Exercise Problem Sets 10

Nov. 24. 2023

Problem 1. Compute $\int_{-\sqrt{\frac{3}{2}}}^{1} \frac{dx}{\sqrt{2-x^2}}$ using the following substitution of variables:

1. $x = \sqrt{2}\sin t$. 2. $x = -\sqrt{2}\sin t$. 3. $x = \sqrt{2}\cos t$. 4. $x = -\sqrt{2}\cos t$.

Problem 2. Find the definite integral $\int_0^{\frac{\pi}{2}} \frac{\sin x dx}{1+\cos^2 x}$.

Solution. Let $u = \cos x$. Then $du = -\sin x dx$; thus

$$\int_0^{\frac{\pi}{2}} \frac{\sin x dx}{1+\cos^2 x} = \int_1^0 \frac{-du}{1+u^2} = \int_0^1 \frac{du}{1+u^2}\,.$$

Let $u = \tan t$. Then $du = \sec^2 t dt$; thus

$$\int_0^{\frac{\pi}{2}} \frac{\sin x dx}{1+\cos^2 x} = \int_0^{\frac{\pi}{4}} \frac{\sec^2 t dt}{1+\tan^2 t} = \frac{\pi}{4}\,,$$

where we have used $1+\tan^2 t = \sec^2 t$ to conclude that last equality. □

Problem 3. Find the following indefinite integrals.

1. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$. 2. $\int \frac{1}{x^3}\cos^2\frac{1}{x^2}\,dx$. 3. $\int \frac{1}{x^2}\cos^3\frac{1}{x}\,dx$. 4. $\int \frac{\sin\sqrt{x}}{\sqrt{x\cos^3\sqrt{x}}}\,dx$.

Solution. 3. Let $u = 1/x$. Then $du = -1/x^2 dx$; thus

$$\begin{aligned}\int \frac{1}{x^2}\cos^3\frac{1}{x}\,dx &= -\int \cos^3 u\,dt = -\int \frac{\cos 3u + 3\cos u}{4}\,du = -\frac{\sin 3u}{12} - \frac{3\sin u}{4} + C \\ &= -\frac{1}{2}\sin\frac{3}{x} - \frac{1}{4}\sin\frac{1}{x} + C\,.\end{aligned}$$

4. Let $u = \cos\sqrt{x}$. Then $du = -\frac{\sin\sqrt{x}}{2\sqrt{x}}dx$; thus

$$\int \frac{\sin\sqrt{x}}{\sqrt{x\cos^3\sqrt{x}}}\,dx = -2\int \frac{du}{\sqrt{u}^3} = -2\int u^{-\frac{3}{2}}\,du = 4u^{-\frac{1}{2}} + C = \frac{4}{\sqrt{\cos\sqrt{x}}} + C\,.$$ □

Problem 4. Find the following indefinite integrals.

1. $\int \tan^3 x\sec^2 x\,dx$, $\int \frac{\sec^2 x}{\tan^2 x}\,dx$, $\int \tan^2 x\sec^2 x\,dx$. Use your experience on these three integrals to find $\int \tan^m x\sec^2 x\,dx$ for $m \neq -1$.

2. $\int \sec^3 x\tan x\,dx$, $\int \sec^5 x\tan x\,dx$, $\int \frac{\tan x}{\sec^3 x}\,dx$. Use your experience on these three integrals to find $\int \sec^m x\tan x\,dx$ for $m \neq 0$.

Problem 5. Compute the indefinite integral $\int \sin^6 x\,dx$ by the following steps:

1. Write $\sin^6 x = \sin^3 x \cdot \sin^3 x$, and use the triple and double angle formula, as well as the product to sum formula, to show that

$$\sin^6 x = -\frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x - \frac{15}{32}\cos 2x + \frac{5}{16}\,.$$

2. Find the indefinite integral $\displaystyle\int \sin^6 x\,dx$ using the identity above.

Problem 6. Compute the indefinite integral $\displaystyle\int \cos^5 x\,dx$ by the following steps:

1. Write $\cos^5 x = \cos^3 x \cdot \cos^2 x$, and use the triple and double angle formula, as well as the product to sum formula, to show that

$$\cos^5 x = \frac{1}{16}\cos 5x + \frac{5}{16}\cos 3x + \frac{5}{8}\cos x\,.$$

2. Find the indefinite integral $\displaystyle\int \cos^5 x\,dx$ using the identity above.

Proof. 1. Since $\cos 3x = 4\cos^3 x - 3\cos x$, we have $\cos^3 x = \dfrac{\cos 3x + 3\cos x}{4}$; thus

$$\begin{aligned}
\cos^5 x = \cos^3 x \cdot \cos^2 x &= \frac{\cos 3x + 3\cos x}{4} \cdot \frac{1 + \cos 2x)}{2} \\
&= \frac{\cos 3x + 3\cos x + \cos 3x \cos 2x + 3\cos x \cos 2x}{8} \\
&= \frac{\cos 3x + 3\cos x + \frac{1}{2}\big[\cos(3x-2x) + \cos(3x+2x)\big] + \frac{3}{2}\big[\cos(2x-x) + \cos(2x+x)\big]}{8} \\
&= \frac{\cos 5x + 5\cos 3x + 10\cos x}{16} = \frac{1}{16}\cos 5x + \frac{5}{16}\cos 3x + \frac{5}{8}\cos x\,.
\end{aligned}$$

2. From 1, we have

$$\int \cos^5 x\,dx = \int \Big(\frac{1}{16}\cos 5x + \frac{5}{16}\cos 3x + \frac{5}{8}\cos x\Big)\,dx = \frac{1}{80}\sin 5x + \frac{5}{48}\sin 3x + \frac{5}{8}\sin x + C\,. \quad \square$$

Problem 7. 1. Let $f : [0,a] \to \mathbb{R}$ be a continuous function. Find $\displaystyle\int_0^a \frac{f(x)}{f(x) + f(a-x)}\,dx$.

2. Find $\displaystyle\int_0^1 \frac{\sin x}{\sin x + \sin(1-x)}\,dx$. 3. Find $\displaystyle\int_0^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{3-x}}\,dx$.

Problem 8. Let $f : [-1,1] \to \mathbb{R}$ be a continuous function.

1. Show that

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx\,.$$

2. Use the identity above to find the integrals $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ and $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$.

Problem 9. Let a and b be positive (rational) numbers. Show that

$$\int_0^1 x^a (1-x)^b\,dx = \int_0^1 x^b (1-x)^a\,dx\,.$$

Problem 10. Let $a_0, a_1, \cdots, a_n$ be real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0 .$$

Show that the polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ has at least one real zero.

Problem 11. Let I be an interval, and $f : I \to \mathbb{R}$ be one-to-one, onto and continuous. Show that if $g : \mathbb{N} \to \mathbb{R}$ is a function satisfying that $\lim\limits_{n\to\infty} f(g(n)) = b$, then $\lim\limits_{n\to\infty} g(n) = f^{-1}(b)$.

Problem 12. Show that the following functions (defined by integrals) are one-to-one and find $(f^{-1})'(0)$.

1. $f(x) = \displaystyle\int_2^x \sqrt{1+t^2}\,dt$.
2. $f(x) = \displaystyle\int_2^x \frac{dt}{\sqrt{1+t^4}}$.

Problem 13. Let f be an one-to-one, twice differentiable function with an inverse function g.

1. Show that g is twice differentiable function and find g''.
2. Show that if in addition f is strictly increasing and the graph of f is concave upward, then the graph of g is concave downward.

Problem 14. Show that for all natural number n,

$$\sum_{k=1}^{2n} \frac{(-1)^{k-1}x^k}{k} \leqslant \ln(1+x) \leqslant \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}x^k}{k} \qquad \forall\, x > 0 .$$

Proof. Define

$$f(x) = \ln(1+x) - \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}x^k}{k} \qquad x > 0$$

and

$$g(x) = \sum_{k=1}^{2n} \frac{(-1)^{k-1}x^k}{k} - \ln(1+x) \qquad x > 0 .$$

Then

$$\begin{aligned} f'(x) &= \frac{1}{1+x} - \sum_{k=1}^{2n-1} (-1)^{k-1}x^{k-1} = \frac{1}{1+x} - \big[1 - x + x^2 - x^3 + \cdots + (-1)^{2n-2}x^{2n-2}\big] \\ &= \frac{1 - \big[1 - x + x^2 - x^3 + \cdots + (-1)^{2n-2}x^{2n-2}\big] - x\big[1 - x + x^2 - x^3 + \cdots + (-1)^{2n-2}x^{2n-2}\big]}{1+x} \\ &= \frac{-x^{2n-1}}{1+x} < 0 . \end{aligned}$$

and

$$\begin{aligned} g'(x) &= \sum_{k=1}^{2n} (-1)^{k-1}x^{k-1} - \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^{2n-1}x^{2n-1} - \frac{1}{1+x} \\ &= \frac{1 - x + x^2 - x^3 + \cdots + (-1)^{2n-1}x^{2n-1} + x\big[1 - x + x^2 - x^3 + \cdots + (-1)^{2n-1}x^{2n-1}\big] - 1}{1+x} \\ &= \frac{-x^{2n}}{1+x} < 0 . \end{aligned}$$

Therefore, f and g are both decreasing on $(0, \infty)$; thus by the continuity of f and g on $[0, \infty)$, we have

$$f(x) < f(0) = 0 \qquad \text{and} \qquad g(x) < g(0) = 0$$

so we obtain the desired inequality. □

Problem 15. Use implicit differentiation to find $\frac{dy}{dx}$, where (x, y) satisfies the relation $4xy + \ln x^2 y = 7$.

Problem 16. Locate any relative extrema and points of inflection of the function $y = x^2 \ln \frac{x}{4}$.

Problem 17. Use the substitution of variable $t = \tan \frac{x}{2}$ to find the integral $\int \csc x \, dx$.

Problem 18. Find the following indefinite integrals.

1. $\int \frac{(\ln x)^2}{x} dx$.
2. $\int \frac{\ln \sqrt{x}}{x} dx$.
3. $\int \frac{dx}{x(\ln x^2)^3}$.
4. $\int \frac{(1 + \ln x)^2}{x} dx$.
5. $\int \frac{\sin(\ln x)}{x} dx$.
6. $\int \frac{\sin 2x}{1 + \cos^2 x} dx$.

Problem 19. Show that $\frac{1}{y} < \frac{\ln x - \ln y}{x - y} < \frac{1}{x}$ for all $0 < x < y$.

Proof. Let $0 < x < y$. By the mean theorem, there exists $z \in (x, y)$ such that

$$\Big(\frac{d}{dt}\Big|_{t=z} \ln t\Big)(x - y) = \ln x - \ln y \qquad \Leftrightarrow \qquad \frac{1}{z} = \frac{\ln x - \ln y}{x - y}.$$

Since $0 < x < z < y$, we have $\frac{1}{y} < \frac{1}{z} < \frac{1}{x}$; thus we conclude the desired inequality. □