ as a function of the two independent variables $x$ and $y$ and the partial derivative exists.

**Problem 14.** Find the value of $\dfrac{\partial x}{\partial z}$ at the point $(1,-1,-3)$ if the equation

$$xz + y\ln x - x^2 + 4 = 0$$

defines $x$ as a function of the two independent variables $y$ and $z$ and the partial derivative exists.

**Problem 15.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f_x(a,b)$ and $f_y(a,b)$ exists. Suppose that $c = f(a,b)$.

1. Using the geometric meaning of partial derivatives, explain what the vectors $(1,0,f_x(a,b))$ and $(0,1,f_y(a,b))$ mean.

2. Suppose that you know that there is a tangent plane (which we have not talked about, but you can roughly imagine what it is) of the graph of $f$ at $(a,b,c)$. What should the equation of the tangent plane be?